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Squares of Fibonacci-like Numbers

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Abstract. Given arbitrary integers G_0 and G_1, not both zero, Fibonacci-like numbers, G_j, are defined for all non-megative integers j by the recurrence relation $G_j = G_{j-1} + G_{j-2}$ $(j \geq 2)$. In this paper, we derive a general identity involving the squares of Fibonacci-like numbers. Closed formulas exist for $\sum_{j=0}^{n} x^j G_j^2$ and $\sum_{j=0}^{n} G_{j+k} G_{j-k}$. We extend these results by providing evaluations for $\sum_{j=0}^{n} x^j G_{j+k}^2$ and $\sum_{j=0}^{n} x^j G_{j+k} G_{j+s}$ for integers n, s and k and a real or complex variable x. Various other properties are developed, including double binomial summation identites. As a bonus unexpected result we derive a generalization of the alternating sum of the products of three consecutive Fibonacci-like numbers.

Key Words: Fibonacci number, Lucas number, Fibonacci-like number, generating function.
2020 MSC: Primary 11B39; Secondary 11B37.

Dedicated to our Professor David E. Dobbs for his 80th Birthday.

1 Introduction

The Fibonacci numbers, F_j, and the Lucas numbers, L_j, $j \in \mathbb{Z}$, are defined by:

$$F_0 = 0,\ F_1 = 1,\ F_j = F_{j-1} + F_{j-2}\ (j \geq 2), \quad F_{-j} = (-1)^{j-1} F_j \tag{1.1}$$

and

$$L_0 = 2,\ L_1 = 1,\ L_j = L_{j-1} + L_{j-2}\ (j \geq 2), \quad L_{-j} = (-1)^j L_j. \tag{1.2}$$

Both $(F_j)_{j\in\mathbb{Z}}$ and $(L_j)_{j\in\mathbb{Z}}$ are examples of a Fibonacci-like sequence. We define a Fibonacci-like sequence (also called a gibonacci sequence), $(G_j)_{j\in\mathbb{Z}}$, as one having the same recurrence relation as the Fibonacci sequence, but with arbitrary initial terms. Thus, given arbitrary integers G_0 and G_1, not both zero, we define

$$G_j = G_{j-1} + G_{j-2}\ (j \geq 2); \tag{1.3}$$

and also extend the definition to negative subscripts by writing the recurrence relation as

$$G_{-j} = G_{-j+2} - G_{-j+1}. \tag{1.4}$$

We have [2, equation (1.5)]

$$G_{-j} = (-1)^j (G_0 L_j - G_j). \tag{1.5}$$

The identity (see Brousseau [4, equation (2)])

$$F_{j-1}^2 + F_{j+2}^2 = 2F_j^2 + 2F_{j+1}^2, \tag{1.6}$$

or, more generally,

$$G_{j-1}^2 + G_{j+2}^2 = 2G_j^2 + 2G_{j+1}^2, \tag{1.7}$$

is well known.

Less familiar are identities such as

$$G_{j+2}^2 + 2G_{j-2}^2 = 3G_{j-1}^2 + 6G_j^2, \tag{1.8}$$

$$3G_{j+3}^2 + G_{j-3}^2 = 16G_{j+1}^2 + 12G_j^2 \tag{1.9}$$

and

$$F_k F_{k+1} G_{j+1}^2 - F_k F_{k-1} G_{j-1}^2 = G_{j+k}^2 - F_{k+1} F_{k-1} G_j^2. \tag{1.10}$$

Our aim in writing this paper is to derive the identity

$$\begin{aligned} F_s F_m F_{m-s} G_{j+k}^2 = F_{m-s} F_{m-k} F_{s-k} G_j^2 + (-1)^{s+k} F_k F_m F_{m-k} G_{j+s}^2 \\ - (-1)^{s+k} F_k F_s F_{s-k} G_{j+m}^2, \end{aligned}$$

of which (1.7), (1.8), (1.9) and (1.10) are particular cases, being evaluations at certain m, k, j and s choices.

Closed formulas are known for $\sum_{j=0}^{n} x^j G_j^2$ and $\sum_{j=0}^{n} G_{j+k} G_{j-k}$. We will extend these results by providing evaluations for $\sum_{j=0}^{n} x^j G_{j+k}^2$ and $\sum_{j=0}^{n} x^j G_{j+k} G_{j+s}$ for integers n, s and k and arbitrary x.

As a bonus surprise result we will discover a generalization (Theorem 4.3) of the following alternating sum of products of three consecutive Fibonacci-like numbers:

$$\begin{aligned} 2\sum_{j=0}^{n} (-1)^j G_{j+k+1} G_{j+k} G_{j+k-1} = (-1)^n G_{n+k+1}^3 + (-1)^{n+1} G_{n+k+2} G_{n+k}^2 \\ + G_k^3 - G_{k+1} G_{k-1}^2. \end{aligned}$$

Finally, we will derive double binomial identities involving the squares of Fibonacci-like numbers.

2 Main identity

The main identity with which we are concerned, proved in Theorem 2.2, utilizes the result stated in Lemma 2.1.

Lemma 2.1. *The following identity holds for arbitrary integers s, k, m and j:*

$$F_{s-k}G_{j+m} = F_{m-k}G_{j+s} + (-1)^{s+k+1}F_{m-s}G_{j+k}.$$

Proof. Let h, n and r be arbitrary integers. Let $(G_j)_{j\in\mathbb{Z}}$ and $(H_j)_{j\in\mathbb{Z}}$ be two Fibonacci-like sequences. It is known [5, Formula (18)] that:

$$H_nG_{n+h+r} = H_{n+h}G_{n+r} + (-1)^{n+1}\left(H_hG_r - H_0G_{h+k}\right).$$

Choosing $(H_j)_{j\in\mathbb{Z}}$ as the Fibonacci sequence gives

$$F_nG_{n+h+r} = F_{n+h}G_{n+r} + (-1)^{n+1}F_hG_r. \tag{2.1}$$

Setting $h = m-s$, $n = s-k$ and $r = j+k$ in (2.1) gives the identity stated in Lemma 2.1. □

Theorem 2.2. If j, k, m and s are integers, then

$$\begin{aligned} F_sF_mF_{m-s}G_{j+k}^2 &= F_{m-s}F_{m-k}F_{s-k}G_j^2 + (-1)^{s+k}F_kF_mF_{m-k}G_{j+s}^2 \\ &\quad - (-1)^{s+k}F_kF_sF_{s-k}G_{j+m}^2. \end{aligned}$$

Proof. Setting $m = 0$ in the identity stated in Lemma 2.1 gives

$$(-1)^kF_{s-k}G_j = F_sG_{j+k} - F_kG_{j+s}, \tag{2.2}$$

from which, by squaring and re-arranging, we get

$$2F_sF_kG_{j+k}G_{j+s} = F_s^2G_{j+k}^2 + F_k^2G_{j+s}^2 - F_{s-k}^2G_j^2. \tag{2.3}$$

The statement of the theorem now follows by squaring the identity stated in Lemma 2.1 and using (2.3) to eliminate the cross-term $G_{j+k}G_{j+s}$ from the right hand side, while making use also of the multiplication formula

$$5F_mF_n = L_{m+n} - (-1)^nL_{m-n}. \tag{2.4}$$

□

3 Partial sums and generating function

Theorem 3.1. The following identity holds for arbitrary x and integers k and n:

$$\begin{aligned} \sum_{j=0}^{n} x^jG_{j+k}^2 &= \frac{xG_{k+1}^2 - (2x-1)G_k^2 - x^2G_{k-1}^2}{x^3-2x^2-2x+1} \\ &\quad - \frac{x^{n+1}\left(xG_{n+k+2}^2 - (2x-1)\,G_{n+k+1}^2 - x^2\,G_{n+k}^2\right)}{x^3-2x^2-2x+1}. \end{aligned} \tag{3.1}$$

Proof. Write $j+k$ for j in (1.7), multiply through by x^j and sum to obtain

$$\sum_{j=0}^{n} x^j G_{j+k-1}^2 + \sum_{j=0}^{n} x^j G_{j+k+2}^2 = 2\sum_{j=0}^{n} x^j G_{j+k}^2 + 2\sum_{j=0}^{n} x^j G_{j+k+1}^2. \tag{3.2}$$

Shifting the summation index in each case gives

$$\sum_{j=0}^{n} x^j G_{j+k-1}^2 = G_{k-1}^2 - x^{n+1} G_{n+k}^2 + xS(x;n,k), \tag{3.3}$$

$$\sum_{j=0}^{n} x^j G_{j+k+1}^2 = -\frac{1}{x} G_k^2 + x^n G_{n+k+1}^2 + \frac{1}{x} S(x;n,k), \tag{3.4}$$

and

$$\sum_{j=0}^{n} x^j G_{j+k+2}^2 = -\frac{1}{x^2} G_k^2 - \frac{1}{x} G_{k+1}^2 + x^{n-1} G_{n+k+1}^2 + x^n G_{n+k+2}^2 + \frac{1}{x^2} S(x;n,k); \tag{3.5}$$

where

$$S(x;n,k) = \sum_{j=0}^{n} x^j G_{j+k}^2.$$

Substituting (3.3), (3.4) and (3.5) into (3.2) and solving for $S(x;n,k)$ gives the identity of Theorem 3.1. □

In particular,

$$\sum_{j=0}^{n} x^j G_j^2 = \frac{xG_1^2 - (2x-1)G_0^2 - x^2(G_1-G_0)^2}{x^3-2x^2-2x+1} - \frac{x^{n+1}\left(xG_{n+2}^2 - (2x-1)G_{n+1}^2 - x^2 G_n^2\right)}{x^3-2x^2-2x+1}; \tag{3.6}$$

so that

$$\sum_{j=0}^{n} x^j F_j^2 = \frac{x-x^2}{x^3-2x^2-2x+1} - \frac{x^{n+1}\left(xF_{n+2}^2 - (2x-1)F_{n+1}^2 - x^2 F_n^2\right)}{x^3-2x^2-2x+1} \tag{3.7}$$

and

$$\sum_{j=0}^{n} x^j L_j^2 = \frac{4-7x-x^2}{x^3-2x^2-2x+1} - \frac{x^{n+1}\left(xL_{n+2}^2 - (2x-1)L_{n+1}^2 - x^2 L_n^2\right)}{x^3-2x^2-2x+1}. \tag{3.8}$$

We also have

$$\sum_{j=0}^{n} G_{j+k}^2 = G_{n+k}G_{n+k+1} - G_{k-1}G_k, \tag{3.9}$$

with the special value

$$\sum_{j=0}^{n} G_j^2 = G_nG_{n+1} - (G_1 - G_0)G_0, \tag{3.10}$$

giving the classical results

$$\sum_{j=0}^{n} F_j^2 = F_nF_{n+1}, \quad \sum_{j=0}^{n} L_j^2 = L_nL_{n+1} + 2. \tag{3.11}$$

Observe that setting $x = -1$ in the identity of Theorem 3.1 makes the right hand side to be an indeterminate form. Application of L'Hospital's rule however provides

$$\begin{aligned}\sum_{j=0}^{n} (-1)^j G_{j+k}^2 = (-1)^n \left(\frac{n+2}{5} G_{n+k+2}^2 - \frac{3n+5}{5} G_{n+k+1}^2 + \frac{n+3}{5} G_{n+k}^2 \right) \\ + \frac{1}{5} G_{k+1}^2 - \frac{2}{5} G_{k+1}G_{k-2}.\end{aligned} \tag{3.12}$$

Theorem 3.2 (Generating function of F_j^2)**.**

$$\sum_{j=0}^{\infty} x^j F_j^2 = \frac{x(1-x)}{1 - 2x - 2x^2 + x^3}, \quad |x| < \frac{3 - \sqrt{5}}{2}. \tag{3.13}$$

Proof. Identity (3.7) as n approaches infinity; with $x^n F_n^2 \to 0$ as n approaches infinity. ☐

Next, we provide an alternative evaluation of $\sum_{j=0}^{n} x^j G_{j+k}^2$ in terms of $\sum_{j=0}^{n} x^j F_j^2$.

Theorem 3.3. The following identity holds for arbitrary $x \neq -1$ and integers k and n:

$$\begin{aligned}\sum_{j=0}^{n} x^j G_{j+k}^2 = \left(G_{k+1}^2 + xG_k^2 + 2(1-x)G_kG_{k+1} \right) \sum_{j=0}^{n} x^j F_j^2 \\ + \left(1 - x^{n+1}F_n^2\right)\left(G_k^2 - 2G_kG_{k+1}\right) \\ + \left(\frac{1 + (-1)^n x^{n+1}}{1+x} \right) 2G_kG_{k+1}.\end{aligned}$$

Proof. Squaring the addition formula $G_{j+k} = F_{j-1}G_k + F_jG_{k+1}$ gives

$$G_{j+k}^2 = G_k^2F_{j-1}^2 + G_{k+1}^2F_j^2 + 2G_kG_{k+1}F_jF_{j-1}. \tag{3.14}$$

But,

$$\begin{aligned}F_jF_{j-1} &= F_{j-1}(F_{j+1} - F_{j-1}) = F_{j-1}F_{j+1} - F_{j-1}^2 \\ &= F_j^2 - F_{j-1}^2 + (-1)^j,\end{aligned} \tag{3.15}$$

by Cassini's identity. Using (3.15) in (3.14), multiplying through by x^j and summing over j yields the identity of the theorem. □

Setting $x = 1$ in the identity of Theorem 3.3 produces

$$\begin{aligned}\sum_{j=0}^{n} G_{j+k}^2 &= F_nF_{n+1}(G_{k+1}^2 + G_k^2) + (F_n^2 - 1)(G_{k+1}^2 - G_{k-1}^2)\\ &\quad + (1 + (-1)^n)G_{k+1}G_k\,;\end{aligned} \tag{3.16}$$

so that

$$\sum_{j=0}^{2n-1} G_{j+k}^2 = F_{2n-1}F_{2n}(G_{k+1}^2 + G_k^2) + (F_{2n-1}^2 - 1)(G_{k+1}^2 - G_{k-1}^2) \tag{3.17}$$

and

$$\begin{aligned}\sum_{j=0}^{2n} G_{j+k}^2 &= F_{2n}F_{2n+1}(G_{k+1}^2 + G_k^2) + (F_{2n}^2 - 1)(G_{k+1}^2 - G_{k-1}^2)\\ &\quad + 2G_{k+1}G_k.\end{aligned} \tag{3.18}$$

In particular,

$$\sum_{j=0}^{n} F_{j+k}^2 = F_nF_{n+1}F_{2k+1} + (F_n^2 - 1)F_{2k} + (1 + (-1)^n)F_kF_{k+1}\,; \tag{3.19}$$

so that

$$\sum_{j=0}^{2n-1} F_{j+k}^2 = F_{2n}F_{2n-1}F_{2k+1} + (F_{2n-1}^2 - 1)F_{2k} \tag{3.20}$$

and

$$\sum_{j=0}^{2n} F_{j+k}^2 = F_{2n}F_{2n+1}F_{2k+1} + (F_{2n}^2 - 1)F_{2k} + 2F_kF_{k+1}. \tag{3.21}$$

4 Sums of products

Theorem 4.1. The following identity holds for integers n, s, k and arbitrary x:

$$\begin{aligned}
&2F_sF_k\sum_{j=0}^{n}x^jG_{j+k}G_{j+s}\\
&\quad=\frac{x\left(F_s^2G_{k+1}^2+F_k^2G_{s+1}^2\right)-(2x-1)\left(F_s^2G_k^2+F_k^2G_s^2\right)-x^2\left(F_s^2G_{k-1}^2+F_k^2G_{s-1}^2\right)}{x^3-2x^2-2x+1}\\
&\qquad-\frac{x^{n+1}\left(x\left(F_s^2G_{n+k+2}^2+F_k^2G_{n+s+2}^2\right)-(2x-1)\left(F_s^2G_{n+k+1}^2+F_k^2G_{n+s+1}^2\right)\right)}{x^3-2x^2-2x+1}\\
&\qquad-\frac{F_{s-k}^2\left(xG_1^2-(2x-1)G_0^2-x^2(G_1-G_0)^2\right)}{x^3-2x^2-2x+1}\\
&\qquad+\frac{x^{n+1}F_{s-k}^2\left(xG_{n+2}^2-(2x-1)G_{n+1}^2-x^2G_n^2\right)}{x^3-2x^2-2x+1}\\
&\qquad+\frac{x^{n+3}\left(F_s^2G_{n+k}^2+F_k^2G_{n+s}^2\right)}{x^3-2x^2-2x+1}.
\end{aligned}\tag{4.1}$$

Proof. Multiply through (2.3) by x^j and sum over j, obtaining

$$2F_sF_k\sum_{j=0}^{n}x^jG_{j+k}G_{j+s}=F_s^2\sum_{j=0}^{n}x^jG_{j+k}^2+F_k^2\sum_{j=0}^{n}x^jG_{j+s}^2-F_{s-k}^2\sum_{j=0}^{n}x^jG_j^2,$$

from which (4.1) follows upon using (3.1) and (3.6). □

In particular, setting $x=1$ in (4.1) produces

$$\begin{aligned}
2F_sF_k\sum_{j=0}^{n}G_{j+k}G_{j+s}&=F_k^2\left(G_{n+s}G_{n+s+1}-G_{s-1}G_s\right)+F_s^2\left(G_{n+k}G_{n+k+1}-G_{k-1}G_k\right)\\
&\quad-F_{s-k}^2\left(G_nG_{n+1}+G_0^2-G_0G_1\right),
\end{aligned}\tag{4.2}$$

which, by shifting the summation index and writing $k+1$ for k and $s+1$ for s, can also be written as

$$\begin{aligned}
&2F_{s+1}F_{k+1}\sum_{j=0}^{n}G_{j+k}G_{j+s}\\
&\quad=F_{k+1}^2\left(G_{n+s+1}G_{n+s+2}-G_sG_{s+1}\right)+F_{s+1}^2\left(G_{n+k+1}G_{n+k+2}-G_kG_{k+1}\right)\\
&\qquad-F_{s-k}^2\left(G_nG_{n+1}+G_0^2-G_0G_1\right)-2F_{s+1}F_{k+1}\left(G_{n+k+1}G_{n+s+1}-G_kG_s\right);
\end{aligned}\tag{4.3}$$

giving, in particular,

$$\sum_{j=0}^{n}G_{j+1}G_{j-2}=(G_n-G_1+G_0)(G_n+G_1-G_0).\tag{4.4}$$

Identity (4.3) subsumes Berzsenyi's results [3].

The next result provides an evaluation of the partial sum of the products of two consecutive Fibonacci-like numbers.

Theorem 4.2. The following identity holds for arbitrary integers n and k and arbitrary non-zero number x:

$$
\begin{aligned}
2\sum_{j=0}^{n} x^j G_{j+k} G_{j+k-1} &= \frac{\left(1-x-x^2\right)\left(xG_{k+1}^2-(2x-1)\,G_k^2-x^2G_{k-1}^2\right)}{x(x^3-2x^2-2x+1)} \\
&\quad - \frac{x^{n+1}\left(1-x-x^2\right)\left(xG_{n+k+2}^2-(2x-1)\,G_{n+k+1}^2-x^2\,G_{n+k}^2\right)}{x\,(x^3-2x^2-2x+1)} \\
&\quad + x^2G_{n+k+1}^2 + x^{n+1}G_{n+k}^2 - \frac{1}{x}G_k^2 - G_{k-1}^2 .
\end{aligned}
\tag{4.5}
$$

In particular,

$$
\begin{aligned}
2\sum_{j=0}^{n} x^j G_j G_{j-1} &= \frac{\left(1-x-x^2\right)\left(xG_1^2-(2x-1)\,G_0^2-x^2(G_1-G_0)^2\right)}{x\,(x^3-2x^2-2x+1)} \\
&\quad - \frac{x^{n+1}\left(1-x-x^2\right)\left(xG_{n+2}^2-(2x-1)\,G_{n+1}^2-x^2G_n^2\right)}{x\,(x^3-2x^2-2x+1)} \\
&\quad + x^nG_{n+1}^2 + x^{n+1}G_n^2 - \frac{1}{x}G_0^2 - (G_1-G_0)^2 ,
\end{aligned}
\tag{4.6}
$$

with the special value

$$
2\sum_{j=0}^{n} G_j G_{j-1} = G_{n+1}G_{n-1} + (G_n-G_0)(G_n+G_0) + (G_1-G_0)(2G_0-G_1). \tag{4.7}
$$

Proof. Squaring and re-arranging the recurrence relation

$$
G_{j+k+1} = G_{j+k} + G_{j+k-1},
$$

and multiplying through by x^j and summing, gives

$$
2\sum_{j=0}^{n} x^j G_{j+k}G_{j+k-1} = \sum_{j=0}^{n} x^j G_{j+k+1}^2 - \sum_{j=0}^{n} x^j G_{j+k}^2 - \sum_{j=0}^{n} x^j G_{j+k-1}^2. \tag{4.8}
$$

Shifting the summation index in each case provides

$$
\sum_{j=0}^{n} x^j G_{j+k-1}^2 = x\sum_{j=0}^{n} x^j G_{j+k}^2 + G_{k-1}^2 - x^{n+1}G_{n+k}^2
$$

and

$$
\sum_{j=0}^{n} x^j G_{j+k+1}^2 = \frac{1}{x}\sum_{j=0}^{n} x^j G_{j+k}^2 - \frac{1}{x}G_k^2 + x^nG_{n+k+1}^2;
$$

which when substituted in (4.8) gives (4.5) in view of (3.1). □

Theorem 4.3. If $(G_j)_{j\in\mathbb{Z}}$ and $(H_j)_{j\in\mathbb{Z}}$ are two Fibonacci-like sequences, then

$$\begin{aligned}2\sum_{j=0}^{n}(-1)^j H_{j+r}G_{j+k}G_{j+k-1} &= (-1)^n H_{n+r}G_{n+k+1}^2 + (-1)^{n+1} H_{n+r+1}G_{n+k}^2\\ &\quad + H_{r-1}G_k^2 - H_rG_{k-1}^2.\end{aligned} \tag{4.9}$$

In particular, we have the alternating sum of the products of three consecutive Fibonacci-like numbers, namely

$$\begin{aligned}2\sum_{j=0}^{n}(-1)^j G_{j+k+1}G_{j+k}G_{j+k-1} &= (-1)^n G_{n+k+1}^3 + (-1)^{n+1} G_{n+k+2}G_{n+k}^2\\ &\quad + G_k^3 - G_{k+1}G_{k-1}^2.\end{aligned} \tag{4.10}$$

Proof. Set $x = -(1+\sqrt{5})/2 = -\alpha$ in (4.5) and multiply through by α^r, where r is an arbitrary integer, to obtain

$$2\sum_{j=0}^{n}(-1)^j \alpha^{j+r}G_{j+k}G_{j+k-1} = (-1)^n \alpha^{n+r}G_{n+k+1}^2 + (-1)^{n+1}\alpha^{n+r+1}G_{n+k}^2 + \alpha^{r-1}G_k^2 - \alpha^rG_{k-1}^2. \tag{4.11}$$

Similarly, $x = -(1-\sqrt{5})/2 = -\beta$ in (4.5) gives

$$2\sum_{j=0}^{n}(-1)^j \beta^{j+r}G_{j+k}G_{j+k-1} = (-1)^n \beta^{n+r}G_{n+k+1}^2 + (-1)^{n+1}\beta^{n+r+1}G_{n+k}^2 + \beta^{r-1}G_k^2 - \beta^rG_{k-1}^2. \tag{4.12}$$

The identity of the theorem now follows by combining (4.11) and (4.12), since

$$H_j = \frac{C\alpha^j - D\beta^j}{\alpha-\beta},$$

where

$$C = H_1 - H_0\beta, \quad D = H_1 - H_0\alpha.$$

□

The next lemma facilitates deriving an alternative version of Theorem 4.1 and the evaluation of the product $\sum_{j=0}^{n} x^jG_{j-k}G_{j+k}$.

Lemma 4.4. *If j, k and s are integers, then*

$$G_{j+k}G_{j+s} = F_{s-1}F_{k-1}G_j^2 + (F_{s-1}F_k + F_sF_{k-1})G_jG_{j+1} + F_sF_kG_{j+1}^2 \tag{4.13}$$

and

$$G_{j-k}G_{j+k} = (-1)^k F_{k-1}F_{k+1}G_j^2 + (-1)^k F_k^2G_jG_{j+1} + (-1)^{k-1}F_k^2G_{j+1}^2. \tag{4.14}$$

Proof. Since, for integers j and m,

$$G_{j+m} = F_{m-1}G_j + F_mG_{j+1},$$

we have

$$G_{j+k}=F_{k-1}G_j+F_kG_{j+1},\tag{4.15}$$

$$G_{j+s}=F_{s-1}G_j+F_sG_{j+1},\tag{4.16}$$

and

$$G_{j-k}=(-1)^kF_{k+1}G_j+(-1)^{k-1}F_kG_{j+1}.\tag{4.17}$$

Multiplication of (4.15) and (4.16) gives (4.13) while multiplication of (4.15) and (4.17) produces (4.14). □

In view of (4.13) we now state an alternative version of Theorem 4.1.

Theorem 4.5. The following identity holds for integers j, s, k and arbitrary x:

$$\begin{aligned}(1-x)&\sum_{j=0}^{n}x^jG_{j+k}G_{j+s}\\&=\left((1-x)F_{s-1}F_{k-1}+F_{s-1}F_k+F_sF_{k-1}+\frac{(1-x)}{x}F_sF_k\right)\sum_{j=0}^{n}x^jG_j^2\\&+(F_{s-1}F_k+F_sF_{k-1})\left(-x^{n+1}G_{n+1}G_n+G_0G_{-1}\right)\\&+F_sF_k\left(-\frac{1-x}{x}G_0^2+x^n(1-x)G_{n+1}^2\right).\end{aligned}\tag{4.18}$$

Proof. Multiply through (4.13) by x^j and sum over j. □

Theorem 4.6. If k and n are integers and x is a variable, then

$$\begin{aligned}\sum_{j=0}^{n}x^jG_{j-k}G_{j+k}=(-1)^k&\left(F_{k-1}F_{k+1}-\frac{F_k^2}{x}\right)\sum_{j=0}^{n}x^jG_j^2+(-1)^k\frac{F_k^2}{x}\sum_{j=0}^{n}x^jG_{j-1}G_j\\&+(-1)^{k-1}F_k^2\left(-\frac{1}{x}G_0^2+x^nG_{n+1}^2\right)\\&+(-1)^k\frac{F_k^2}{x}\left(-G_{-1}G_0+x^{n+1}G_nG_{n+1}\right),\end{aligned}\tag{4.19}$$

where $\sum_{j=0}^{n}x^jG_j^2$ and $\sum_{j=0}^{n}x^jG_{j-1}G_j$ are as given in (3.6) and (4.6).

Proof. Multiply (4.14) by x^j and sum. □

In particular,

$$\begin{aligned}\sum_{j=0}^{n}G_{j-k}G_{j+k}=(-1)^k\frac{F_k^2}{2}&\left(G_n^2-G_{n+1}G_{n-1}+G_0G_2-G_1^2\right)\\&+G_nG_{n+1}-G_{-1}G_0;\end{aligned}\tag{4.20}$$

on account of (3.10) and (4.7).

Corollary 4.7 (Generating function of $G_{j+k}G_{j-k}$).

$$\begin{aligned}\sum_{j=0}^{\infty} x^j G_{j-k}G_{j+k} &= (-1)^k\left(F_{k-1}F_{k+1}-\frac{F_k^2}{x}\right)\frac{xG_1^2-(2x-1)\,G_0^2-x^2(G_1-G_0)^2}{x^3-2x^2-2x+1}\\ &\quad+(-1)^k F_k^2\frac{\left(1-x-x^2\right)}{2x^2}\frac{\left(xG_1^2-(2x-1)\,G_0^2-x^2(G_1-G_0)^2\right)}{(x^3-2x^2-2x+1)}\\ &\quad-\frac{G_0^2}{x}\left((-1)^{k-1}F_k^2+\frac{1}{2}\right)-\frac{1}{2}(G_1-G_0)^2\\ &\quad-(-1)^k\frac{F_k^2}{x}G_{-1}G_0;\end{aligned}\tag{4.21}$$

valid for $|x|<(3-\sqrt{5})/2$.

5 Double binomial sums

Lemma 5.1 ([1, Lemma 5]). *Let* (X_r) *be any arbitrary sequence,* X_r *satisfying a four-term recurrence relation* $hX_r = f_1X_{r-a} + f_2X_{r-b} + f_3X_{r-c}$*, where* h*,* f_1*,* f_2 *and* f_3 *are arbitrary non-vanishing functions and* a*,* b *and* c *are integers. Then, the following identities hold:*

$$\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{n}{j}\binom{j}{i}f_3^{n-j}f_2^{n+j-i}f_1^{i}X_{r-cn+(c-b)j+(b-a)i}=h^nf_2^nX_r\,,\tag{5.1}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{n}{j}\binom{j}{i}f_2^{n-j}f_3^{n+j-i}f_1^{i}X_{r-bn+(b-c)j+(c-a)i}=h^nf_3^nX_r\,,\tag{5.2}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{n}{j}\binom{j}{i}f_1^{n-j}f_3^{n+j-i}f_2^{i}X_{r-an+(a-c)j+(c-b)i}=h^nf_3^nX_r\,,\tag{5.3}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^i\binom{n}{j}\binom{j}{i}h^if_3^{n-j}f_2^{j-i}X_{r-(c-a)n+(c-b)j+bi}=(-f_1)^nX_r\,,\tag{5.4}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^i\binom{n}{j}\binom{j}{i}h^if_3^{n-j}f_1^{j-i}X_{r-(c-b)n+(c-a)j+ai}=(-f_2)^nX_r\tag{5.5}$$

and

$$\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^i\binom{n}{j}\binom{j}{i}h^if_2^{n-j}f_1^{j-i}X_{r-(b-c)n+(b-a)j+ai}=(-f_3)^nX_r.\tag{5.6}$$

Theorem 5.2. The following identities hold for non-negative integer n and integers s, k, m, r:

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^{i+(s+k+1)j}\binom{n}{j}\binom{j}{i}F_s^{n-j+i}F_k^{n+j}F_m^{2n-i}F_{m-s}^{n-j}F_{m-k}^{n+j-i}F_{s-k}^{i}G_{r+kn+(s-k)j+(m-s)i}^2\\ &\quad=(F_mF_kF_{m-k}^2F_{m-s}F_{s-k})^nG_r^2\,,\end{aligned}\tag{5.7}$$

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^{j+(s+k)(i+j)}\binom{n}{j}\binom{j}{i}F_s^{n+j}F_k^{n-j+i}F_m^{2n-i}F_{m-s}^{n+j-i}F_{m-k}^{n-j}F_{s-k}^{i}G^2_{r+sn+(k-s)j+(m-k)i}\\&\quad=(-1)^{(s+k-1)n}(F_mF_sF_{m-s}^2F_{m-k}F_{s-k})^nG_r^2,\end{aligned}\tag{5.8}$$

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^{(s+k)(i+j)+i}\binom{n}{j}\binom{j}{i}F_s^{n+j-i}F_k^{i}F_m^{n+j}F_{m-s}^{n+j-i}F_{m-k}^{i}G^2_{r+mn+(k-m)j+(s-k)i}\\&\quad=(-1)^{(s+k)n}(F_mF_sF_{m-k}F_{s-k}F_{m-s}^2)^nG_r^2.\end{aligned}\tag{5.9}$$

Proof. Change j to r and re-arrange the identity of Theorem 2.2 as

$$\begin{aligned}F_{m-s}F_{m-k}F_{s-k}G_r^2&=(-1)^{s+k}F_kF_sF_{s-k}G^2_{r+m}\\&\quad+(-1)^{s+k+1}F_kF_mF_{m-k}G^2_{r+s}\\&\quad+F_sF_mF_{m-s}G^2_{r+k}.\end{aligned}$$

In Lemma 5.1 with $X_r=G_r^2$, set $h=F_{m-s}F_{m-k}F_{s-k}$, $f_1=(-1)^{s+k}F_kF_sF_{s-k}$, $f_2=(-1)^{s+k+1}F_kF_mF_{m-k}$, $f_3=F_sF_mF_{m-s}G^2_{r+k}$, $a=-m$, $b=-s$ and $c=-k$ in identities (5.1) – (5.6). ☐

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