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Author(s):

Kajtaz H. Bllaca, Jawher Khmiri, Kamel Mazhouda, and Bouchaïb Sodaïgui

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On superzeta functions of the first kind on function fields

Kajtaz H. Bllaca[1], Jawher Khmiri[2], Kamel Mazhouda[3] and Bouchaïb Sodaïgui[4]

[1] Department of Mathematics, University of Prishtina, Mother Theresa, no. 5, 10000 Prishtina, Kosovo
e-mail: kajtaz.bllaca@uni-pr.edu
[2] University of Monastir, Faculty of Science of Monastir, 5000 Monastir, Tunisia
e-mail: Jawherkhmiri11@gmail.com
[3] University of Sousse, Higher Institute of Applied Sciences and Technology, 4003 Sousse, Tunisia
e-mail: kamel.mazhouda@fsm.rnu.tn
[4] Univ. Polytechnique Hauts-De-France, INSA Hauts-De-France, FR CNRS 2037,
CERAMATHS, F-59313 Valenciennes, France

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Abstract. In a recent article [2], we studied the superzeta functions on function fields. In this paper, we continue our investigation on the superzeta functions of the first kind. We prove results concerning their link with the Euler-Stieltjes constants and we give formulas for the r-depth regularized product of their zeros.
Key Words: Function fields, Superzeta functions, Riemann hypothesis, Li coefficients.
2020 MSC: Primary 11M41; Secondary 11M38.

Dedicated to our Professor David E. Dobbs for his 80$^{\text{th}}$ Birthday.

1 Introduction

1.1 Background

Functions defined by using the zeros of the Riemann zeta function as the building block of new Dirichlet series were studied firstly by Mellin in [6] (see an English translation in [12, Appendix D, p. 139]). Voros in [12] gave the name *Superzeta functions* and introduced three types of such functions, and in [12, Chapter 10, p. 91] discussed possible extensions to other classes of zeta functions.

The purpose of the present paper is to continue our study on superzeta functions of the first kind on function fields started in [2].

For the convenience of the reader, let us recall the definition and some proprieties of the zeta function of a function field as stated in [2, §1.1] (for more details see [11, Chapter 5, Section 5.1 and Section 5.2, pp. 185–218] or [9]).

Let K denote an algebraic function field of genus g whose field of constants is the finite field $\mathbb{F}_q$. Let $Z_K(X)$ be the following power series

$$Z_K(X)=\sum_{n=0}^{+\infty}C_nX^n=\prod_{D \text{ prime}}\left(1-X^{\deg(D)}\right)^{-1},$$

where $C_n = \#\{D \in Div(K);\ D \geq 0,\ \deg(D) = n\}$; $Z_K(X)$ is actually a rational function

$$Z_K(X) = \frac{L(X)}{(1-X)(1-qX)}, \tag{1}$$

where $L(X)$ factors in $\mathbb{C}[X]$ as

$$L(X) = \prod_{j=1}^{2g}(1-\alpha_j X) \in \mathbb{Z}[X]. \tag{2}$$

The special value $L(1) = \prod_{i=1}^{2g}(1-\alpha_i)$ is the class number of K, denoted by h_K. The complex numbers $\alpha_1,\dots,\alpha_{2g}$ are algebraic integers and can be arranged so that $\alpha_j\alpha_{g+j} = q$ holds for $j = 1,\dots,g$. Since the Riemann hypothesis for function fields (abbreviated to RH) proved by A. Weil [13] states that α_i, $i = 1,\dots 2g$ have absolute value $q^{1/2}$, we may order the indices $j \in \{1,\dots,g\}$ so that $\alpha_{g+j} = \overline{\alpha_j}$, and we then can write $\alpha_j = q^{1/2}\exp(i\theta_j)$ with $\theta_j \in [0,\pi]$.

Now, we define the (classical) zeta function ζ_K of K as follows: for $s \in \mathbb{C}$, we substitute X with q^{-s} in $Z_K(X)$ to get the function

$$\zeta_K(s) := Z_K(q^{-s}) = \sum_{n=0}^{+\infty} C_n q^{-ns}, \tag{3}$$

which converges for $\mathrm{Re}(s) > 1$. We define the following completed zeta function

$$\xi_K(s) := q^s(1-q^{-s})(1-q^{1-s})q^{(g-1)s}\zeta_K(s) = q^{gs}L(q^{-s}), \tag{4}$$

which is an entire function of order one, whose zeros coincide with the zeros of ζ_K. Moreover, ξ_K satisfies the functional equation

$$\xi_K(s) = \xi_K(1-s). \tag{5}$$

Let us recall that all zeros of the zeta function ζ_K lie in the critical strip $0 \leq \mathrm{Re}(s) \leq 1$, and they are symmetric with respect to the real axis and the line $\mathrm{Re}(s) = 1/2$. Note that the RH in this context is equivalent to saying that the zeros of ζ_K lie on the line $\mathrm{Re}(s) = 1/2$. Let $\mathbb{Z}(K)$ be the set of the zeros ρ of ζ_K. Using (1) and (2), we obtain

$$\mathbb{Z}(K) = \left\{\frac{1}{2} \pm i\frac{\theta_j}{\log q} + i\frac{2k\pi}{\log q},\ j \in \{1,\dots,g\},\ k \in \mathbb{Z}\right\}.$$

For an integer $n \neq 0$, the nth Li coefficient for the function field K is defined as the sum

$$\lambda_K(n) = \sum_{\rho\in\mathbb{Z}(K)}{}^{*}\left(1-\left(1-\frac{1}{\rho}\right)^n\right) := \lim_{T\longrightarrow+\infty}\sum_{\rho\in\mathbb{Z}(K);|\mathrm{Im}(\rho)|\leq T}\left(1-\left(1-\frac{1}{\rho}\right)^n\right). \tag{6}$$

From the definition of $\mathbb{Z}(K)$, we deduce that $\mathbb{Z}(K) = 1 - \mathbb{Z}(K) = 1 - \overline{\mathbb{Z}(K)}$. Hence, the Li coefficients $\lambda_K(n)$ are real and $\lambda_K(-n) = \overline{\lambda_K(n)} = \lambda_K(n)$, for all $n \in \mathbb{N}$. The Li criterion for ζ_K states: the zeros of ζ_K lie on the line $\mathrm{Re}(s) = 1/2$ if and only if the Li coefficients $\lambda_K(n)$ are nonnegative for all $n \geq 1$. Let us recall that

$$\frac{\xi_K'(z)}{\xi_K(z)} = \sum_{\rho\in\mathbb{Z}(K)}{}^{*}\frac{1}{z-\rho} \tag{7}$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}(K)$ (here $\sum^*$ is defined as in (6)). One has the following Hadamard product

$$\xi_K(z) = \prod_{\rho \in \mathbb{Z}(K)}^{*} \left(1 - \frac{z}{\rho}\right) := \lim_{T \longrightarrow +\infty} \prod_{\substack{\rho \in \mathbb{Z}(K) \\ |\mathrm{Im}(\rho)| \le T}} \left(1 - \frac{z}{\rho}\right). \tag{8}$$

Using (6) and (8) we obtain

$$\frac{d}{dz} \log\left(\xi_K\left(\frac{1}{1-z}\right)\right) = \sum_{n=1}^{+\infty} \lambda_K(n) z^{n-1}. \tag{9}$$

The Li coefficients $\lambda_K(n)$ associated to the function field K can be expressed for any positive integer n as follows (see, [5, Proposition 3.1])

$$\lambda_K(n) = -\frac{1}{(n-1)!} \frac{d^n}{dz^n} \left((z-1)^{n-1} \log \xi_K(z)\right)\Big|_{z=0}.$$

1.2 Superzeta functions of the first kind on function fields

In this subsection, we recall definitions and previous results on superzeta functions of the first kind on function fields (we refer to [2] for more details).

Following the ideas of Voros in [12, Chapter 10, 10.2, p. 93], the authors in [2] define two types of superzeta functions on K. The superzeta functions of the first kind are

$$\mathrm{Z}_K(s,z) = \sum_{\rho} \frac{1}{(z-\rho)^s}, \; Re(s) > 1, \tag{10}$$

where the sum is taken over the zeros ρ of the function ζ_K and $z \in X_K = \{z \in \mathbb{C}, \; \forall \rho \in \mathbb{Z}(K), \; z - \rho \notin \mathbb{R}_-\}$. Note that these functions are well-defined in the half-plane $Re(s) > 1$, which can be seen if we put $(z-w)^{-s} := e^{-s\log(z-w)}$ for $w \in C \setminus [z, +\infty[$, using the convention that $\log(z-w)$ is real-valued for real $w < z$. The authors in [2, Theorem 1.1] proved that for fixed $z \in X_K \setminus]-\infty, 1]$, the superzeta function $\mathrm{Z}_K(s,z)$ of the first kind has a holomorphic continuation to all $s \in \mathbb{C}$. Furthermore, they expressed $\mathrm{Z}_K(s,z)$ in terms of the Hurwitz zeta functions which provides a holomorphic continuation of it to the whole s-plane. An expression for the zeta regularized product associated to the superzeta functions of the first kind is also given.

1.3 Main results

In this subsection we give the main results of the paper.

In Section 2, we study special values of the derivative of the superzeta functions of the first kind using Seri's results (see Proposition 2.1 below from [10]).

Let us define $\mathrm{Z}_{K,*}(n) := \mathrm{Z}_K(n,1)$. Let $\gamma_K(n)$ be the coefficients appearing in the Taylor series expansion of the function $\frac{\xi_K'(z)}{\xi_K(z)}$ around $z = 1$ so that

$$\frac{\xi_K'(z)}{\xi_K(z)} = g \log q + \sum_{n=0}^{+\infty} \gamma_K(n)(z-1)^n. \tag{11}$$

The coefficients $\gamma_K(n)$ are called the Euler-Stieltjes constants of the second kind on K.

In the following theorem, we consider some special values of $\mathrm{Z}_K(s,z)$ and relate them to $\gamma_K(n)$.

Theorem 1.1. Let n be a positive integer. Then

$$\mathrm{Z}_{K,*}(n) = \begin{cases} g \log q + \gamma_K(0) & \text{if} \quad n = 1, \\ (-1)^{n-1} \gamma_K(n-1) & \text{if} \quad n \ge 2. \end{cases} \tag{12}$$

In Section 3, we give in Theorem 3.1 and Proposition 3.3 two different expressions for the r-depth regularized product associated to the superzeta functions of the first kind which are generalizations of Proposition 3.1 in [2] for $r \geq 1$.

2 The superzeta functions of the first kind and the Euler-Stieltjes constants of the second kind

2.1 Special values of the derivative of the superzeta functions of the first kind

In this subsection, we only give an idea of how to obtain an expression for $Z_K^{(m)}(0,z)$ and $Z_K^{(m)}(-n,z)$, as the formulas used (16) and (14) below are complicated. However, these expressions can be evaluated numerically for the first m values, as Seri has done in [10].

The Hurwitz zeta function $\zeta_H(s,z)$ is defined for $\Re(s) > 1$ and $z \in \mathbb{C} \setminus \{-\mathbb{N}\}$ by the absolutely convergent series $\zeta_H(s,z) = \sum_{n=0}^{+\infty} \frac{1}{(n+z)^s}$. Let us recall the following formulas.

Proposition 2.1. *For large $|a|$ and $|\arg(a)| < \pi$, we have (see [3, p. 3223] recalled in [10, Equation (2)])*

$$\zeta_H'(z+1,a) = -\left(\frac{1}{z} + \ln a\right)\zeta_H(z+1,a) + \frac{1}{2z}a^{-z-1} + \frac{1}{z}\sum_1(z,a), \tag{13}$$

where

$$\sum_1(z,a) := \sum_{k=2}^{\infty} B_k a^{-z-k} \sum_{j=0}^{k-1} \frac{(z)_k}{j!(k-j)}$$

and for $m \geq 2$, (see [10, Equation (4)])

$$\begin{aligned} &\zeta_H^{(m)}(z+1,a) \\ = \quad & \frac{(-1)^m\Gamma(m+1,z\log a)}{z^{m+1}} + \frac{(-1)^m}{2}\log^m(a)\, a^{-z-1} + \frac{1}{z}\sum_m(z,a) \\ & + \frac{1}{z}\sum_{i=0}^{m-1}\sum_i(z,a)\left\{C_{m,m-i} + \sum_{l=1}^{m-i-1}\sum_{1\leq k_0<k_1<\ldots<k_{l-1}<m-i}\left[\prod_{j=0}^{l-1} C_{m-k_{j-1},k_j-k_{j-1}}\right] C_{m-k_{l-1},m-i-k_{l-1}}\right\}, \end{aligned} \tag{14}$$

where

$$\sum_m(z,a) = \sum_{j_0=2}^{+\infty} B_{j_0} a^{-z-j_0}\left\{\sum_{j_1=0}^{j_0-1}\frac{1}{j_0-j_1}\sum_{j_2=0}^{j_1-1}\frac{1}{j_1-j_2}\cdots\sum_{j_m=0}^{j_{m-1}-1}\frac{(z)_{j_m}}{j_m!(j_{m-1}-j_m)}\right\},$$

the B_k are the Bernoulli numbers and $(n)_k := n(n+1)...(n+k-1) = \frac{\Gamma(n+k)}{\Gamma(n)}$ and for $m \geq 1$ and $1 \leq j \leq m$, $C_{m,j} := C_{m,j}(z,a) := -\binom{m}{j}\left(\frac{j}{z} + \log a\right)\log^{j-1} a$.

Let denote

$$b_j^{\pm} = \frac{1}{2} \pm \frac{i\theta_j}{\log q} \text{ and } a_j^{\pm}(z) = \frac{(z-\frac{1}{2})\log q}{2i\pi} \pm \frac{\theta_j}{2\pi}.$$

From [2, Proposition 2.1], for $z \in X'_K = \{z \in X_K \text{ and } \frac{(z-\frac{1}{2})\log q}{2i\pi} \pm \frac{\theta_j}{2\pi} \notin \mathbb{Z}\}$ one has

$$\begin{aligned} Z_K(s,z) &= -\sum_{j=1}^{g} \frac{1}{\left(z-\frac{1}{2} \pm \frac{i\theta_j}{\log q}\right)^s} + \left(\frac{2i\pi}{\log q}\right)^{-s} \sum_{j=1}^{g} \zeta_H\left(s, \frac{(z-\frac{1}{2})\log q}{2i\pi} \pm \frac{\theta_j}{2\pi}\right) \\ &\quad + \left(-\frac{2i\pi}{\log q}\right)^{-s} \sum_{j=1}^{g} \zeta_H\left(s, \frac{-(z-\frac{1}{2})\log q}{2i\pi} \pm \frac{\theta_j}{2\pi}\right) \end{aligned} \tag{15}$$

for $\mathfrak{Re}(s) > 1$ and by analytic continuation for all $s \in \mathbb{C}$. The m derivative of (15) with respect to the variable s gives

$$\begin{aligned} Z_K^{(m)}(s,z) &= \sum_{j=1}^{g} (-1)^{m+1} \frac{\log^m(z-b_j^{\pm})}{\left(z-b_j^{\pm}\right)^s} \\ &\quad + \sum_{j=1}^{g}\sum_{k=0}^{m} \left[(-1)^{m-k}\binom{m}{k}\left(\frac{2i\pi}{\log q}\right)^{-s} \log^{m-k}\left(\frac{2i\pi}{\log q}\right)\zeta_H^{(k)}(s,a_j^{\pm}(z))\right] \\ &\quad + \sum_{j=1}^{g}\sum_{k=0}^{m} \left[(-1)^{m-k}\binom{m}{k}\left(-\frac{2i\pi}{\log q}\right)^{-s} \log^{m-k}\left(-\frac{2i\pi}{\log q}\right)\zeta_H^{(k)}(s,-a_j^{\pm}(z))\right]. \end{aligned} \tag{16}$$

Now, substituting s by 0 and $-n$ in (16), for large q and $|a_j^{\pm}(z)| < \pi$, we get an expression for $Z_K^{(m)}(0,z)$, where the terms $\zeta_H^{(k)}(0,a_j^{\pm}(z))$ (resp. $\zeta_H^{(k)}(-n,a_j^{\pm}(z))$) can be obtained from Proposition 2.1 by substituting z by -1 and a by $a_j^{\pm}(z)$ (rep. z by $-n-1$ and a by $a_j^{\pm}(z)$). Since they are very long formulas we will write them only for $m = 1$ and $s = 0$.

Example 2.2. For $m = 1$ and $s = 0$ one has

$$\begin{aligned} Z'_K(0,z) &= \sum_{j=1}^{g} \log(z-b_j^{\pm}) + \sum_{j=1}^{g}\left(\zeta'_H(0,a_j^{\pm}(z)) - \log\left(\frac{2i\pi}{\log q}\right)\zeta_H(0,a_j^{\pm}(z))\right) \\ &\quad + \sum_{j=1}^{g}\left(\zeta'_H(0,-a_j^{\pm}(z)) - \log\left(-\frac{2i\pi}{\log q}\right)\zeta_H(0,-a_j^{\pm}(z))\right). \end{aligned} \tag{17}$$

Let us recall that

$$\zeta_H(-n,z) = -\frac{B_{n+1}(z)}{n+1},$$

where $n \in \mathbb{N}$ and B_n denotes the n-th Bernoulli polynomial. If $n = 0$, using $B_1(z) = z - 1/2$, we get $\zeta_H(0,z) = -B_1(z) = 1/2 - z$, then $\zeta_H(0,-z) = z + 1/2$. Furthermore, by (13), we have

$$\begin{aligned} \zeta'_H(0,a) &= (1-\ln a)\zeta_H(0,a) - \frac{1}{2} - \sum_{k=2}^{\infty} B_k a^{1-k} \sum_{j=0}^{k-1} \frac{(-1)_k}{j!(k-j)} \\ &= (1-\ln a)\left(\frac{1}{2}-a\right) - \frac{1}{2} - \sum_{k=2}^{\infty} B_k a^{1-k} \sum_{j=0}^{k-1} \frac{(-1)_k}{j!(k-j)}. \end{aligned}$$

Hence, we get

$$Z'_K(0,z) = \sum_{j=1}^{g}\left[\log\left(\frac{(z-b_j^{\pm})(a_j^{\pm}(z))^{a_j^{\pm}(z)-1/2}}{(-a_j^{\pm}(z))^{a_j^{\pm}(z)+1/2}}\right) + \sum_{k=2}^{\infty}\left(1+(-1)^{1+k}\right)B_k \times (a_j^{\pm}(z))^{1-k} \sum_{j=0}^{k-1}\frac{(-1)_k}{j!(k-j)}\right].$$

□

2.2 Superzeta functions of the first kind and the Euler-Stieltjes constants

In this subsection, we will consider special values of $Z_K(s,z)$ when s is a positive integer and relate them to the Euler-Stieltjes constants of the second kind $\gamma_K(n)$.

Proof of Theorem 1.1. By differentiating the logarithm of the Hadamard product formula (8), we obtain: for $z \in X_K$ and $n \geq 1$

$$Z_K(n,z) = \frac{(-1)^{n-1}}{(n-1)!}(\log \xi_K(z))^{(n)}.$$

Since the zeros are invariant by $\rho \longmapsto 1-\rho$, one has

$$Z_K(n,z) = (-1)^n Z_K(n, 1-z).$$

Using $\xi_K(z) = \xi_K(1-z)$, we get

$$Z_{K,*}(n) = \frac{(-1)^{n-1}}{(n-1)!}\left[(\log \xi_K(z))^{(n)}\right]_{z=1} = \frac{-1}{(n-1)!}\left[\left(\frac{\xi_K'(1-z)}{\xi_K(1-z)}\right)^{(n-1)}\right]_{z=0}. \tag{18}$$

From equations (18) and (11), we obtain the theorem. □

Proposition 2.3. *Let n be a positive integer. We have*

$$Z_{K,*}(n) = \begin{cases} g\log q + \gamma_K(0) & \text{if} \quad n = 1, \\ \displaystyle\sum_{j=1}^{n}\sum_{l=1}^{j}(-1)^{j+1}\binom{n}{j}\binom{j}{l}\gamma_K(l-1) & \text{if} \quad n \geq 2. \end{cases}$$

Proof. From [2, p. 19, line 8], one has

$$\sum_{\rho \in \mathbb{Z}(K)} \frac{1}{\rho^n} = \sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}\lambda_K(j).$$

Since $\mathbb{Z}(K) = 1 - \mathbb{Z}(K)$, then

$$Z_{K,*}(n) = \sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}\lambda_K(j).$$

Hence, the proposition follows from the following formula (see [1, Theorem 2])

$$\lambda_K(j) = jg\log q + \sum_{l=1}^{j}\binom{j}{l}\gamma_K(l-1)$$

and

$$g\log q\sum_{j=1}^{n}(-1)^{j+1}j\binom{n}{j} = \begin{cases} g\log q & \text{if} \quad n = 1, \\ 0 & \text{if} \quad n \geq 2. \end{cases}$$

□

3 Superzeta functions of the first kind and the r-depth regularized product

In this section, we derive an expression for the r-depth regularized product associated to the superzeta functions of the first kind.

The zeta regularized product of a complex sequence $(b_n)_{n\in I}$ is defined by

$$\prod_{n\in I} b_n := \exp\left(-\frac{d}{ds}\zeta_b(s)\Big|_{s=0}\right),$$

where $\zeta_b(s) := \sum_{n\in I} b_n^{-s}$ is the zeta function attached to $(b_n)_{n\in I}$, $b_n^{-s} := \exp(-s\log b_n)$, with log being the principal branch of the logarithm; we assume that ζ_b converges absolutely in some right half-plane, has a meromorphic continuation to a region containing the origin and is holomorphic at the origin.

Let r be a positive integer and consider the function

$$\phi_{K,r}(z) := \exp\left(-\frac{d}{ds}Z_K(s,z)\Big|_{s=1-r}\right). \tag{19}$$

For $r=1$, the right-hand side of (19) is the zeta regularized product of the sequence $(z-\rho)_{\rho\in Z(K)}$. For $r>1$, following [4] and [7] we call the function $\phi_{K,r}(z)$ an r-depth regularized product of the sequence $(z-\rho)_{\rho\in Z(K)}$.

Let us consider the Milnor gamma function of depth r defined by

$$\Gamma_r(z) := \exp\left(\frac{d}{ds}\zeta_H(s,z)\Big|_{s=1-r}\right).$$

For $r=1$, the authors proved in [2, Proposition 3.1] that

$$\begin{aligned}\phi_{K,1}(z) &= \left(\frac{2i\pi}{\log q}\right)^{g\left(\frac{1}{2}-\frac{(z-1/2)\log q}{2i\pi}\right)}\left(-\frac{2i\pi}{\log q}\right)^{g\left(\frac{(z-1/2)\log q}{2i\pi}-\frac{1}{2}\right)}\\ &\quad\times\prod_{j=1}^{g}\left[(z-b_j^{\pm})\left(\frac{\Gamma(a_j^{\pm}(z))\Gamma(-a_j^{\pm}(z))}{2\pi}\right)\right]^{-1}.\end{aligned} \tag{20}$$

In the following theorem we generalize (20).

Theorem 3.1. Let r be a positive integer. We have

$$\begin{aligned}\phi_{K,r}(z) &= \prod_{j=1}^{g}\Bigg[(z-b_j^{\pm})^{-(z-b_j^{\pm})^{r-1}}\left(\frac{2i\pi}{\log q}\right)^{-\frac{1}{r}\left(\frac{2i\pi}{\log q}\right)^{r-1}B_r(a_j^{\pm}(z))}\left(-\frac{2i\pi}{\log q}\right)^{\frac{1}{r}\left(\frac{2i\pi}{\log q}\right)^{r-1}\left\{B_r(a_j^{\pm}(z))+r(a_j^{\pm}(z))^{r-1}\right\}}\\ &\quad\times\Gamma_r(a_j^{\pm}(z))^{-\left(\frac{2i\pi}{\log q}\right)^{r-1}}\Gamma_r(-a_j^{\pm}(z))^{-\left(-\frac{2i\pi}{\log q}\right)^{r-1}}\Bigg].\end{aligned} \tag{21}$$

Proof. The first derivative of (15) yields

$$\begin{aligned}-\frac{d}{ds}Z_K(s,z) &= -\sum_{j=1}^{g}\log(z-b_j^{\pm})(z-b_j^{\pm})^{-s}\\ &\quad+\log\left(\frac{2i\pi}{\log q}\right)\left(\frac{2i\pi}{\log q}\right)^{-s}\sum_{j=1}^{g}\zeta_H(s,a_j^{\pm}(z))-\left(\frac{2i\pi}{\log q}\right)^{-s}\sum_{j=1}^{g}\zeta_H'(s,a_j^{\pm}(z))\\ &\quad+\log\left(-\frac{2i\pi}{\log q}\right)\left(-\frac{2i\pi}{\log q}\right)^{-s}\sum_{j=1}^{g}\zeta_H(s,-a_j^{\pm}(z))-\left(-\frac{2i\pi}{\log q}\right)^{-s}\sum_{j=1}^{g}\zeta_H'(s,-a_j^{\pm}(z)).\end{aligned}$$

Recall that $\zeta_H(1-r,w) = -\frac{B_r(w)}{r}$, then

$$\begin{aligned} -\frac{d}{ds}Z_K(s,z)\Big|_{1-r} &= -\sum_{j=1}^{g}\log(z-b_j^{\pm})(z-b_j^{\pm})^{r-1} \\ &\quad -\frac{1}{r}\log\left(\frac{2i\pi}{\log q}\right)\left(\frac{2i\pi}{\log q}\right)^{r-1}\sum_{j=1}^{g}B_r(a_j^{\pm}(z)) - \left(\frac{2i\pi}{\log q}\right)^{r-1}\sum_{j=1}^{g}\log\left(\Gamma_r(a_j^{\pm}(z))\right) \\ &\quad -\frac{1}{r}\log\left(-\frac{2i\pi}{\log q}\right)\left(-\frac{2i\pi}{\log q}\right)^{r-1}\sum_{j=1}^{g}B_r(-a_j^{\pm}(z)) - \left(-\frac{2i\pi}{\log q}\right)^{r-1}\sum_{j=1}^{g}\log\left(\Gamma_r(-a_j^{\pm}(z))\right). \end{aligned}$$

Using $B_r(-x) = (-1)^r B_r(x) + (-1)^r r x^{r-1}$, we obtain

$$\begin{aligned} -\frac{d}{ds}Z_K(s,z)\Big|_{s=1-r} &= -\sum_{j=1}^{g}\log(z-b_j^{\pm})(z-b_j^{\pm})^{r-1} \\ &\quad -\frac{1}{r}\log\left(\frac{2i\pi}{\log q}\right)\left(\frac{2i\pi}{\log q}\right)^{r-1}\sum_{j=1}^{g}B_r(a_j^{\pm}(z)) \\ &\quad -\frac{1}{r}\log\left(-\frac{2i\pi}{\log q}\right)\left(-\frac{2i\pi}{\log q}\right)^{r-1}\sum_{j=1}^{g}\left\{(-1)^r B_r(a_j^{\pm}(z)) + (-1)^r r(a_j^{\pm}(z))^{r-1}\right\} \\ &\quad -\left(\frac{2i\pi}{\log q}\right)^{r-1}\sum_{j=1}^{g}\log\left(\Gamma_r(a_j^{\pm}(z))\right) - \left(-\frac{2i\pi}{\log q}\right)^{r-1}\sum_{j=1}^{g}\log\left(\Gamma_r(-a_j^{\pm}(z))\right). \end{aligned}$$

Therefore (21) follows from $\phi_{K,r}(z) := \exp\left(-\frac{d}{ds}Z_K(s,z)\Big|_{s=1-r}\right)$. □

Remark 3.2. The expression of $\phi_{K,r}(z)$ can be written differently using the following identity (see [8, Equation (14)]) in equation (21) :

$$\begin{aligned} \zeta_H'(1-r,w) &= -\frac{1}{r^2}\left(B_r(w) + \frac{r}{2}w^{r-1}\right) \\ &\quad +\frac{1}{r}B_r(w)\log w - \frac{1}{r}\sum_{k=2}^{+\infty}\frac{B_k}{k!}\left[\frac{d}{ds}(s)_k\right]_{s=-r} w^{r-k}, \end{aligned}$$

where $\frac{d}{ds}(s)_k = \sum_{l=0}^{k-1}\frac{(s)_k}{s+1}$ and $(-r)_k = (-1)^k r!/(r-k)!$. □

Let r be a positive integer. Let us consider the polylogarithm function Li_r of degree r which is defined for $|z| < 1$ by $Li_r(z) := \sum_{m=1}^{+\infty}\frac{z^m}{m^r}$. Put $K_r(z) := \exp(-Li_r(z))$ and let $\zeta_K^{(r)}$ be the *poly-zeta-function* defined by

$$\zeta_K^{(r)}(s) = \prod_D K_r\left(\frac{1}{q^{sdeg(D)}}\right)^{-(\log D)^{1-r}}, \quad \Re(s) > 1. \tag{22}$$

Note that when $r = 1$, since $Li_1(z) = -\log(1-z)$ and $K_1(z) = (1-z)$, we have $\zeta_K^{(1)} = \zeta_K$.

In the following proposition, we give another expression for $\phi_{K,r}(z)$ using the poly-zeta-function $\zeta_K^{(r)}$ defined above (similar definition is given by Kurokawa et al. in [4, Section 4.2]).

Proposition 3.3. *For $r \geq 1$ and $\mathfrak{Re}(z) > 1$, we have*

$$\phi_{K,r}(z) = (z-1)^{(z-1)^{1-r}} z^{z^{1-r}} \left(\frac{2\pi i}{\log q}\right)^{-\frac{1}{r}\left(\frac{2\pi i}{\log q}\right)^{r-1}\left[B_r\left(\frac{z\log q}{2\pi i}\right)+B_r\left(\frac{(z-1)\log q}{2\pi i}\right)\right]}$$
$$\times\left(-\frac{2\pi i}{\log q}\right)^{-\frac{(-1)^r}{r}\left(-\frac{2\pi i}{\log q}\right)^{r-1}\left\{B_r\left(\frac{z\log q}{2\pi i}\right)+B_r\left(\frac{(z-1)\log q}{2\pi i}\right)+r\left(-\frac{z\log q}{2\pi i}\right)^{r-1}+r\left(-\frac{(z-1)\log q}{2\pi i}\right)^{r-1}\right\}}$$
$$\times\left[\Gamma_r\left(\frac{z\log q}{2\pi i}\right)\Gamma_r\left(\frac{(z-1)\log q}{2\pi i}\right)\right]^{-\left(\frac{2\pi i}{\log q}\right)^{r-1}}$$
$$\times\left[\Gamma_r\left(-\frac{z\log q}{2\pi i}\right)\Gamma_r\left(-\frac{(z-1)\log q}{2\pi i}\right)\right]^{-\left(-\frac{2\pi i}{\log q}\right)^{r-1}}$$
$$\times\left(\zeta_K^{(r)}(z)\right)^{(-1)^r(r-1)!}.$$

Proof. By [2, Proposition 4.1], we have

$$\begin{aligned}
&(-1)^r(r-1)!\log\zeta_K^{(r)}(z)\\
= \;& \frac{\log(z-1)}{(z-1)^{1-r}}+\frac{\log z}{z^{1-r}}\\
&+\frac{1}{r}\left(\frac{2\pi i}{\log q}\right)^{r-1}\log\left(\frac{2\pi i}{\log q}\right)\left[B_r\left(\frac{z\log q}{2\pi i}\right)+B_r\left(\frac{(z-1)\log q}{2\pi i}\right)\right]\\
&+\frac{(-1)^r}{r}\left(-\frac{2\pi i}{\log q}\right)^{r-1}\log\left(-\frac{2\pi i}{\log q}\right)\left\{B_r\left(\frac{z\log q}{2\pi i}\right)+B_r\left(\frac{(z-1)\log q}{2\pi i}\right)+r\left(-\frac{z\log q}{2\pi i}\right)^{r-1}+r\left(-\frac{(z-1)\log q}{2\pi i}\right)^{r-1}\right\}\\
&+\left(\frac{2\pi i}{\log q}\right)^{r-1}\log\left(\Gamma_r\left(\frac{z\log q}{2\pi i}\right)\Gamma_r\left(\frac{(z-1)\log q}{2\pi i}\right)\right)\\
&+\left(-\frac{2\pi i}{\log q}\right)^{r-1}\log\left(\Gamma_r\left(-\frac{z\log q}{2\pi i}\right)\Gamma_r\left(-\frac{(z-1)\log q}{2\pi i}\right)\right)\\
&-\frac{d}{ds}Z_K(s,z)\Big|_{s=1-r}.
\end{aligned}$$

We finish the proof by using $\phi_{K,r}(z) = \exp\left(-\frac{d}{ds}Z_K(s,z)\Big|_{s=1-r}\right)$. □

Remark 3.4. Let us note that for $r = 1$ and $\mathfrak{Re}(z) > 1$ we get (see [2, Corollary 4.4])

$$\phi_K(z) = z(z-1)\left(\frac{2\pi i}{\log q}\right)^{1-\frac{(2z-1)\log q}{2\pi i}}\left(-\frac{2\pi i}{\log q}\right)^{1+\frac{(2z-1)\log q}{2\pi i}}\left[\left(\frac{\Gamma\left(\frac{\pm z\log q}{2\pi i}\right)}{\sqrt{2\pi}}\right)\left(\frac{\Gamma\left(\frac{\pm(z-1)\log q}{2\pi i}\right)}{\sqrt{2\pi}}\right)\zeta_K(z)\right]^{-1}.$$

□

4 Concluding remarks

The zeros of the function ζ_K are denoted by

$$\rho_{k,j}^{\pm} = \frac{1}{2}+i\tau_{k,j}^{\pm}, \quad \text{where} \quad \tau_{k,j}^{\pm} = \frac{\pm\theta_j + 2k\pi}{\log q}, \; j = 1,..,g \text{ and } k \in \mathbb{Z}.$$

As in [2] and [12, Chapter 8 and 9], the superzeta functions of the second kind $\mathcal{Z}_K$ and the third kind $\mathfrak{Z}_K$ are defined by

$$\mathcal{Z}_K(s,t) = \sum_{k\in\mathbb{Z}}\sum_{j=1}^{g}\frac{1}{\left((\tau_{k,j}^{\pm})^2+t^2\right)^s}, \quad Re(s) > \frac{1}{2},$$

where $t \in \mathbb{C}$ such that $t^2 + (\tau^{\pm}_{k,j})^2 \notin \mathbb{R}_-$ for all k, and

$$\mathfrak{Z}_K(s,\tau) = \sum_{k\in\mathbb{Z}} \sum_{j=1}^{g} \frac{1}{(\tau^{\pm}_{k,j} + \tau)^s}, \quad Re(s) > 1,$$

where $\tau \in \mathbb{C}$ such that $\tau + \tau^{\pm}_{k,j} \notin \mathbb{R}_-$ for all $j = 1, .., g$.

As Voros did in [12, (9.3) and (9.4), pp. 87-88] for the classical zeta function, a perspective of this paper is to describe the family of $\mathfrak{Z}_K$ using that of Z_K and get similar results for superzeta functions of the third kind as for superzeta functions of the first kind and the second kind (see [2]).

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