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Weakly S-2-prime ideals of commutative rings

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Abstract. The objective of this paper is to introduce and investigate the concept of weakly S-2-prime ideals which are extensions of weakly 2-prime ideals in commutative rings. Let R be a commutative ring with identity and S be a multiplicative subset of R with $1 \in S$. A proper ideal Q of R with $Q \cap S = \emptyset$ is called a weakly S-2-prime ideal of R if there exists an $s \in S$ such that for all $\alpha, \beta \in R$ with $0 \neq \alpha\beta \in Q$, we have $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. Various characterizations of weakly S-2-prime ideals are given and the relationship between weakly S-2-prime ideals and other classical ideals are illustrated by a diagram. For this relationship, a myriad of supporting examples and counter examples are presented. Moreover, this class of ideals is analyzed in idealization rings and amalgamated duplication along an ideal. Besides, the rings over which every weakly S-2-prime ideal is S-2-prime ideal is examined.

Key Words: 2-prime ideals, S-2-prime ideals, weakly 2-prime ideals, weakly S-2-prime ideals.
2020 MSC: Primary 13A15, 13C05; Secondary 13A99.

Dedicated to our Professor David E. Dobbs for his 80th Birthday.

1 Introduction

In this article, we only take into consideration on commutative rings with identity and unitary modules. Unless otherwise stated, R will always describe such a ring and M will be such an R-module. The concept of prime ideals and its generalizations have a crucial role in commutative algebra as they have many applications in different research areas. For instance, prime ideals and their generalizations were used to characterize specific rings such as fields, Von-Neumann regular rings, and Dedekind domains. As it is well-known a proper ideal Q of R is called a prime ideal if $\alpha, \beta \in R$ such that $\alpha\beta \in Q$ refers that $\alpha \in Q$ or $\beta \in Q$. In 2003, Anderson and Smith [2] introduced the notion of weakly prime ideals which is a generalization of prime ideals in order to examine factorization in commutative rings with zero divisors. A proper ideal Q of R is called weakly prime if $0 \neq \alpha\beta \in Q$ for some $\alpha, \beta \in R$ implies that $\alpha \in Q$ or $\beta \in Q$. Afterwards, many studies generalizing prime ideals have been done. One of them was the work entitled S-prime ideals by Hamed and Malek [7]. Recall that a subset S of R is called multiplicatively closed subset (in briefly m.c.s) if S is closed under multiplication and $1 \in S$. Let S be a m.c.s of R and Q be an ideal with $Q \cap S = \emptyset$. According to their paper [7], Q is called an S-prime ideal of R if there exists an $s \in S$ such that for all α, $\beta \in R$ with $\alpha\beta \in Q$, we have $s\alpha \in Q$ or $s\beta \in Q$. Moreover, Almahdi et. al. [1] described weakly S-prime ideals so as to characterize S-Noetherian rings and S-principal ideal rings. An ideal Q disjoint with S is said to be a weakly S-prime ideal if there exists an element $s \in S$ such that for all α, $\beta \in R$ whenever $0 \neq \alpha\beta \in Q$ implies $s\alpha \in Q$ or $s\beta \in Q$. On the other hand, Beddani and Messirdi [4] introduced and studied 2-prime ideals which is a different generalization of prime ideals and they used this notion to present specific characterizations of valuation rings. A proper ideal Q of R is called a 2-prime ideal if $\alpha, \beta \in R$ such that $\alpha\beta \in Q$, then either $\alpha^2 \in Q$ or $\beta^2 \in Q$. This class of ideals is examined by Nikandish et.al. as well [12]. Furthermore, Koç [10] defined weakly 2-prime ideals and examined compactly packedness and coprimely packedness on trivial extention. Moreover, Issoual et.al. [8] further investigated properties of this concept of ideals. Q is called a weakly 2-prime ideal of R if whenever $0 \neq \alpha\beta \in Q$ for some $\alpha, \beta \in R$, then either $\alpha^2 \in Q$ or $\beta^2 \in Q$. As a recent research, in [14] the concept of S-2-prime ideals is defined. A proper ideal Q of R with $Q \cap S = \emptyset$ is called an S-2-prime ideal of R if there exists an $s \in S$ such that for all α, $\beta \in R$ with $\alpha\beta \in Q$, we have $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. As usual, for any proper ideal Q of R, the radical of Q is defined by the intersection of all prime ideals containing Q, denoted by $\sqrt{Q}$ which is equivalent to the set $\{r \in R : r^n \in Q \text{ for some } n \in \mathbb{N}\}$ in commutative rings. In particular, $\sqrt{0_R} = Nil(R)$.

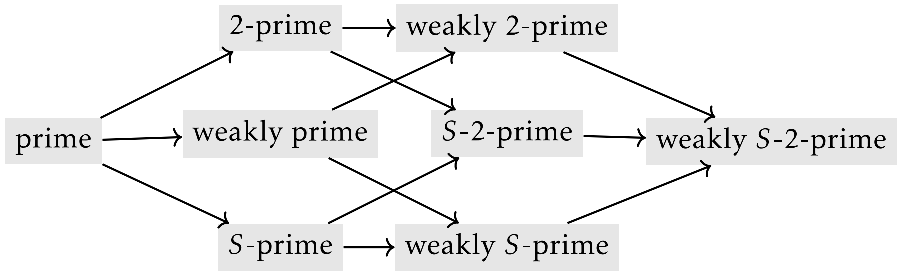
Our aim of this study is to introduce weakly S-2-prime ideals which are generalization of S-2-prime ideals. A proper ideal Q of R disjoint with S is called a weakly S-2-prime ideal if there exists an $s \in S$ such that for all α, $\beta \in R$ with $0 \neq \alpha\beta \in Q$, we have $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. We give a chart which demonstrate the relationship among weakly S-2-prime ideals and other classical ideals such as 2-prime ideal, S-prime ideal, weakly S-prime ideal etc. with several supporting examples. Besides, we investigate the behavior of weakly S-2-prime ideals under homomorphism, in the direct product of rings and in localization rings (Theorems 2.9-2.12, Proposition 2.13). Furthermore, we discuss weakly S-2-prime ideals in trivial extention rings and amalgamated duplication along an ideal which is defined by [3, 6, 11] (Theorems 2.14-2.15). Finally, with the help of [5, 9, 13], we give the rings over which every weakly S-2-prime ideal is S-2-prime ideal (Propositions 2.16, 2.17).

2 Properties of Weakly S-2-Prime Ideals

Definition 2.1. Let S be a m.c.s of a ring R and Q be a proper ideal of R with $Q \cap S = \emptyset$. Then, Q is called a weakly S-2-prime ideal of R if there exists an $s \in S$ such that for all α, $\beta \in R$ with $0 \neq \alpha\beta \in Q$, we have $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. In this case, Q is called associated to s.

The diagram below shows the relationship among weakly S-2-prime ideals with other classical ideals.

Every weakly 2-prime ideal of R disjoint with S is weakly S-2-prime ideal of R, where S is a m.c.s of R. If S consists of units of R, then weakly 2-prime ideals and weakly S-2-prime ideals are coincide. Otherwise, these are distinct notions. The following examples are given to verify that the rows above are not reversible. (see also [1, Examples 2-3], [2, Example 20], [7, Example 1], [10, Examples 1-2], [12, Example of the ideal $9\mathbb{Z}$ of the ring $\mathbb{Z}$] and [14, Example 3]).



Example 2.2. (Weakly S-2-prime ideal which is not weakly 2-prime) Let $R = \mathbb{Z}[X]$ and $S = \{5^n \mid n \in \mathbb{N}\}$. Consider $Q = 5X\mathbb{Z}[X]$. It is clear that $Q \cap S = \emptyset$ and Q is a weakly S-2-prime ideal of R. Indeed, if $0 \neq fg \in Q = 5X\mathbb{Z}[X]$ for all $f, g \in R$, then since $Q \subset X\mathbb{Z}[X]$ and $X\mathbb{Z}[X]$ is a prime ideal of R, we have $X|f$ or $X|g$. Hence, $5f^2 \in Q$ or $5g^2 \in Q$ for $s := 5$. However, Q is not a weakly 2-prime ideal since $0 \neq 5 \cdot X \in Q$ but $5^2 \notin Q$ and $X^2 \notin Q$.

Example 2.3. (S-2-prime ideal which is not 2-prime) Consider $R = \mathbb{Z}$, $Q = 24\mathbb{Z}$ and $S = \{3^n \mid n \in \mathbb{N}\}$. Then, Q is not a 2-prime ideal of R since $6 \cdot 4 \in Q$ but $6^2, 4^2 \notin Q$. However, Q is an S-2-prime ideal of R. Indeed, it is clear that $Q \cap S = \emptyset$. Let $\alpha, \beta \in \mathbb{Z}$ such that $\alpha\beta \in Q$. Since $\alpha\beta \in 8\mathbb{Z}$ and $8\mathbb{Z}$ is a primary ideal of R, we have either $\alpha \in 8\mathbb{Z}$ or $\beta \in \sqrt{8\mathbb{Z}} = 2\mathbb{Z}$. Put $s := 3$. In the former case, we conclude that $s\alpha^2 \in Q$. Suppose that the latter case holds. If $\beta \in 2\mathbb{Z}\backslash 4\mathbb{Z}$, then it is clear that $\alpha \in 4\mathbb{Z}$, and we have $s\alpha^2 \in Q$. If $\beta \in 4\mathbb{Z}$, then $s\beta^2 \in Q$, so we are done.

Example 2.4. (Weakly S-2-prime ideal which is not S-2-prime) Suppose that p, q are distinct prime numbers and $R = \mathbb{Z}_{pq^2}$, $S = \{\overline{s}^n \mid n \in \mathbb{N}$ and s is a prime number with $s \neq p, q\}$ and consider $Q = (\overline{0})$. Then, Q is a weakly S-2-prime ideal of R. However, Q is not an S-2-prime ideal of R since $\overline{p} \cdot \overline{q}^2 \in Q$ but $s \cdot \overline{p}^2 \notin Q$ and $s \cdot \overline{q}^4 \notin Q$ for all $s \in S$.

Note that if R is a Von-Neumann regular ring, then every weakly S-2-prime ideal is weakly S-prime ideal. However, these are different concepts.

Example 2.5. (Weakly S-2-prime ideal which is not weakly S-prime) Suppose that K is a field and $R = K[X, Y]$, where X and Y are indeterminates. Let $Q = (XY, Y^2)$ and S be the set of constant polynomials from K. Suppose that $0 \neq g(X,Y)h(X,Y) \in Q \subseteq \sqrt{Q} = (Y)$. This implies that either $Y|g(X,Y)$ or $Y|h(X,Y)$. Thus, $sg^2(X,Y) \in Q$ or $sh^2(X,Y) \in Q$ for all $s \in S$, and so Q is a weakly S-2-prime ideal. However, Q is not a weakly S-prime ideal of R as $0 \neq Y \cdot Y \in Q$ but $s \cdot Y \notin Q$ for all $s \in S$.

Theorem 2.6. Let S be a m.c.s of R and Q be a weakly S-2-prime ideal of R. If Q is not an S-2-prime ideal of R, then $Q^2 = (0)$. In this case, $\sqrt{Q} = \sqrt{0_R}$.

Proof. Assume that $Q^2 \neq (0)$. We claim that Q is an S-2-prime ideal of R. Let $x, y \in R$ such that $xy \in Q$. If $0 \neq xy \in Q$, then there exists an $s \in S$ such that $sx^2 \in Q$ or $sy^2 \in Q$, as desired. If $0 = xy \in Q$, we have three cases. Firstly, if $xQ \neq 0$, then there exists an $m \in Q$ such that $0 \neq xm$, and so $0 \neq xm = x(m+y) \in Q$. Therefore, there exists an $s \in S$ such that $sx^2 \in Q$ or $s(m+y)^2 \in Q$. Thus, we have $sx^2 \in Q$ or $sy^2 \in Q$. Secondly, if $yQ \neq 0$, we achieve a similar result. Finally, let $xQ = 0$ and $yQ = 0$. Because of $Q^2 \neq 0$, then there exist $m, n \in Q$ such that $mn \neq 0$. Therefore, $0 \neq mn = (x+m)(y+n) \in Q$. Then, there exists an $s \in S$ such that $s(x+m)^2 \in Q$ or $s(y+n)^2 \in Q$. Thus, we have $sx^2 \in Q$ or $sy^2 \in Q$. From all of cases, we conclude that Q is an S-2-prime ideal of R. □

Consequently, one can observe that any weakly S-2-prime ideal Q of R with $Nil(R) \supsetneq Q$ is an S-2-prime ideal of R. In particular, if R is a reduced ring, then a proper ideal Q disjoint with S is a weakly S-2-prime ideal of R if and only if $Q = (0)$ or Q is an S-2-prime ideal of R. The next example is given to show that the situation $Q^2 = (0)$ in the Theorem 2.6 does not guarantee that Q is a weakly S-2-prime ideal of R.

Example 2.7. Let p, q be distinct prime numbers and $R = \mathbb{Z}_{p^2q}$, $S = \{\overline{1}, \overline{s} : s$ is a prime number with $s \neq p, q\}$ and $Q = (\overline{pq})$. Then, $Q^2 = (0)$. Nevertheless, Q is not a weakly S-2-prime ideal of R since $0 \neq \overline{p} \cdot \overline{q} \in Q$ but $s \cdot \overline{p}^2 \notin Q$ and $s \cdot \overline{q}^2 \notin Q$ for all $s \in S$.

Let S be a m.c.s of R and Q be an ideal of R with $Q \cap S = \emptyset$. As it is shown in [14] that if Q is an S-2-prime ideal of R, then $\sqrt{Q}$ is an S-prime ideal of R. A similar statement does not hold for weakly S-2-prime ideals.

Example 2.8. Let p, q be distinct prime numbers and $R = \mathbb{Z}_{p^2q^2}$, $S = \{\overline{1}, \overline{s} : s$ is a prime number with $s \neq p, q\}$. Then $Q = (\overline{0})$ is a weakly S-2-prime ideal of R, but $\sqrt{Q} = (\overline{pq})$ is not a weakly S-prime ideal of R as $0 \neq \overline{p} \cdot \overline{q} \in \sqrt{Q}$ but $s \cdot \overline{p} \notin \sqrt{Q}$ and $s \cdot \overline{q} \notin \sqrt{Q}$ for all $s \in S$.

Theorem 2.9. Suppose that $f : R \to R'$ is a ring homomorphism and S is a m.c.s of R. Then, the following statements are provided:

1. If f is an epimorphism and Q is a weakly S-2-prime ideal of R containing $ker(f)$, then $f(Q)$ is a weakly $f(S)$-2-prime ideal of R'.

2. If f is a monomorphism and Q' is a weakly $f(S)$-2-prime ideal of R', then $f^{-1}(Q')$ is a weakly S-2-prime ideal of R.

Proof. (1) Let $r \in f(S) \cap f(Q)$. Then, $r = f(q) = f(s)$ for some $q \in Q$ and $s \in S$. We have $s - q \in ker(f) \subseteq Q$, that is $s \in Q$ which contradicts with $Q \cap S = \emptyset$. Hence, $f(S) \cap f(Q) = \emptyset$. Now, let $0 \neq \alpha'\beta' \in f(Q)$. Then, there is $\alpha, \beta \in R$ such that $f(\alpha) = \alpha'$ and $f(\beta) = \beta'$ with $0 \neq f(\alpha\beta) = \alpha'\beta' \in f(Q)$. Since $ker(f) \subseteq Q$, we get $0 \neq \alpha\beta \in Q$. From our assumption, there exists an $s \in S$ such that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. This implies that $f(s\alpha^2) = f(s)(\alpha')^2 \in f(Q)$ or $f(s\beta^2) = f(s)(\beta')^2 \in f(Q)$. Hence, $f(Q)$ is a weakly $f(S)$-2-prime ideal of R'.

(2) Suppose that Q' is a weakly $f(S)$-2-prime ideal of R'. Clearly, $f^{-1}(Q') \cap S = \emptyset$. Let $\alpha, \beta \in R$ with $0 \neq \alpha\beta \in f^{-1}(Q')$. Because of $ker(f) = \{0\}$, we have $0 \neq f(\alpha\beta) = f(\alpha)f(\beta) \in Q'$. Then, there exists $f(s) \in f(S)$ such that $f(s)(f(\alpha))^2 \in Q'$ or $f(s)(f(\beta))^2 \in Q'$. Thus, $f(s\alpha^2) \in Q'$ or $f(s\beta^2) \in Q'$, that is $s\alpha^2 \in f^{-1}(Q')$ or $s\beta^2 \in f^{-1}(Q')$. Hence, $f^{-1}(Q')$ is a weakly S-2-prime ideal of R. □

Let S be a m.c.s of R and Q be an ideal of R such that $Q \cap S = \emptyset$. Let $s \in S$. By $\overline{s}$, we denote the class of s in R/Q and set $\overline{S} = \{\overline{s} : s \in S\}$. It is clear that $\overline{S}$ is a m.c.s of R/Q.

Corollary 2.10. *Let S be a m.c.s of R. Then*

1. *Let $J \subseteq K$ are two ideals of R. If K is a weakly S-2-prime ideal of R, then K/J is a weakly $\overline{S}$-2-prime ideal of R/J.*

2. *If R is a subring of R' and K' is a weakly S-2-prime of R' with $R \not\subseteq K'$, then $K' \cap R$ is a weakly S-2-prime ideal of R.*

Proof. (1) Consider the epimorphism $f : R \to R/J$, described by $f(r) = r + J$ for each $r \in R$. Moreover, $ker(f) = J \subseteq K$ and by Theorem 2.9 (1.), we have $f(K) = K/J$ is a weakly $\overline{S}$-2-prime ideal of R/J.

(2) It is clear that $(K' \cap R) \cap S = \emptyset$. Consider the monomorphism $h : R \to R'$, described by $h(r) = r$ for each $r \in R$. Since K' is a weakly S-2-prime of R' by Theorem 2.9 (2.), $h^{-1}(K') = K' \cap R$ is a weakly S-2-prime ideal of R. □

Theorem 2.11. Let $R = R_1 \times R_2$ and $S = S_1 \times S_2$, where S_1 is a m.c.s of R_1 and S_2 is a m.c.s of R_2. Suppose that Q_1 and Q_2 are non-zero proper ideals of R_1 and R_2, respectively. The following statements are equivalent:

1. $Q = Q_1 \times Q_2$ is a weakly S-2-prime ideal of R.
2. Either Q_1 is an S_1-2-prime ideal of R_1 with $S_2 \cap Q_2 \neq \emptyset$ or Q_2 is an S_2-2-prime ideal of R_2 with $S_1 \cap Q_1 \neq \emptyset$.
3. $Q = Q_1 \times Q_2$ is an S-2-prime ideal of R.

Proof. (1)⇒(2) Let $0 \neq (g,h) \in Q$ for all $g \in R_1$ and $h \in R_2$. Then, $0 \neq (g,h) = (g,1)(1,h) \in Q$. From our assumption, there exists an $s = (s_1,s_2) \in S$ such that $(s_1,s_2)(g,1)^2 = (s_1g^2,s_2) \in Q$ or $(s_1,s_2)(1,h)^2 = (s_1,s_2h^2) \in Q$. Therefore, we have $S_1 \cap Q_1 \neq \emptyset$ or $S_2 \cap Q_2 \neq \emptyset$. Let $S_2 \cap Q_2 \neq \emptyset$. Since $Q \cap S = \emptyset$, we have $S_1 \cap Q_1 = \emptyset$. We will indicate that Q_1 is an S_1-2-prime ideal of R_1. Now, let $gg_1 \in Q_1$ for all g, $g_1 \in R_1$. Because of $S_2 \cap Q_2 \neq \emptyset$, then there is $0 \neq m \in S_2 \cap Q_2$ such that $0 \neq (g,m)(g_1,1) \in Q$. From assumption, there exists an $s = (s_1,s_2) \in S = S_1 \times S_2$ such that $(s_1,s_2)(g,m)^2 = (s_1g^2,s_2m^2) \in Q$ or $(s_1,s_2)(g_1,1)^2 = (s_1g_1^2,s_2) \in Q$. Hence, $s_1g^2 \in Q_1$ or $s_1g_1^2 \in Q_1$, and so Q_1 is an S_1-2-prime ideal of R_1. Similarly, if $S_1 \cap Q_1 \neq \emptyset$, then we could prove that Q_2 is an S_2-2-prime ideal of R_2.

(2)⇒(3) Follows from [[14], Lemma 1].

(3)⇒(1) The proof is straightforward. □

Theorem 2.12. Let $n \geq 2$, $R = R_1 \times R_2 \times \cdots \times R_n$ and $S = S_1 \times S_2 \times \cdots \times S_n$ where S_i is a m.c.s of R_i and Q_i is a non-zero proper ideal of R_i for $i = 1,2,...,n$, respectively. The following statements are equivalent:

1. $Q = Q_1 \times Q_2 \times \cdots \times Q_n$ is a weakly S-2-prime ideal of R.
2. Q_m is an S_m-2-prime ideal of R_m for some $m \in \{1,2,...,n\}$ and $Q_i \cap S_i \neq \emptyset$ for each $i \in \{1,2,...,n\} - \{m\}$.
3. $Q = Q_1 \times Q_2 \times \cdots \times Q_n$ is an S-2-prime ideal of R.

Proof. (1)⇒(2) In order to prove the claim, we will use induction on n. If $n = 2$, then the result follows from the Theorem 2.11. Suppose that the claim is true for all $k < n$. Let $R' = R_1 \times R_2 \times \cdots \times R_{n-1}$, $S' = S_1 \times S_2 \times \cdots \times S_{n-1}$ and $Q' = Q_1 \times Q_2 \times \cdots \times Q_{n-1}$. Since $0 \neq Q = Q' \times Q_n$ is a weakly $S = S' \times S_n$-2-prime ideal of $R = R' \times R_n$, by Theorem 2.11, we have Q_n is an S_n-2-prime of R_n and $Q' \cap S' \neq \emptyset$ or Q' is an S'-2-prime ideal of R' and $Q_n \cap S_n \neq \emptyset$. If $Q' \cap S' \neq \emptyset$ and Q_n is an S_n-2-prime of R_n, then proof is complete. So, suppose that $Q_n \cap S_n \neq \emptyset$ and Q' is an S'-2-prime ideal of R'. In this case, using the induction hypothesis for $k = n-1$, we have Q_j is an S_j-2-prime ideal of R_j for some $j \in \{1,2,...,n-1\}$ and $Q_i \cap S_i \neq \emptyset$ for each $i \in \{1,2,...,n-1\} - \{j\}$, which completes the proof.

(2)⇒(3) Follows from [[14], Theorem 3].

(3)⇒(1) The proof is straightforward. □

Let Q be an ideal of R. By $Z_Q(R)$, we mean $Z_Q(R) = \{\alpha \in R : a\beta \in Q$ for some $\beta \notin Q\}$. Besides, if $Q = (0)$, then $Z_Q(R)$ is the set of all zero divisors of R. In this case, we can use $zd(R)$ in the place of $Z_Q(R)$. Moreover, we will describe the set of all regular elements of R, that is they are non-zero divisors, with $reg(R) = R - zd(R)$. [10]

Proposition 2.13. *Suppose that S, $S' \subseteq R$ are m.c.ss of R and Q is an ideal of R. If Q is a weakly S'-2-prime ideal of R with $Q \cap S = \emptyset$, then $S^{-1}Q$ is a weakly $S^{-1}S'$-2-prime ideal of $S^{-1}R$. The converse of this implication also holds provided that $Z_Q(R) \cap S = \emptyset$ and $S \subseteq reg(R)$.*

Proof. It is clear that $S^{-1}S'$ is a m.c.s of $S^{-1}R$ as S' is a m.c.s of R. Also, $S^{-1}Q \cap S^{-1}S' = \emptyset$ because of $Q \cap S = \emptyset$. Let m, $n \in R$ and s, $t \in S$ such that $0 \neq \frac{m}{s}\frac{n}{t} = \frac{mn}{st} \in S^{-1}Q$. Then, $0 \neq \widetilde{s}(mn) = (\widetilde{s}m)n \in Q$ for some $\widetilde{s} \in S$. By assumption, there exists an $s' \in S'$ such that $s'(\widetilde{s}m)^2 \in Q$ or $s'n^2 \in Q$. This yields that $\frac{s'}{1}(\frac{m}{s})^2 = \frac{s'}{1}\frac{(\widetilde{s}m)^2}{(\widetilde{s}s)^2} \in S^{-1}Q$ or $\frac{s'}{1}(\frac{n}{t})^2 \in S^{-1}Q$. Hence, $S^{-1}Q$ is a weakly $S^{-1}S'$-2-prime ideal of $S^{-1}R$. Conversely, it is clear that $Q \cap S' = \emptyset$. Suppose that $0 \neq mn \in Q$ for all m, $n \in R$. Then, we have $0 \neq \frac{m}{1}\frac{n}{1} \in S^{-1}Q$ because of $S \subseteq reg(R)$. From assumption, there exists an $\frac{s'}{s} \in S^{-1}S'$ such that $\frac{s'}{s}(\frac{m}{1})^2 \in S^{-1}Q$ or $\frac{s'}{s}(\frac{n}{1})^2 \in S^{-1}Q$. Then, there exists $\widetilde{s} \in S$ such that $\widetilde{s}s'm^2 \in Q$ or $\widetilde{s}s'n^2 \in Q$. Since $Z_Q(R) \cap S = \emptyset$, we have $s'm^2 \in Q$ or $s'n^2 \in Q$ for $s' \in S'$. Thus, Q is a weakly S'-2-prime ideal of R. □

Let R be a commutative ring with identity and M be a unitary R-module. The trivial extention of R in M denoted by $R(+)M = \{(\widetilde{r}, m) : \widetilde{r} \in R,\ m \in M\}$ is a commutative ring with usual addition and the multiplication $(\widetilde{r_1}, m_1)(\widetilde{r_2}, m_2) = (\widetilde{r_1}\widetilde{r_2}, \widetilde{r_1}m_2 + \widetilde{r_2}m_1)$ for all $(\widetilde{r_1}, m_1)$, $(\widetilde{r_2}, m_2) \in R(+)M$ [3, 11]. Clearly, if S is a m.c.s of R, then $S(+)0$ and $S(+)M$ are m.c.ss of $R(+)M$.

Theorem 2.14. Let R and M be the above statement and S be a m.c.s of R. For a proper ideal Q disjoint with S, the following statements are equivalent:

1. $Q(+)M$ is a weakly $(S(+)M)$-2-prime ideal of $R(+)M$.
2. $Q(+)M$ is a weakly $(S(+)0)$-2-prime ideal of $R(+)M$.
3. Q is a weakly S-2-prime ideal of R associated to $s \in S$, and if there exist α, $\beta \in R$ with $\alpha\beta = 0$, but $s\alpha^2 \notin Q$ and $s\beta^2 \notin Q$, then $\alpha \in ann_R(M)$ and $\beta \in ann_R(M)$.

Proof. (1)⇒(3) Let α, $\beta \in R$ with $0 \neq \alpha\beta \in Q$. Then, $(0,0) \neq (\alpha,0)(\beta,0) \in Q(+)M$. From assumption, there exists $(s,m) \in S(+)M$ such that $(s,m)(\alpha,0)^2 = (s\alpha^2, m\alpha^2) \in Q(+)M$ or $(s,m)(\beta,0)^2 = (s\beta^2, m\beta^2) \in Q(+)M$. Therefore, $s\alpha^2 \in Q$ or $s\beta^2 \in Q$, and so Q is a weakly S-2-prime ideal of R associated to that $s \in S$. Now, let $\alpha\beta = 0$ but $s\alpha^2 \notin Q$ and $s\beta^2 \notin Q$. Suppose $\alpha \notin ann_R(M)$. Then, there is $m' \in M$ with $\alpha m' \neq 0$. So, we have $(0,0) \neq (\alpha,0)(\beta,m') \in Q(+)M$. From assumption, there exits $(s,m) \in S(+)M$ such that $(s,m)(\alpha,0)^2 = (s\alpha^2, m\alpha^2) \in Q(+)M$ or $(s,m)(\beta,m')^2 = (s\beta^2, 2s\beta m' + m\beta^2) \in Q(+)M$. Thus, $s\alpha^2 \in Q$ or $s\beta^2 \in Q$, a contradiction. So, $\alpha \in ann_R(M)$, and similarly we can obtain $\beta \in ann_R(M)$.

(3)⇒(2) Let (α,m), $(\beta,m') \in R(+)M$ with $(0,0) \neq (\alpha,m)(\beta,m') \in Q(+)M$. If $\alpha\beta \neq 0$, then $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. So, $(s,0)(\alpha,m)^2 \in Q(+)M$ or $(s,0)(\beta,m')^2 \in Q(+)M$. If $\alpha\beta = 0$ but $s\alpha^2 \notin Q$ and $s\beta^2 \notin Q$, then α, $\beta \in ann_R(M)$. Hence, we have $(\alpha,m)(\beta,m') = (0,0)$ and it is a contradiction because of our assumption. We conclude that $Q(+)M$ is a weakly $(S(+)0)$-2-prime ideal of $R(+)M$.

(2)⇒(1) The proof is clear. □

Recall from [6] that the amalgamated duplication of R along an ideal I of R is defined as follow:

$$R \bowtie I = \{(\widetilde{r}, \widetilde{r} + i) : \widetilde{r} \in R, i \in I\}$$

$R \bowtie I$ is the subring of $R \times R$.

Theorem 2.15. Suppose that I is an R-module, S is a m.c.s of R and Q is a proper ideal of R with $Q \cap S = \emptyset$. Then, the following statements are equivalent:

1. $Q \bowtie I$ is a weakly $(S \bowtie I)$-2-prime ideal of $R \bowtie I$.
2. $Q \bowtie I$ is a weakly $(S \bowtie 0)$-2-prime ideal of $R \bowtie I$.
3. Q is a weakly S-2-prime ideal of R associated to $s \in S$, and if there exist α, $\beta \in R$ with $\alpha\beta = 0$, but $s\alpha^2 \notin Q$ and $s\beta^2 \notin Q$, then α, $\beta \in ann_R(I)$ and $I^2 = (0)$.

Proof. (1)⇒(3) Let $Q \bowtie I$ be a weakly $(S \bowtie I)$-2-prime ideal of $R \bowtie I$ associated to $(s, s+i)$ and $0 \neq \alpha\beta \in Q$ for all α, $\beta \in R$. Then, $(0,0) \neq (\alpha,\alpha)(\beta,\beta) \in Q \bowtie I$. Hence, $(s,s+i)(\alpha,\alpha)^2 \in Q \bowtie I$ or $(s,s+i)(\beta,\beta)^2 \in Q \bowtie I$. So, $s\alpha^2 \in Q$ or $s\beta^2 \in Q$ and Q is a weakly S-2-prime ideal of R associated to $s \in S$. Now, suppose $\alpha\beta = 0$, but $s\alpha^2 \notin Q$ and $s\beta^2 \notin Q$. Let $\alpha \notin ann_R(I)$. Then, there is $\acute{i} \in I$ such that $\alpha\acute{i} \neq 0$. Therefore, we have $(0,0) \neq (\alpha,\alpha)(\beta,\beta+\acute{i}) \in Q \bowtie I$. Thus, $(s,s+i)(\alpha,\alpha)^2 \in Q \bowtie I$ or $(s,s+i)(\beta,\beta+\acute{i})^2 \in Q \bowtie I$ which contradicts with our assumption. Hence, $\alpha \in ann_R(I)$ and similarly $\beta \in ann_R(I)$. Let $m,n \in I$. Then, $(\alpha,\alpha+m)(\beta,\beta+n) = (0,mn) \in Q \bowtie I$. If $mn \neq 0$, then $(\alpha,\alpha+m)(\beta,\beta+n) \neq (0,0)$. Hence, $(s,s+i)(\alpha,\alpha+m)^2 \in Q \bowtie I$ or $(s,s+i)(\beta,\beta+n)^2 \in Q \bowtie I$, again a contradiction. Thus, $I^2 = (0)$.

(3)⇒(2) Let $(0,0) \neq (\alpha,\alpha+m)(\beta,\beta+n) \in Q \bowtie I$ for all $(\alpha,\alpha+m)$, $(\beta,\beta+n) \in R \bowtie I$. If $\alpha\beta \neq 0$, then $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. So, we have $(s,s)(\alpha,\alpha+m)^2 \in Q \bowtie I$ or $(s,s)(\beta,\beta+n)^2 \in Q \bowtie I$. If $\alpha\beta = 0$ but $s\alpha^2 \notin Q$ and $s\beta^2 \notin Q$, then α, $\beta \in ann_R(I)$ and $I^2 = (0)$. So, $(\alpha,\alpha+m)(\beta,\beta+n) = (0,0)$ and it is a contradiction from assumption. Hence, $Q \bowtie I$ is a weakly $(S \bowtie 0)$-2-prime ideal of $R \bowtie I$.

(2)⇒(1) The proof is clear. □

Finally, we will examine the rings over which every weakly S-2-prime ideal is S-2-prime ideal. Note that every weakly S-2-prime ideal is an S-2-prime ideal of R if and only if (0) is an S-2-prime ideal of R. Recall from [[5], Definition 1.4.3] a commutative ring with identity R is called an S-integral domain if there exists an $s \in S$ such that for all α, $\beta \in R$ with $\alpha\beta = 0$, we have $s\alpha = 0$ or $s\beta = 0$ where S is a m.c.s of R.

Proposition 2.16. *Let R be an S-integral domain and M be an R-module where S is a m.c.s of R. Then, an ideal Q of $R(+)M$ is a weakly $(S(+)0)$-2-prime ideal of $R(+)M$ if and only if Q is an $(S(+)0)$-2-prime ideal of $R(+)M$.*

Proof. To verify that weakly $(S(+)0)$-2-prime ideals and $(S(+)0)$-2-prime ideals coincide in $R(+)M$, it is enough to show that the zero ideal of $R(+)M$ is an $(S(+)0)$-2-prime ideal of $R(+)M$. Since $0 \notin S$, it is clear that $((0,0)) \cap (S(+)0) = \emptyset$. Let $(\alpha,\sigma),(\beta,\gamma) \in R(+)M$ such that $(\alpha,\sigma)(\beta,\gamma) = (\alpha\beta, \alpha\gamma+\beta\sigma) = (0,0)$. Then, $\alpha\beta = 0$, and since R is an S-integral domain, there exists an $s \in S$ such that $s\alpha = 0$ or $s\beta = 0$. Hence, $(s,0)(\alpha,\sigma)^2 = (s\alpha^2, 2s\alpha\sigma) = (0,0)$ or $(s,0)(\beta,\gamma)^2 = (s\beta^2, 2s\beta\gamma) = (0,0)$ for $s' := (s,0) \in S(+)0$. Thus, the zero ideal is an $(S(+)0)$-2-prime ideal of $R(+)M$ associated to s', as required. □

A ring R is called a Von-Neumann regular ring if for every $\alpha \in R$, there exists $r \in R$ such that $\alpha = \alpha^2 r$ [13]. Moreover, it is shown that in a Von-Neumann regular ring every ideal Q of R is a semiprime ideal or $Q = \sqrt{Q}$ [[9], Theorem 1].

Proposition 2.17. *If R is an S-integral domain, then every weakly S-2-prime ideal is S-2-prime ideal of R, where S is a m.c.s of R. The converse also holds if R is a Von-Neumann regular ring.*

Proof. Let R be an S-integral domain. Suppose that Q is a weakly S-2-prime ideal of R and $\alpha\beta \in Q$ for all $\alpha, \beta \in R$. If $\alpha\beta \neq 0$, then there exists an $s \in S$ such that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$, as desired. If $\alpha\beta = 0$, then we have $s\alpha = 0$ or $s\beta = 0$ as R is an S-integral domain. Hence, $s\alpha^2 = 0 \in Q$ or $s\beta^2 = 0 \in Q$ and Q is an S-2-prime ideal of R. Conversely, assume that R is a Von-Neumann regular ring and every weakly S-2-prime ideal is S-2-prime ideal of R. Then, (0) is an S-2-prime ideal of R. Let $\alpha, \beta \in R$ with $\alpha\beta = 0$. From assumption, there exists an $s \in S$ such that $s\alpha^2 = 0$ or $s\beta^2 = 0$. Then, $\alpha \in \sqrt{(0 : s)}$ or $\beta \in \sqrt{(0 : s)}$. Since R is a Von-Neumann regular ring, we have $s\alpha = 0$ or $s\beta = 0$, and so R is an S-integral domain. □

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