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Rings in which every S-prime ideal is prime

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Abstract. In this paper, we study the S-α-ring property, that is a ring in which every $S-$prime ideal is prime. We investigate the possible transfer of this property to various context of commutative ring extensions including homomorphic image, localization, trivial ring extensions and amalgamation rings. Our results provide new classes of commutative rings satisfying the above property.

Key Words: S-prime ideal, prime ideal, S-α-ring, localization, trivial ring extension, amalgamation ring.
2020 MSC: 13F05, 13A15, 13E05, 13F20, 13C10, 13C11, 13F30, 13D05.

Dedicated to our Professor David E. Dobbs for his 80th Birthday.

1 Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are unital. The notion of prime ideal plays an important role in commutative ring theory. Several generalizations of the concept of a prime ideal are studied in the literature for example, almost prime, strongly prime, weakly prime and $S-$prime. In this paper, we are interested with the $S-$prime ideal property introduced by Ahmed Hamed and Achraf Malek [18]. Let A be a ring. Recall that a nonempty set S of A is said to be a multiplicative set if: $0 \notin S$, $1 \in S$ and $ab \in S$ for all $a, b \in S$. A proper ideal P of A disjoint with S is called $S-$prime if there exists $s \in S$ such that for all $a, b \in A$ with $ab \in P$, then $sa \in P$ or $sb \in P$. Equivalently, if there exists $s \in S$ such that for all ideals I, J of A, if $IJ \subset P$, then $sI \subset P$ or $sJ \subset P$ (see [18]). A proper ideal P of A disjoint with S is called S-weakly prime if there exists an $s \in S$ such that for all $a, b \in A$ if $0 \neq ab \in P$, then $sa \in P$ or $sb \in P$ (see [24]). Clearly, every prime ideal is S-prime, however the converse is not true in general. There is no investigation on the following natural question: when are every $S-$prime ideal is prime? To answer this question we will study a specific class of rings that satisfy this condition.

Let A be a ring and E an $A-$module. The trivial ring extension of A by E (also called the idealization of E over A) is a ring defined as follows:

$$R := A \propto E := \{(a, e) | a \in A \ \ and \ \ e \in E\}$$

under the usual addition and the multiplication defined by $(a_1, e_1)(a_2, e_2) = (a_1a_2, a_1e_2 + a_2e_1)$ for all $(a_1, e_1), (a_2, e_2) \in A \propto E$. It is clear that $(1, 0)$ is the identity of $A \propto E$, and if S is a multiplicative subset of A, then $S \propto E$, and $S \propto 0$ are multiplicative subsets of $A \propto E$. Trivial ring extensions play an important role in commutative ring theory due to their effectiveness of producing new classes of examples and counter examples of rings subject to various ring theoretic properties (see counter-examples [4, 5, 6, 8, 21, 22, 23]).

In [11], M. D'Anna considered a different type of construction, obtained involving a ring A and an ideal $I \subseteq A$, which is denoted by $A \bowtie I$ and defined as the following subring of $A \times A$.

$$A \bowtie I := \{(a, a+i) | a \in A, \quad i \in I\}$$

.

This extension has been studied, in the general case, and from the different point of view of pull-backs, by D'anna and Fontana [10]. One main difference of this construction with respect to the idealization is that the ring $A \bowtie I$ is reduced whenever A is reduced.

Let A and B be two rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A \ \ and \ j \in J\}$$

called the amalgamation of A and B along J with respect to f. This construction is a generalization of the amalgamated duplication of a ring along an ideal denoted $A \bowtie^f J$ (introduced and studied by D'Anna and Fontana in [10]). Motivations and additional applications of the amalgamation are discussed in detail in [9, 10, 12, 13, 14, 15, 16].

In this paper, we introduce and study the class of rings in which every $S-$prime ideal is prime (S-α-ring). We next study the possible transfer of the property that every $S-$prime ideal is prime in direct product, homomorphic image, localization, trivial ring extensions, and the amalgamated algebra along an ideal, in order to give new examples.

2 Main Results

Let R be a ring and S be a multiplicative set of R. Recall from [18] that an ideal P of R disjoint with S is said to be an S-prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in P$, then $sa \in P$ or $sb \in P$.

Definition 2.1. A ring R with multiplicative set S is called S-α-ring if every S-prime ideal of R disjoint with S is prime.

Example 2.2. Let $R = D$ be a integral domain and $S = D \setminus \{0\}$ be a multiplicative set of R, then R is S-α-ring.

Let R be a ring, $S \subseteq R$ be a multiplicative set and I an ideal of R. Assume that $I \cap S = \emptyset$. Notice that $\overline{S} = \{s + I / \ s \in S\}$ is a multiplicative subset of R/I.

Remark 2.3. Let R be a ring and $S \subseteq R$ be a multiplicative set.

1. If $S \subseteq U(R)$, then R is an S-α-ring.
2. Let P an ideal of R disjoint with S. If $Z(R/P) \cap \overline{S} = \emptyset$, then the $S-$prime ideals of R are exactly the prime ideals of R.

3. Let $S_1 \subseteq S_2$ be two multiplicative subsets of R, and P be an ideal of R disjoint with S_2. If P is a S_1-prime of R, then P is S_2-prime. However, the converse is not true in general. Indeed, let $R = \mathbb{Z}[X]$, $S = \{2^n / n \in N\}$ and $T = \{1\} \subseteq U(R)$. Then $T \subseteq S$ are two multiplicative subsets of R. Let $P = 4X\mathbb{Z}[X]$. By [18, Example 1], P is an $S-$prime ideal of R. Note that P is not a T-prime ideal of R because $4X \in P$ but neither $1.4 \in P$ nor $1.X \in P$.

Proof. 1. Trivial.

2. We can use the same proof as in [18, Example 2].

□

Proposition 2.4. *Let A be a ring and $S_1 \subseteq S_2$ be two multiplicatively closed subsets of A. If A is an S_2-α-ring, then A is an S_1-α-ring.*

Proof. Assume that A is an S_2-α-ring. Let I be an S_1-prime ideal of A. Then I is S_2-prime ideal of A by Remark 2.3, and so I is a prime ideal of A. Hence, A is an S_1-α-ring, as desired. □

Proposition 2.5. *Let R be a ring, $S \subseteq R$ be a multiplicative set and I an ideal of R disjoint with S. If R is an S-α-ring, then R/I is an $\overline{S}$-α-ring.*

To prove the previous proposition, we need the following lemma.

Lemma 2.6. *[18, Proposition 3]*
Let R be a commutative ring with identity, $S \subset R$ be a multiplicative set and I be an ideal of R disjoint with S. Let P be a proper ideal of R containing I such that $P/I \cap \overline{S} = \emptyset$. Then P is an S-prime ideal of R if and only if P/I is an $\overline{S}$-prime ideal of R/I .

Proof. Assume that R is an S-α-ring. Let P/I be an ideal of R/I is $\overline{S}$-prime such that $P/I \cap \overline{S} = \emptyset$. By Lemma 2.6, P is an S-prime ideal of R. Since A is an S-α-ring, then P is a prime ideal and hence P/I is a prime ideal of R/I, as desired. □

Proposition 2.7. *Let $R = R_1 \times R_2$ be a ring, where R_1 and R_2 are two rings and $S = S_1 \times S_2$ be a multiplicative set of R, with S_1 and S_2 are multiplicative sets of R_1 and R_2 respectively. If R is an S-α-ring, then R_1 is S_1-α-ring and R_2 is S_2-α-ring.*

Proof. If $R = R_1 \times R_2$ is an S-α-ring, then by Proposition 2.5, R_1 is an S_1-α-ring and R_2 is S_2-α-ring, as $R_1 \simeq R/(0 \times R_2)$ and $R_2 \simeq R/(R_1 \times 0)$. □

The converse of Proposition 2.7 is not true in general, as shown in the following example.

Example 2.8. Let D be an integral domain, $R_1 = R_2 = D$ and $S_1 = S_2 = D \setminus \{0\}$, where S_i is a multiplicative set of A_i for $i = 1, 2$. Then R_1 is an S_1-α-ring and R_2 is an S_2-α-ring. However, $R = R_1 \times R_2$ is not an $(S_1 \times S_2)$-α-ring.

Proof. By Example 2.2, R_1 is an S_1-α-ring and R_2 is an S_2-α-ring. Let P_1 be an S_1-prime ideal of R_1. Then P_1 is prime, since R_1 is an S_1-α-ring and consider an S_2-prime ideal P_2 of R_2. Likewise. P_2 is prime. So, $P = P_1 \times P_2$ is an $(S_1 \times S_2)$-prime ideal of $R = R_1 \times R_2$ by [28, Lemma 2.13]. However, P is not always a prime ideal, as desired. □

Recall that the prime ideals of a decomposable ring $R_1 \times R_2$ are $P_1 \times R_2$ and $R_1 \times P_2$ where P_1 (resp., P_2) is a prime ideal of R_1 (resp., R_2).

Theorem 2.9. Let $R = R_1 \times R_2$ be a ring, where R_1 and R_2 are two rings and $S = (S_1 \times S_2) \subseteq R$ be a multiplicative set. If R is an S-α-ring, then $S_1 \subseteq U(R_1)$ and $S_2 \subseteq U(R_2)$ (i.e, $S = S_1 \times S_2 \subseteq U(R_1 \times R_2)$).

Proof. Assume that $S_1 \not\subseteq U(R_1)$. Then there exists $a \in S_1 \setminus U(R_1)$, and let P_2 be a prime ideal of R_2. Set $P = aR_1 \times P_2$, as $(P : (a,1)) = R_1 \times P_2$ is a prime ideal of R, we get that P is an S−prime of R by [18, Proposition 1]. Thus P is a prime ideal of R since R is S-α-ring, a contradiction since $aR_1 \neq R_1$ and $P_2 \neq R_2$, as desired. □

Let R be a ring and S be a multiplicative set of R. Recall from [24] that an ideal P of R disjoint with S is said to be an S-weakly prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $0 \neq ab \in P$, then $sa \in P$ or $sb \in P$.

Remark 2.10. Let R be an integral domain, and S be a multiplicative subset of R. Then R is an S-α-ring if and only if every S−weakly prime ideal of R is prime.

Proof. Assume that R is an S-α-ring. Since R is a domain, then every S−weakly prime ideal of R is S−prime, and so every S-prime ideal is prime, as R is an S-α-ring. Consequently, every S-weakly prime ideal of R is prime. The converse is straightforward, as desired. □

Proposition 2.11. *Let (A, M) be a zero-dimensional local ring which is not a field, and S be a multiplicative subset of A. Then A is an S-α-ring if and only if M is the only S-prime ideal of A.*

Proof. Assume that A is an S-α-ring. Let I be an S−prime ideal of A. Then I is a prime ideal of A. If $I \subseteq M$, then we get $dim(A) \geq 1$, which is a contradiction. The converse is trivial. □

Let S be a multiplicative subset of R, $S^* = \{r \in R | \ \frac{r}{1} \ is \ unit \ in \ S^{-1}A\}$ denotes the saturation of S. Observe that S^* is a multiplicative subset containing S. A multiplicative subset S of A is called saturated if $S = S^*$. It is clear that S^* is always a saturated multiplicative subset of R.

Proposition 2.12. *Let A be a ring, S be a multiplicative subset of R. Then A is an S-α-ring if and only if A is an S^*-α-ring.*

To prove the previous proposition, we need the following lemmas.

Lemma 2.13. *Let $S_1 \subseteq S_2$ be multiplicative subsets of A such that for any $s \in S_2$, there is an element $t \in S_2$ satisfying $st \in S_1$. If P is an S_2-prime ideal of A, then P is an S_1-prime ideal of A.*

Proof. Let $a, b \in A$ such that $ab \in P$. So, there is $s \in S_2$ such that $sa \in P$ or $sb \in P$. From assumption, $s' = st \in S_1$ for some $t \in S_2$, and so $s'a \in P$ or $s'b \in P$. Consequently, P is an S_1-prime ideal of A, as desired. □

Lemma 2.14. *[7, Proposition 5.1]*
Let A be a ring, S be a multiplicative subset of A and P be an ideal of A disjoint with S. Then P is an S-prime ideal of A if and only if P is an S^*-prime ideal.

Proof. Assume that A is an S^*-α-ring. Then A is an S-α-ring by Proposition 2.4. Conversely, assume that A is an S-α-ring. Let P be an S^*-prime ideal of A. Then P is an S-prime ideal of A by Lemma 2.14. Therefore, P is a prime ideal of A. Hence, A is an S^*-α-ring, as desired. □

Let $A \propto E$ be the trivial ring extension of a ring A by an A−module E. Note that if S is a multiplicative set of A, then $S \propto E$ and $S \propto 0$ are multiplicative subsets of $A \propto E$.

Theorem 2.15. Let A be a ring, E be an A−module, $R = A \propto E$ be the trivial ring extension of A by E and $S \subseteq A$ be a multiplicative set of A. Then the following statements hold:

(1) If R is an $(S \propto E)$-α-ring, then A is an S-α-ring.

(2) Assume that A is an integral domain, $K = qf(A)$, E be a $K-$vector space, and set $R = A \propto E$ be the trivial ring extension of A by E. Then the following assertions are equivalent:

i) R is an $(S \propto 0)$-α-ring.

ii) A is an S-α-ring.

iii) R is an $(S \propto E)$-α-ring.

To prove this theorem, we need the following lemmas.

Lemma 2.16. *[4, Corollary 3.4]*
Let R be an integral domain and E be an $R-$module. Then the following conditions are equivalent:

(1) Every ideal of $R \propto E$ is comparable to $0 \propto E$.

(2) Every ideal of $R \propto E$ has the form $I \propto E$ or $0 \propto F$ for some ideal I of R or sub-module F of E.

(3) Every ideal of $R \propto E$ is homogeneous.

(4) E is divisible.

Lemma 2.17. *Let A be an integral domain, $K = qf(A)$, E be a $K-$vector space, $R = A \propto E$ be the trivial ring extension of A by E and $S \subseteq A$ be a multiplicative set of A. Consider an ideal J of R. Then J is an $(S \propto E)-$prime ideal of R if and only if $J = I \propto E$ where I is an $S-$prime ideal of A.*

Proof. Assume that J is an $(S \propto E)-$prime ideal of R. Then by Lemma 2.16, two cases are then possible:
Case 1: $J = I \propto E$ with I an ideal of A. Then we show that I is an S-prime ideal of A. Let $a, b \in A$ such that $ab \in I$. Hence, $(a,0)(b,0) = (ab,0) \in I \propto E$ and so, there exists $(s,e) \in S \propto E$ such that $(s,e)(a,0) \in I \propto E$ or $(s,e)(b,0) \in I \propto E$, then either $sa \in I$ or $sb \in I$ and hence I is $S-$prime.
Case 2: $J = 0 \propto F$ with $F \neq E$ sub-module of E. In this case, we show that J is not $(S \propto E)-$prime. Assume by the way of contradiction that J is $(S \propto E)-$prime. Then there exists $(s,t) \in S \propto E$ for every $(a,e),(b,f) \in R$ such that $(a,e)(b,f) \in J$, then $(s,t)(a,e) \in J$ or $(s,t)(b,f) \in J$. Let $x \in E \setminus F$. Since $sE = E$, then $x = sy$ for some $y \in E$. On the other hand, $(0,y)(0,y) = (0,0) \in J$, since J is $(S \propto E)$-prime, so $(s,t)(0,y) = (0,sy) = (0,x) \in J$ thus $x \in F$, which is absurd.
Conversely, assume that $J = I \propto E$ where I is an $S-$prime ideal of A. Let $(a,e),(b,f) \in A \propto E$ such that $(a,e)(b,f) = (ab, af + be) \in J$. Thus, $ab \in I$ and so either $sa \in I$ or $sb \in I$ for some $s \in S$. Consequently, $(s,0)(a,e) \in I \propto E$ or $(s,0)(b,f) \in I \propto E$, making J, an $(S \propto E)-$prime ideal of R, as desired. □

Proof of Theorem 2.15

(1) Assume that R is an $(S \propto E)$-α-ring. Let I be an $S-$prime ideal of A and $(a,e),(b,f) \in A \propto E$ such that $(a,e)(b,f) = (ab, af + be) \in I \propto E$. Then $ab \in I$ and so either $sa \in I$ or $sb \in I$ for some $s \in S$. Consequently, $(s,0)(a,e) \in I \propto E$ or $(s,0)(b,f) \in I \propto E$. Therefore, $I \propto E$ is an $(S \propto E)-$prime ideal of $A \propto E$, and so $I \propto E$ is a prime ideal of R. Now, let $a, b \in A$ with $ab \in I$. Then $(a,0)(b,0) \in I \propto E$, and so $(a,0) \in I \propto E$ or $(b,0) \in I \propto E$. Consequently, $a \in I$ or $b \in I$, making I, a prime ideal of A. Hence, A is an S-α-ring.

(2) $(i) \Rightarrow (ii)$. Trivial.

$(ii) \Rightarrow (iii)$. Assume that A is an S-α-ring. By Lemma 2.16, every ideal of $A \propto E$ has the form $I \propto E$ or $0 \propto F$ for some ideal I of A or sub-module F of E.
Let J be an $(S \propto E)$-prime ideal of R. Then by Lemma 2.17, $J = I \propto E$ for some ideal I of A. Let $a, b \in A$ such that $ab \in I$. Hence $(a,0)(b,0) = (ab,0) \in I \propto E$, and so there exists $s \in S$ such

that $(s,0)(a,0) \in I \propto E$ or $(s,0)(b,0) \in I \propto E$. Therefore, either $sa \in I$ or $sb \in I$ and hence I is an $S-$prime ideal of A. So, I is a prime ideal of A, as A is an S-α-ring. Let $(a,e),(b,f) \in A \propto E$ with $(a,e)(b,f) = (ab, af + be) \in I \propto E$. Then $ab \in I$, and so $a \in I$ or $b \in I$. Consequently, $(a,e) \in I \propto E$ or $(b,f) \in I \propto E$, making J, a prime ideal of R.

$(iii) \Rightarrow (i)$. This follows from Proposition 2.4, as $S \propto 0 \subseteq S \propto E$. □

Example 2.18. Let $A = D$ be an integral domain, $S = D \setminus \{0\}$ be a multiplicative set of A and E be an A-module. Since A is an S-α-ring, then $R = D \propto E$ is an $(S \propto E)$-α-ring by assertion (2) of Theorem 2.15.

Theorem 2.19. Let (D,M) be a local domain with a prime element a of D, $S \subseteq D$ be a multiplicative set of D with $S \not\subset U(D)$, E be an $D-$module such that $ME = 0$, and let $R = D \propto E$ be the trivial ring extension of D by E. Then $J = (a,1)R$ is an $(S \propto E)-$prime ideal of R which is not prime. In particular, R is not an $(S \propto E) - \alpha-$ring.

Proof. We first claim that J is an $(S \propto E)-$prime ideal of R. Indeed, consider a non-unit element $(s,t) \in S \propto E$. Let $(x,e),(y,f) \in R$ such that $(x,e)(y,f) \in J$. If $x \notin M$, then (x,e) is a unit and so $(y,f) \in J$ and therefore, $(s,t)(y,f) \in J$. Similarly, if $y \notin M$, then $(s,t)(x,e) \in J$. On the other hand, $(x,e)(y,f) = (xy,0) = (a,1)(b,d)$ hence $xy = ab \in aD$, so $x \in aD$ or $y \in aD$. Without loss of generality, we may assume that $x \in aD$. Thus $x = ad$ hence $(s,t)(x,e) = (sx,0) = (sad,0) = (sd,0)(a,1) \in J$. Hence, $J = (a,1)R$ is $(S \propto E)-$prime. Next, we claim that J is not a prime ideal of R. Indeed, from [23, Example 2.5], it follows that $J = (a,1)R$ is not homogeneous, and so $Nil(R) \not\subset J = (a,1)R$. Thus J is not prime ideal of R. In particular, R is not an $(S \propto E) - \alpha-$ring, as desired. □

The next example illustrates Theorem 2.19.

Example 2.20. Let $(D,M) = (\mathbb{Z}_2, 2\mathbb{Z}_2)$ be a local domain, $S \subseteq D$ be a multiplicative set, with $S \not\subset U(D)$, and $E = D/M$ a $D-$module. Then $R = D \propto E$ is not an $(S \propto E) - \alpha-$ring by Theorem 2.19.

Let S be a multiplicative set of a ring A. Notice that $S' = \{(s, f(s)) \ / \ s \in S\}$. Observe that S' is a multiplicative set of $A \bowtie^f J$. Also, if $0 \notin f(S)$ then $f(S)$ is a multiplicative set of B. Set $T = \{I \bowtie^f J \ / \ I \ \ ideal \ \ of \ A\}$. Our next result establishes the transfer of the S-α-ring property to amalgamation rings.

Theorem 2.21. Let (A,B) be a pair of rings, J be an ideal of B, $f : A \longrightarrow B$ be a ring homomorphism, and S be a multiplicative set of A. Then the following statements hold :

(1) If $A \bowtie^f J$ is an S'-α-ring, then A is an S-α-ring.

(2) Assume that $f^{-1}(J) = \{0\}$. Then $A \bowtie^f J$ is an S'-α-ring if and only if $f(A)+J$ is an $f(S)$-α-ring.

(3) A is an S-α-ring if and only if every $S'-$prime ideal of $A \bowtie^f J$ in T is prime.

Proof. (1) Assume that $A \bowtie^f J$ is an S'-α-ring. Let I be an S-prime ideal of A. We first show that $I \bowtie^f J$ is an S'-prime ideal of $A \bowtie^f J$. Let $(a, f(a)+i), (b, f(b)+j) \in A \bowtie^f J$ with $(a, f(a)+i)(b, f(b)+j) \in I \bowtie^f J$. Then $ab \in I$ which is S-prime and so there exists $s \in S$ such that $sa \in I$ or $sb \in I$. One can easily check that $(s, f(s))(a, f(a)+i) \in I \bowtie^f J$ or $(s, f(s))(b, f(b)+j) \in I \bowtie^f J$. Consequently, $I \bowtie^f J$ is an S'-prime ideal of $A \bowtie^f J$ which is prime, as $A \bowtie^f J$ is an S'-α-ring. Let $a,b \in A$ such that $ab \in I$. Then $(a, f(a))(b, f(b)) \in I \bowtie^f J$, and so $(a, f(a)) \in I \bowtie^f J$ or $(b, f(b)) \in I \bowtie^f J$. It follows that $a \in I$ or $b \in I$. Finally, I is a prime ideal of A, making A, an S-α-ring.

(2) This due to the fact that $A \bowtie^f J \simeq f(A)+J$, as $f^{-1}(J) = 0$.

(3) Assume that A is an S-α-ring. Let $I \bowtie^f J$ be an S'−prime ideal of $A \bowtie^f J$ in T. We claim that $I \bowtie^f J$ is a prime ideal of $A \bowtie^f J$. Let $a, b \in A$ with $ab \in I$. Then $(a, f(a))(b, f(b)) \in I \bowtie^f J$. So, there exists $(s, f(s)) \in S'$ such that $(s, f(s))(a, f(a)) \in I \bowtie^f J$ or $(s, f(s))(b, f(b)) \in I \bowtie^f J$. It follows that $sa \in I$ or $sb \in I$. Therefore, I is a prime ideal of A since A is an S-α-ring. Let $(a, f(a)+i), (b, f(b)+j) \in A \bowtie^f J$ with $(a, f(a)+i)(b, f(b)+j) = (ab, f(a)j + f(b)i + ij) \in I \bowtie^f J$. Then $ab \in I$ and so $a \in I$ or $b \in I$. Consequently, it follows that $(a, f(a)+i) \in I \bowtie^f J$ or $(b, f(b)+j) \in I \bowtie^f J$, making $I \bowtie^f J$, a prime ideal of $A \bowtie^f J$. The converse is straightforward.

□

Next, let I be a proper ideal of A. The (amalgamated) duplication of A along I is a special amalgamation given by

$$A \bowtie I := \{(a, a+i) \mid a \in A, i \in I\}$$

Note that if S is a multiplicative set of A, then $S' = \{(s, s) \mid s \in S\}$ is a multiplicative set of $A \bowtie I$. Set $T' = \{K \bowtie I \ / \ K \ \ ideal \ \ of \ A\}$. The next corollary is an immediate consequence of assertion (3) of Theorem 2.21.

Corollary 2.22. *Let A be a ring, I an ideal of A and S be a multiplicative set of A. Then A is an S-α-ring if and only if every S'− prime ideal of $A \bowtie I$ in T' is prime.*

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