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Some New Fixed point theorems for asymptotically nonexpansive type mappings in $CAT(k)$ spaces

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Abstract. In this paper we prove a convergence theorem of the Mann iteration scheme for a asymptotically nonexpansive type mappings in $CAT(k)$ spaces with $k > 0$. We also obtain a convergence theorem of the Ishikawa iteration scheme for asymptotically nonexpansive mappings.

Key Words: Mann and Ishikawa iteration; strong convergence; $CAT(k)$ space; common fixed point; R−convex space.
2020 MSC: 39B05; 39B22; 46B22; 20K30.

Dedicated to our Professor David E. Dobbs for his 80th Birthday.

1 Introduction and preliminaries

Roughly speaking, $CAT(k)$ spaces are geodesic spaces of bounded curvature and generalization of Riemanian manifold of sectional curvature bounded above. The letter C,A and T stand for Cartav, Alexandrv and Toponogov who have made important contributions to the understanding of curvature via inequalities for the distance function and k is a real number that we impose it as the curvature bound of the space. Fixed point theory in $CAT(k)$ spaces was first studied by Kirk [1]-[2]. Since any $CAT(k)$ space is a $CAT(k')$ space for $k' > k$, so, all results for $CAT(0)$ spaces immediately apply to any $CAT(k)$ spaces with $k \leq 0$. However, there are only a few articles that contain fixed point results in the setting of $CAT(k)$ spaces with $k > 0$, because in this case the proof seems to be more complicated. The interplay between the geometry of Banach spaces and fixed point theory has been very strong and fruitful. In particular, geometric properties play a Key role in metric fixed point problems.
Gromov [9], introduced the notation of $CAT(0)$ spaces as follows:

Definition 1.1. Let C be a nonempty subset of metric space (X, d). A mapping $T : C \to C$ is said to be
(i) Lipschitzian if $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in C, k \geq 0$,
(ii) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$
(iii) asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [0, 1)$ with $lim_{n\to\infty} k_n = 1$ such that $d(T^n x, T^n y) \leq k_n d(x, y)$ for all $x, y \in C$. Class of asymptotically nonexpansive mappings includes a class of nonexpansive mapping as proper subclass and both the mappings are Lipschitzian.

In 1974 , Kirk, substantially weaken the assumption of asymptotic nonexpansive ness of T by replacing it with assumption, which may hold even if none of the iterates of T is lipschitzian. A mapping $T : C \to C$ is said to be asymptotically nonexpansive type if for each $y \in C$ the following inequality holds:

$$lim_{n\to\infty} Sup\{Sup_{x\in C}\{d(T^n x, T^n y) - d(x,y)\} \leq 0.$$

Every asymptotically nonexpansive mapping is asymptotically nonexpansive type but converse need not be true. The concept of asymptotically nonexpansive type mappings is more of asymptotically nonexpansive mappings. Iterative approximation of fixed points of asymptotically nonexpansive and asymptotically nonexpansive type mapping have been studied by various author in the setting of Hilbert spaces, Banach spaces and convex metric spaces (see [9] and reference therein).

Definition 1.2. Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map C from a closed interval $[0,l] \subset R$ to X such that $C(0) = x, C(l) = y$ and $d(C(t), C(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, C is an isometry and $d(x,y) = l$. The image of C is called geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x,y]$.

Definition 1.3. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if Y includes every geodesic segment joining any two of its point.

Definition 1.4. A subset C of X is said to be bounded if $diam(C) = sup\{d(x,y) : x, y \in C\} < \infty$.

Definition 1.5. Given $k \in R$, we denote by M_k^n the following metric spaces:
(i) if $k = 0$ then M_k^n is the Euclidean space E^n;
(ii) if $k > 0$ then M_k^n is obtained from the spherical space S^n by multiplying the distance function by the constant $\frac{1}{\sqrt{k}}$;
(iii) if $k < 0$ then M_k^n is obtained from the hyperbolic space H^n by multiplying the distance function by the constant $\frac{1}{\sqrt{-k}}$.

Definition 1.6. If x, y_1, y_2 are points in a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}\left\{d^2(x, y_1) + d^2(x, y_2) - \frac{1}{2}d^2(y_1, y_2)\right\},$$

the above inequality named as $CN-$inequality. In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the $CN-$inequality. $CAT(0)$ spaces may be regarded as a metric version of Hilbert spaces. For example, in any Hilbert space H we have the following extended version of parallelogram law:

$$||z - (\alpha x + (1-\alpha)y)||^2 = \alpha||z - x||^2 + (1-\alpha)||z - y||^2 - \alpha(1-\alpha)||x - y||^2$$

for any $\alpha \in [0,1]$ and $x, y, z \in H$.
$CN-$ inequality is extended by Dhom Pongsa and Panyanak [4] as CN^*-inequality such as :

$$d^2(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d^2(x,z) + (1-\alpha)d^2(z,y) - \alpha(1-\alpha)\alpha d^2(x,y)$$

for any $\alpha \in [0,1]$ and $x, y, z \in X$. If $\alpha = \frac{1}{2}$, then the above inequality becomes the $CN-$inequality. Infact , If X is a geodesic space then the following statements are equivalent:
(i) X is a $CAT(0)$ space
(ii) X satisfies $CN-$inequality
(iii) X satisfies CN^*-inequality.

Definition 1.7. Let $R \in (0,2]$, a geodesic space (X,d) is said to be $R-$convex for R if for any three points $x,y,z \in X$ we have

$$d^2(z,\alpha x \oplus (1-\alpha)y) \leq \alpha d^2(x,z) + (1-\alpha)d^2(z,y) - \frac{R\alpha(1-\alpha)d^2(x,y)}{2}.$$

It follows from CN^*-inequality that a geodesic space (X,d) is a $CAT(0)$ space if and only if (X,d) is $R-$convex for $R=2$.

Lemma 1.8. *Let k be an arbitrary positive real number and (X,d) be a $CAT(k)$ space with $diam(X) \leq \frac{\pi-2\eta}{2\sqrt{k}}$ for some $\eta \in \left(0,\frac{\pi}{2}\right)$. Then (X,d) is $R-$convex for $R=(\pi-\eta)tan\eta$.*

Lemma 1.9. *Let (X,d) be a $CAT(0)$ space, then*
(i) for $x,y \in X$ and $t \in [0,1]$ there exists a unique point $z \in [x,y]$ such that

$$d(z,x) = td(x,y) \text{ and } d(y,z) = (1-t)d(x,y),$$

we use the notation $(1-t)x \oplus ty$ for the unique point z satisfying the above equalities.
(ii) for $x,y,z \in X$ and $t \in [0,1]$, we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Definition 1.10. Let C be nonempty subset of a $CAT(k)$ space (X,d) and $T: C \to C$ be a mapping. We denote by $F(T)$ the set of all fixed points of T, i.e., $F(T) = \{x \in C : x = Tx\}$. Then T is said to
(i) be completely continuous if T is continuous and for any bounded sequence $\{x_n\}_{n-=1}^{+\infty}$ in C, $\{Tx_n\}_{n-=1}^{+\infty}$ has a convergent subsequence in C.
(ii) be uniformly $L-$Lipschitzian if there exists a constant $L>0$ such that

$$d(T^n x, T^n y) \leq Ld(x,y) \text{ for all } x,y \in C \text{ and all } n \in N.$$

(iii) be asymptotically demicontractive if $F(T) \neq \phi$ and there exists $k \in [0,1)$ and a sequence $\{a_n\}$ with $lim_{n\to\infty} a_n = 1$ such that

$$d^2(T^n x, p) \leq a_n^2 d^2(x,p) + kd^2(x,T^n x) \text{ for all } x \in C,\, p \in F(T) \text{ and all } n \in N.$$

(iv) be asymptotically hemicontractive if $F(T) \neq \phi$ and there exists a sequence $\{a_n\} \in [0,1)$ with $lim_{n\to\infty} a_n = 1$ such that

$$d^2(T^n x, p) \leq a_n d^2(x,p) + d^2(x,T^n x) \text{ for all } x \in C\ ,\, p \in F(T) \text{ and all } n \in N.$$

Remark 1.11. It follows from the definition that every asymptotically demicontractive mapping is asymptotically hemicontractive.

Definition 1.12. A mapping $P: X \to C$ is said to be retraction if $C \subset X$ and P restricted to C is the identity, i.e., $Px = x$ for any $x \in C$. Clearly $P^2 = P$, the set C is called a retract of X.

Lemma 1.13. *Let C be a convex subset of X which is complete in the induced metric. Then, for every $x \in X$, there exists a unique point $p(x) \in C$ such that $d(x,p(x)) = d(x,C) = inf_{y\in C} d(x,y)$. Moreover, the map $x \longrightarrow px$ is a nonexpansive retract from X onto C.*

Lemma 1.14. *Let $\{s_n\}_{n=1}^{+\infty}, \{t_n\}_{n=1}^{+\infty} \in R^+$ be sequences satisfying $s_{n+1} \leq s_n + t_n$ for all $n \in N$. If $\sum_{n=1}^{\infty} t_n < \infty$ and $\{s_n\}_{n=1}^{+\infty}$ has a subsequence converging to 0 , then $lim_{n\to\infty} s_n = 0$.*

Definition 1.15. Let C be a nonempty convex subset of a $CAT(k)$ space (X,d) and $T: C \to C$ be a mapping. Given $x_1 \in C$ and the sequence $\{x_n\}_{n-=1}^{+\infty}$ is defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n y_n \\ y_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n. \end{cases}$$

is called an Ishikawa iterative sequence. If $\beta_n = 0$ for all $n \in N$, then the above algorithm reduces to the following:

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n$$

is called a Mann iterative sequence.

Definition 1.16. Let C be a nonempty subset of a $CAT(0)$ space X. A map $T: C \to X$ is said to be asymptotically nonexpansive type, if for each $y \in C$

$$lim_{n\to\infty} Sup\{Sup_{x\in C}\{d(T(PT)^{n-1}x, T(PT)^{n-1}y) - d(x,y)\}\} \le 0.$$

where P is the nonexpansive retraction of X onto C.

Definition 1.17. Let $\{x_n\}_{n=1}^{+\infty}$ be bounded sequence in a $CAT(0)$ space X and C be a subset of X. In the rest of the present paper, we use the following notations:
(i) $r(x, \{x_n\}_{n=1}^{+\infty}) = lim_{n\to\infty} sup d(x, x_n)$.
(ii) $r(\{x_n\}_{n=1}^{+\infty}) = inf_{x\in X} r(x, \{x_n\}_{n=1}^{+\infty})$.
(iii) $r_C(\{x_n\}_{n=1}^{+\infty}) = inf_{x\in C} r(x, \{x_n\}_{n=1}^{+\infty})$.
(iv) $A(\{x_n\}_{n=1}^{+\infty}) = \{x \in X : r(x, \{x_n\}_{n=1}^{+\infty}) = r(\{x_n\}_{n=1}^{+\infty})$.
(v)$A_C(\{x_n\}_{n=1}^{+\infty}) = \{x \in X : r(x, \{x_n\}_{n=1}^{+\infty}) = r_C(\{x_n\}_{n=1}^{+\infty})$.
Note that $x \in X$ is called an asymptotic center of $\{x_n\}$ if $x \in A(\{x_n\}_{n=1}^{+\infty})$.

Definition 1.18. Let (X,d) be a $CAT(0)$ space. A sequence $\{x_n\}_{n=1}^{+\infty}$ in X is said to be $\Delta-$convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}_{n=1}^{+\infty}$ for every subsequence $\{u_n\}_{n=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$, i.e., $A(\{u_n\}_{n=1}^{+\infty}) = \{x\}$ for every subsequence $\{u_n\}_{n=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$. In this case, we write $\triangle - lim x_n = x$ and x is called $\triangle - limit$ of $\{x_n\}_{n=1}^{+\infty}$.

Definition 1.19. A map T from a subset K of a metric space (X,d) into itself is said to be semi-compact if every bounded sequence $\{x_n\}_{n=1}^{+\infty} \subset K$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a strongly convergent subsequence.

Definition 1.20. A nonself mapping $T: K \longrightarrow X$ with $F(T) \neq \emptyset$ is said to satisfy the condition (I) if there exists a non-decreasing function $f: [0,\infty) \longrightarrow [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $d(x, Tx) \ge f(d(x, F(T))$ for all $x \in K$.

Lemma 1.21. *Let $k > 0$ and (X,d) be a complete $CAT(k)$ space with $diam(X) \le \frac{\pi-2\eta}{2\sqrt{k}}$ for some $\eta \in \left(0, \frac{\pi}{2}\right)$. Then the following statements hold:*
(i) Every sequence in X has a $\triangle-$convergent subsequence;
(ii) if $\{x_n\}_{n=1}^{+\infty} \subseteq X$ and $\triangle - lim_{n\to\infty} x_n = x$ then $x \in \bigcap_{k=1}^{\infty} \overline{conv}\{x_k, x_{k+1}, \ldots\}$, where $\overline{conv}(A) = \bigcap_{B\supseteq A} B$ where B is closed and convex.

Lemma 1.22. *Let $k > 0$ and (X,d) be a complete $CAT(k)$ space with $diam(X) \le \frac{\pi-2\eta}{2\sqrt{k}}$ for some $\eta \in \left(0, \frac{\pi}{2}\right)$. If $\{x_n\}_{n=1}^{+\infty}$ is a sequence in X with $A(\{x_n\}_{n=1}^{+\infty}) = \{x\}$, $\{u_n\}_{n=1}^{+\infty}$ is a subsequences of $\{x_n\}$ with $A\left(\{u_n\}_{n=1}^{+\infty}\right) = \{u\}$ and the sequence $\{d(x_n, u)\}_{n=1}^{+\infty}$ converges then $x = u$.*

The concept of asymptotically nonexpansive type mappings is more general than that of asymptotically nonexpansive mappings. Iterative approximation of fixed points of asymptotically nonexpansive and asymptotically nonexpansive type map- ping have been studied by various author in the setting of Hilbert spaces, Banach spaces and convex metric spaces (see [3]-[17] and reference therein).

2 Main Results

Theorem 2.1. Let $k > 0$ and (X,d) be a $CAT(k)$ space with $diam(X) \leq \frac{\pi-2\eta}{2\sqrt{k}}$ for some $\eta \in \left(0, \frac{\pi}{2}\right)$. Let C be a nonempty convex subset of X and $T : C \to C$ be asymptotically nonexpansive type and $\{\alpha_n\}_{n=1}^{+\infty}, \{\beta_n\}_{n=1}^{+\infty} \in [0,1]$. Given $x_1 \in C$, define the iteration scheme $\{x_n\}_{n=1}^{+\infty}$ by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n y_n \\ y_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n. \end{cases}$$

then

$$d(x_n, Tx_n) \leq d(x_n, T^n x_n) + 3d(x_{n-1}, T^{n-1}x_{n-1}) + \frac{2}{n^2}.$$

Proof. Let $C_n = d(x_n, T^n x_n)$, the we have

$$\begin{aligned} d(x_{n-1}, y_{n-1}) &= d(x_{n-1}, (1-\beta_{n-1})x_{n-1} \oplus \beta_{n-1} T^{n-1}x_{n-1}) \\ &\leq \beta_{n-1} d(x_{n-1}, T^{n-1}x_{n-1}) \\ &= \beta_{n-1} C_{n-1}. \end{aligned} \tag{1}$$

Also

$$\begin{aligned} d(x_{n-1}, T^{n-1}y_{n-1}) &\leq d(x_{n-1}, T^{n-1}x_{n-1}) + d(T^{n-1}x_{n-1}, T^{n-1}y_{n-1}) \\ &\leq d(x_{n-1}, T^{n-1}x_{n-1}) + d(x_{n-1}, y_{n-1}) \\ &+ \{d(T^{n-1}x_{n-1}, T^{n-1}y_{n-1}) - d(x_{n-1}, y_{n-1})\} \\ &\leq d(x_{n-1}, T^{n-1}x_{n-1}) + d(x_{n-1}, y_{n-1}) \\ &+ Sup_{k\geq n}[Sup_{x_{n-1}\in C}\{d(T^{n-1}x_{n-1}, T^{n-1}y_{n-1}) - d(x_{n-1}, y_{n-1})\}]. \end{aligned} \tag{2}$$

Now, since T is asymptotically nonexpansive type we can choose n_0 such that $n \geq n_0$ implies that

$$Sup_{k\geq n}[Sup_{x_{n-1}\in C}\{d(T^{n-1}x_{n-1}, T^{n-1}y_{n-1}) - d(x_{n-1}, y_{n-1})\}] \leq \frac{1}{n^2},$$

therefore

$$d(x_{n-1}, T^{n-1}y_{n-1}) \leq C_{n-1} + \beta_{n-1} + \frac{1}{n^2}.$$

So, by the above inequalities we conclude that

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, T^n x_n) + d(T^n x_n, Tx_n) \\ &\leq C_n + d(T^n x_n, T^n x_{n-1}) + d(T^n x_{n-1}, Tx_n) \\ &\leq C_n + [d(T^n x_n, T^n x_{n-1}) - d(x_n, x_{n-1})] + d(x_n, x_{n-1}) + d(T^n x_{n-1}, Tx_n) \\ &\leq C_n + Sup_{k\geq n}[Sup_{x\in C}\{d(T^k x_n, T^k x_{n-1}) - d(x_n, x_{n-1})\}] \\ &+ d(x_n, x_{n-1}) + d(T^n x_{n-1}, Tx_n) \\ &\leq C_n + \frac{1}{n^2} + d(x_n, x_{n-1}) + d(T^n x_{n-1}, Tx_n), \end{aligned} \tag{3}$$

and then we have

$$\begin{aligned} d(x_n, Tx_n) &\leq C_n + \frac{1}{n^2} + d(x_n, x_{n-1}) + d(T^n x_{n-1}, Tx_n) \\ &\leq C_n + \frac{1}{n^2} + d(x_n, x_{n-1}) + d(T^{n-1}x_{n-1}, x_n) \\ &+ [d(T(T^{n-1})x_{n-1}, Tx_n) - d(T^{n-1}x_{n-1}, x_n)] \\ &\leq C_n + d(x_n, x_{n-1}) + \frac{1}{n^2} \\ &+ Sup_{k\geq n}[Sup_{x\in C}\{d(T(T^{k-1})x_{n-1}, Tx_n) - d(T^{k-1}x_{n-1}, x_n)\}] \\ &+ d(T^{n-1}x_{n-1}, x_n). \end{aligned} \tag{4}$$

Thus

$$d(x_n, Tx_n) \leq C_n + C_{n-1} + \frac{2}{n^2} + 2d(x_n, x_{n-1}),$$

and finally we have

$$\begin{aligned} d(x_n, Tx_n) &\leq C_n + C_{n-1} + \frac{2}{n^2} + 2d((1-\alpha_{n-1})x_{n-1} \oplus \alpha_{n-1}T^{n-1}(y_{n-1}), x_{n-1}) \\ &\leq C_n + C_{n-1} + \frac{2}{n^2} + 2\alpha_{n-1}d(T^{n-1}(y_{n-1}), x_{n-1}) \\ &\leq C_n + 3C_{n-1} + \frac{2}{n^2} \end{aligned} \tag{5}$$

□

Theorem 2.2. Let $k > 0$ and (X, d) be a $CAT(k)$ space with $diam(X) \leq \frac{\pi - 2\eta}{2\sqrt{k}}$ for some $\eta \in \left(0, \frac{\pi}{2}\right)$. Let C be a nonempty closed convex subset of X, $T : C \to C$ be a completely continuous asymptotically nonexpansive type and $\{\alpha_n\}_{n=1}^{\infty} \in \left[\varepsilon, \frac{R}{2} - k - \varepsilon\right]$ for some $\varepsilon > 0$ where $R = (\pi - 2\eta)tan\eta$. Given $x_1 \in C$, let define the iteration scheme $\{x_n\}_{n=1}^{\infty}$ by $x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n$, then the sequence $\{x_n\}_{n=1}^{\infty}$ convergence strongly to a fixed point of T.

Proof. Let $p \in F(T)$, since (X, d) is a $R-convex$ for R, then we have

$$\begin{aligned} &d^2(x_{n+1}, p) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(T^n x_n, p) - \frac{R\alpha_n(1-\alpha_n)d^2(x_n, T^n x_n)}{2} \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n[d^2(T^n x_n, T^n p) - d^2(x_n, p)] \\ &+\alpha_n d^2(x_n, p) - \frac{R\alpha_n(1-\alpha_n)d^2(x_n, T^n x_n)}{2} \\ &\leq d^2(x_n, p) + \alpha_n Sup_{k\geq n}[Sup_{x_n\in C}\{d^2(T^k x_n, T^k p) - d^2(x_n, p)\}] \\ &+\alpha_n d^2(x_n, p) - \frac{R\alpha_n(1-\alpha_n)d^2(x_n, T^n x_n)}{2} \\ &\leq d^2(x_n, p) + \alpha_n Sup_{k\geq n}[Sup_{x_n\in C}\{d(T^k x_n, T^k p) - d(x_n, p)\} \times \{d(T^k x_n, T^k p) + d(x_n, p)\}] \\ &+\alpha_n d^2(x_n, p) - \frac{R\alpha_n(1-\alpha_n)d^2(x_n, T^n x_n)}{2}. \end{aligned} \tag{6}$$

So we have

$$d^2(x_{n+1}, p) \leq d^2(x_n, p) + \frac{\alpha_n}{n^2}\frac{\pi^2}{2k} - \varepsilon^2 d^2(x_n, T^n x_n),$$

and therefore

$$\varepsilon^2 d^2(x_n, T^n x_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p) + \frac{\alpha_n}{n^2}\frac{\pi^2}{2k},$$

and since $\sum_{n=1}^{\infty} \frac{\pi^2}{4kn^2} < \infty$ then $\sum_{n=1}^{\infty} d^2(x_n, T^n x_n) < \infty$ which implies that $lim_{n\to\infty} d(x_n, T^n x_n) = 0$. So, by theorem 2.1, we have $lim_{n\to\infty} d(x_n, Tx_n) = 0$. On the other hand, since T is completely continuous, so $\{Tx_n\}_{n=1}^{\infty}$ has a convergent subsequence in C and therefore by the last equality, $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence, say $x_{n_k} \to q \in C$. Moreover

$$d(q, Tq) \leq d(q, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, Tq) \to 0 \text{ as } k \to \infty,$$

and it means that $q \in F(T)$. On the other hand $d^2(x_{n+1}, p) \leq d^2(x_n, p) + \frac{\alpha_n}{n^2}\frac{\pi^2}{2k}$ and since $\sum_{n=1}^{\infty} \frac{\pi^2}{4kn^2} < \infty$, so by lemma 1.14 , we have $\{x_n\}_{n=1}^{\infty} \to q$. This completes the proof. ☐

Corollary 2.3. *Let $k > 0$ and (X, d) be a $CAT(k)$ space with $diam(X) \leq \frac{\pi-2\eta}{2\sqrt{k}}$ for some $\eta \in \left(0, \frac{\pi}{2}\right)$. Let C be a nonempty closed convex subset of X and $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping. Also, $\{\alpha_n\}_{n=1}^{\infty}$ be a sequences in $\left[\varepsilon, \frac{R}{2} - k - \varepsilon\right]$ for some $\varepsilon > 0$ where $R = (\pi - 2\eta)tan\eta$. Given $x_1 \in C$ and we define the iteration scheme $\{x_n\}_{n=1}^{\infty}$ by $x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ convergence strongly to a fixed point of T.*

Proof. We know that every asymptotically nonexpansive mapping is assymptotically nonexpansive mapping type and it is well known that every convex subset of a $CAT(0)$ space equipped with the induced metric is a $CAT(0)$ space. Then (C, d) is a $CAT(0)$ space and hence it is a $CAT(k)$ space for all $k > 0$. Notice, also that C is $R-$convex for $R = 2$. Since C is bounded, we can choose $\eta \in \left(0, \frac{\pi}{2}\right)$ and $k > 0$ so that $diam(X) \leq \frac{\pi-2\eta}{2\sqrt{k}}$ and then he conclusion follows from the previous theorem. ☐

Now, we prove the strong convergence of Ishikawa iteration for asymptotically nonexpansive type mapping.

Theorem 2.4. Let $k > 0$ and (X, d) be a $CAT(k)$ space with $diam(X) \leq \frac{\pi-2\eta}{2\sqrt{k}}$ for some $\eta \in \left(0, \frac{\pi}{2}\right)$. Let $R = (\pi - 2\eta)tan(\eta)$, C be a nonempty closed convex subset of X, $T : C \to C$ is asymptotically nonexpansive type mapping and $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \in [0, 1]$. Also, let $x_1 \in C$ and define the iteration scheme $\{x_n\}_{n=1}^{\infty}$ by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n T^n y_n \\ y_n = (1 - \beta_n)x_n \oplus \beta_n T^n x_n. \end{cases}$$

Then the following inequality holds :

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2((1 - \alpha_n)x_n \oplus \alpha_n T^n y_n, p) \\ &\leq d^2(x_n, p) + \alpha_n \pi \left(\frac{1 + \pi\beta_n}{4n^2\sqrt{k}}\right) \\ &- \frac{R\alpha_n\beta_n(1 - \beta_n)d^2(x_n, T^n x_n)}{2} - \frac{R\alpha_n(1 - \alpha_n)d^2(x_n, T^n y_n)}{2} \end{aligned} \tag{7}$$

Proof. Let $p \in F(T)$, since (X, d) is $R-$convex, so we have

$$d^2(x_{n+1}, p) \leq (1 - \alpha_n)d^2(x_n, p) + \alpha_n d^2(T^n y_n, p) - \frac{R\alpha_n(1 - \alpha_n)d^2(x_n, T^n y_n)}{2} \tag{8}$$

and

$$d^2(y_n,p) \leq (1-\beta_n)d^2(x_n,p) + \beta_n d^2(T^n x_n,p) - \frac{R\beta_n(1-\beta_n)d^2(x_n,T^n x_n)}{2}. \tag{9}$$

On the other hand, since T is asymptotically nonexpansive type mapping, so we obtain that

$$\begin{aligned} d^2(T^n y_n,p) &= [d^2(T^n y_n,T^n p) - d^2(y_n,p)] + d^2(y_n,p) \\ &= [d(T^n y_n,T^n p) - d(y_n,p)] \times [d(T^n y_n,T^n p) + d(y_n,p)] + d^2(y_n,p) \\ &\leq Sup_{k\geq n}[Sup_{x\in C} d(T^k y_n,T^k p) - d(y_n,p)][d(T^n y_n,T^n p) + d(y_n,p)] + d^2(y_n,p) \\ &\leq \frac{\pi}{4n^2\sqrt{k}} + d^2(y_n,p). \end{aligned} \tag{10}$$

In a similar way we have

$$d^2(T^n x_n,p) \leq \frac{\pi}{4n^2\sqrt{k}} + d^2(x_n,p) \tag{11}$$

It follows from (9) and (11) that

$$\begin{aligned} d^2(y_n,p) &\leq (1-\beta_n)d^2(x_n,p) + \beta_n\left(\frac{\pi}{4n^2\sqrt{k}} + d^2(x_n,p)\right) \\ &- \frac{R\beta_n(1-\beta_n)d^2(x_n,T^n x_n)}{2} \\ &= d^2(x_n,p) + \beta_n\left(\frac{\pi}{4n^2\sqrt{k}}\right) \\ &- \frac{R\beta_n(1-\beta_n)d^2(x_n,T^n x_n)}{2}, \end{aligned} \tag{12}$$

on the other hand, by substituting (12) in (10), we get

$$\begin{aligned} d^2(T^n y_n,p) &\leq \frac{\pi}{4n^2\sqrt{k}} + d^2(x_n,p) + \beta_n\left(\frac{\pi}{4n^2\sqrt{k}}\right) \\ &- \frac{R\beta_n(1-\beta_n)d^2(x_n,T^n x_n)}{2}. \end{aligned} \tag{13}$$

Substituting (13) in (8) , we get

$$\begin{aligned} d^2(x_{n+1},p) &\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n\Bigg(\frac{\pi}{4n^2\sqrt{k}} + d^2(x_n,p) + \beta_n\left(\frac{\pi}{4n^2\sqrt{k}}\right) \\ &- \frac{R\beta_n(1-\beta_n)d^2(x_n,T^n x_n)}{2}\Bigg) - \frac{R\alpha_n(1-\alpha_n)d^2(x_n,T^n y_n)}{2}, \end{aligned}$$

so

$$\begin{aligned} d^2(x_{n+1},p) &\leq d^2(x_n,p) + \alpha_n\pi\left(\frac{1+\beta_n}{4n^2\sqrt{k}}\right) \\ &- \frac{R\alpha_n\beta_n(1-\beta_n)d^2(x_n,T^n x_n)}{2} - \frac{R\alpha_n(1-\alpha_n)d^2(x_n,T^n y_n)}{2}, \end{aligned}$$

and the proof is completed. □

Theorem 2.5. Let $k > 0$ and (X,d) be a $CAT(k)$ space with $diam(X) \le \frac{\pi-2\eta}{2\sqrt{k}}$ for some $\eta \in \left(0, \frac{\pi}{2}\right)$. Also, let $R = (\pi - 2\eta)tan(\eta)$, C is a nonempty closed convex subset of X, $T : C \longrightarrow C$ be asymptotically nonexpansive type mapping and $\{\alpha_n\}_{n=1}^{+\infty}, \{\beta_n\}_{n=1}^{+\infty} \in [\varepsilon, b]$ for some $\varepsilon > 0$ and $b \in (0,1)$. Assume that $x_1 \in C$ and define the iteration scheme x_n by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n y_n \\ y_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n. \end{cases}$$

then $lim_{n\longrightarrow\infty} d(x_n, Tx_n) = 0$.

Proof. First of all, we prove that $lim_{n\to\infty} d(x_n, T^n x_n) = 0$. By Theorem 2.4, we have

$$\begin{aligned} d^2(x_{n+1}, p) &\le d^2(x_n, p) + \alpha_n \pi \left(\frac{1+\beta_n}{4n^2\sqrt{k}}\right) \\ &- \frac{R\alpha_n\beta_n(1-\beta_n)d^2(x_n, T^n x_n)}{2}, \end{aligned} \tag{14}$$

and so $D = R(1-b)$ and this implies that

$$\frac{R(1-\beta_n)}{2} > \frac{R(1-b)}{2} > 0 \tag{15}$$

for all $n \in N$. Suppose that $lim_{n\to\infty} d(x_n, T^n x_n) \neq 0$, then there exists $\varepsilon_0 > 0$ and a subsequence $\{x_{n_i}\}_{i=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$ such that

$$d^2(x_{n_i}, T^{n_i} x_{n_i}) \ge \varepsilon_0. \tag{16}$$

Without loss of generality, let $n_1 > N$ and then we have

$$\alpha_n\beta_n\left[\frac{R}{2}(1-\beta_n)\right]d^2(x_n, T^n x_n) \le d^2(x_n, p) - d^2(x_{n+1}, p) + \alpha_n\pi\left(\frac{1+\beta_n}{4n^2\sqrt{k}}\right),$$

then

$$\begin{aligned} \sum_{l=1}^{i} \alpha_{n_l}\beta_{n_l}\left[\frac{R}{2}(1-\beta_{n_l})\right]d^2(x_{n_l}, T^{n_l}x_{n_l}) &= \sum_{m=n_1}^{n_i} \alpha_m\beta_m\left[\frac{R}{2}(1-\beta_m)\right]d^2(x_m, T^m x_m) \\ &\le \sum_{m=n_1}^{n_i} d^2(x_m, p) - d^2(x_{m+1}, p) + \alpha_m\pi\left(\frac{1+\beta_m}{4m^2\sqrt{k}}\right) \end{aligned} \tag{17}$$

from this, together (15) and (16) and this fact that $\varepsilon \le \alpha_n, \beta_n$ for every $n \in N$, we obtain that

$$i.\varepsilon^2.\frac{D}{2}.\varepsilon_0 \le \sum_{m=n_1}^{n_i} d^2(x_m, p) - d^2(x_{m+1}, p) + \alpha_m\pi\left(\frac{1+\beta_m}{4m^2\sqrt{k}}\right).$$

If we take $i \to \infty$, then the right hand side of the above inequality is bounded while the left side is unbounded and this is contradiction. Therefore, $lim_{n\to\infty} d(x_n, T^n x_n) = 0$ and hence $lim_{n\to\infty} d(x_n, Tx_n) = 0$ by theorem 2.1. □

Theorem 2.6. Let $k > 0$ and (X,d) be a $CAT(k)$ space with $diam(X) \le \frac{\pi-2\eta}{2\sqrt{k}}$ for some $\eta \in \left(0, \frac{\pi}{2}\right)$. Let $R = (\pi - 2\eta)tan(\eta)$, C be a nonempty closed convex subset of X, $T : C \to C$ be a completely continuous and asymptotically nonexpansive type mapping and $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \in [\varepsilon, b]$ for some $\varepsilon > 0$ and $b \in (0,1)$. Given $x_1 \in C$ and define the iteration scheme $\{x_n\}_{n=1}^{\infty}$ by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n y_n \\ y_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n. \end{cases}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Proof. Since T is completely continuous , $\{Tx_n\}_{n=1}^{\infty}$ has a convergent subsequence in C. By using previous theorem, we can show that $\{x_n\}_{n=1}^{\infty}$ has a convergence subsequence, say $\{x_{n_k}\}_{n=1}^{\infty} \to q \in C$. Hence $q \in F(T)$ and the continuity of T. It follows from the inequalities using in the proof of theorem 3.4 that,

$$d^2(x_{n+1}, p) \leq d^2(x_n, p) + \alpha_n \pi \left(\frac{1+\beta_n}{4n^2\sqrt{k}} \right).$$

On the other hand, since $\sum_{n=1}^{\infty} \alpha_n \pi \left(\frac{1+\beta_n}{4n^2\sqrt{k}} \right) < \infty$, by lemma 1.14 we have $\{x_n\}_{n=1}^{\infty} \to q$. This completes the proof. □

Theorem 2.7. Let $k > 0$ and (X,d) be a complete $CAT(k)$ space with $diam(X) \leq \frac{\pi - 2\eta}{2\sqrt{k}}$ for some $\eta \in \left(0, \frac{\pi}{2}\right)$. Also, let K be a nonempty closed convex subset of X, P be non-expansive retraction of X on to K and $T : K \to X$ be a uniformly continuous and asymptotically nonexpansive type mapping with $F(T) \neq \emptyset$. If $\{x_n\}_{n=1}^{+\infty} \subseteq K$ such that $lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - lim_{n\to\infty} x_n = w$ then $w \in K$ and $Tw = w$.

Proof. By lemma 1.21, $w \in K$. Now, we define $\psi(u) = limsup_{n\to\infty} d(x_n, u)$ for each $u \in K$. Since $lim_{n\to\infty} d(x_n, Tx_n) = 0$, so by induction on m, we can prove

$$lim_{n\to\infty} d(x_n, T(PT)^{m-1}x_n) = 0 \tag{18}$$

for all $m \in N$. Because of uniform continuity of TP, we have

$$lim_{n\longrightarrow\infty} d(T(PT)^{m-1}x_n, T(PT)^m x_n) = 0$$

and then

$$d(x_n, T(PT)^m x_n) \leq d(x_n, T(PT)^{m-1}x_n) + d(T(PT)^{m-1}x_n, T(PT)^m x_n) \to 0,$$

as $n \longrightarrow \infty$ and equation (18) is proved. This implies that

$$\psi(u) = limSup_{n\to\infty} d(T(PT)^{m-1}x_n, u) \tag{19}$$

for each $u \in K$ and $m \in N$. Now, taking $u = T(PT)^{m-1}w$ in (19), we have

$$\begin{aligned} \psi(T(PT)^{m-1}w) &= \lim Sup_{n\to\infty} d(T(PT)^{m-1}x_n, T(PT)^{m-1}w) \\ &= \lim Sup_{n\to\infty} \{d(T(PT)^{m-1}x_n, T(PT)^{m-1}w) - d(x_n, w)\} + d(x_n, w) \\ &\leq \lim Sup_{m\to\infty} \{Sup_{x_n \in K} \{d(T(PT)^{m-1}x_n, T(PT)^{m-1}w) - d(x_n, w)\}\} \\ &+ \quad d(x_n, w). \end{aligned}$$

Now, since T is asymptotically nonexpansive type mapping we have

$$\lim sup_{m\to\infty} \psi(T(PT)^{m-1}w) \leq d(x_n, w) \leq \psi(w). \tag{20}$$

Furthermore, for any $n, m \in N$ it follows from the CN^*−inequality with $\alpha = \frac{1}{2}$,

$$\begin{aligned} d^2\left(x_n, \frac{1}{2}w \oplus \frac{1}{2}T(PT)^{m-1}w\right) &\leq \frac{d^2(x_n, w)}{2} + \frac{d^2(x_n, T(PT)^{m-1}w)}{2} \\ &- \frac{Rd^2(w, T(PT)^{m-1}w)}{8}, \end{aligned}$$

and since $\triangle - lim_{n\to\infty} x_n = w$ so by letting $n \to \infty$ in the above inequality, we get

$$\begin{aligned} \psi^2(w) &\leq \psi^2\left(\frac{1}{2}w \oplus \frac{1}{2}T(PT)^{m-1}w\right) \\ &\leq \frac{\psi^2(w)}{2} + \frac{\psi^2(T(PT)^{m-1}w)}{2} - \frac{Rd^2(w, T(PT)^{m-1}w)}{8}, \end{aligned}$$

which yields that

$$d^2(w, T(PT)^{m-1}w) \leq \frac{4[\psi^2(T(PT)^{m-1}w) - \psi^2(w)]}{R}. \tag{21}$$

On the other hand, by (20) and (21) we have $lim_{n\to\infty} d(w, T(PT)^{m-1}w) = 0$. So, in view of the continuity of TP we obtain that $w = lim_{m\to\infty} T(PT)^m w = lim_{m\to\infty} TP(T(PT)^{m-1}w) = TPw = Tw$. This completes the proof. □

Corollary 2.8. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space (X,d), P be a nonexpansive retraction of X onto K and $T : K \to X$ be a uniformly continuous and asymptotically nonexpansive type mapping . If $\{x_n\}_{n=1}^{+\infty} \subseteq K$ such that $lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\triangle - lim_{n\to\infty} x_n = w$ then $w \in K$ and $Tw = w$.*

Proof. It is well known that every convex subset of a $CAT(0)$ space, equipped with the induced metric is a $CAT(0)$ space. Then (K,d) is a $CAT(0)$ space and hence it is a $CAT(k)$ space for all $k > 0$. Notice also that K is $R-convex$ for $R = 2$ and since K is bounded, we can choose $\eta \in \left(0, \frac{\pi}{2}\right)$ and $k > 0$ so that $diam(K) \leq \frac{\pi - 2\eta}{2\sqrt{k}}$. The conclusion follows from theorem 2.6 . □

Theorem 2.9. Let $k > 0$ and (X,d) be a complete $CAT(k)$ space with $diam(X) \leq \frac{\pi-2\eta}{2\sqrt{k}}$ for some $\eta \in \left(0, \frac{\pi}{2}\right)$, K be a nonempty closed convex subset of X, P be nonexpansive retraction of X onto K and $T : K \to X$ be a uniformly continuous and asymptotically nonexpansive type mapping. Also, let $\{x_n\}_{n=1}^{+\infty} \subseteq K$ defined by

$$\begin{cases} x_1 \in K \\ y_n = P((1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n) \\ x_{n+1} = P((1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n) \end{cases}, \tag{22}$$

where $\{\alpha_n\}_{n=1}^{+\infty}, \{\beta_n\}_{n=1}^{+\infty} \subseteq (0,1)$ such that

$$\begin{cases} liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0 \\ liminf_{n\to\infty} \beta_n(1-\beta_n) > 0 \end{cases}.$$

If $F(T) \neq \emptyset$ then $\{x_n\}_{n=1}^{+\infty}$ is $\triangle$−convergent to a fixed point of T.

Proof. We divide our proof into three steps.

Step 1: First we prove that $\lim_{n\to\infty} d(x_n,q)$ exist for each $q\in F(T)$. For this, let $q\in F(T)$ and since T is a asymptotically nonexpansive type mapping, so by (22) we have

$$\begin{aligned} d(y_n,q) &= d(P((1-\beta_n)x_n\oplus\beta_n T(PT)^{n-1}x_n),q)\\ &= d(P((1-\beta_n)x_n\oplus\beta_n T(PT)^{n-1}x_n),P(q))\\ &\le d((1-\beta_n)x_n\oplus\beta_n T(PT)^{n-1}x_n),q)\\ &\le (1-\beta_n)d(x_n,q)+\beta_n d(T(PT)^{n-1}x_n,q)\\ &= (1-\beta_n)d(x_n,q)+\beta_n d(T(PT)^{n-1}x_n,T(PT)^{n-1}q) \qquad (23)\\ &\le (1-\beta_n)d(x_n,q)+\beta_n\{sup_{x_n\in K}d(T(PT)^{m-1}x_n,T(PT)^{m-1}q)\}\\ &\le (1-\beta_n)d(x_n,q)+\beta_n\left(\frac{1}{n^2}+d(x_n,q)\right)=d(x_n,q)+\frac{\beta_n}{n^2}. \end{aligned}$$

and this implies that

$$\begin{aligned} d(x_{n+1},q) &= d(P((1-\alpha_n)T(PT)^{n-1}x_n\oplus\alpha_n T(PT)^{n-1}y_n),q)\\ &= d(P((1-\alpha_n)T(PT)^{n-1}x_n\oplus\alpha_n T(PT)^{n-1}y_n),P(q)) \qquad (24)\\ &\le d((1-\alpha_n)T(PT)^{n-1}x_n\oplus\alpha_n T(PT)^{n-1}y_n)\\ &\le (1-\alpha_n)d(T(PT)^{n-1}x_n,q)+\alpha_n d(T(PT)^{n-1}y_n,q)\\ &\le (1-\alpha_n)\left[\frac{1}{n^2}+d(x_n,q)\right]+\alpha_n\left[d(y_n,q)+\frac{1}{n^2}\right]. \end{aligned}$$

Now, by the above inequality and (23) we obtain that

$$d(x_{n+1},q)\le d(x_n,q)+\frac{\beta_n\alpha_n+1}{n^2} \tag{25}$$

Step 2: Now, we prove that $lim_{n\to\infty}d(x_n,Tx_n)=0$. Since $\{x_n\}_{n=1}^{+\infty}$ is bounded, so there exists $R'>0$ such that $\{x_n\}_{n=1}^{+\infty},\{y_n\}_{n=1}^{+\infty}\subset B(q,R')$ for all $n\in N$ with $R'<\frac{D_k}{2}$. Therefore we have

$$\begin{aligned} d^2(y_n,q) &= d^2(P((1-\beta_n)x_n\oplus\beta_n T(PT)^{n-1}x_n),q)\\ &= d^2(P((1-\beta_n)x_n\oplus\beta_n T(PT)^{n-1}x_n),P(q))\\ &\le d^2((1-\beta_n)x_n\oplus\beta_n T(PT)^{n-1}x_n),q)\\ &\le (1-\beta_n)d^2(x_n,q)+\beta_n d^2(T(PT)^{n-1}x_n,q)-\frac{R\beta_n(1-\beta_n)}{2}d^2(T(PT)^{n-1}x_n,x_n)\\ &\le (1-\beta_n)d^2(x_n,q)\\ &+ \beta_n\{sup_{x_n\in K}\{d^2(T(PT)^{m-1}x_n,T(PT)^{m-1}q)-d^2(x_n,q)\}\}+\beta_n d^2(x_n,q)\\ &\le (1-\beta_n)d^2(x_n,q)+\beta_n\{sup_{x_n\in K}\{[d(T(PT)^{m-1}x_n,T(PT)^{m-1}q)\\ &- d(x_n,q)][d(T(PT)^{m-1}x_n,T(PT)^{m-1}q)+d(x_n,q)]\}\}+\beta_n d^2(x_n,q)\\ &\le (1-\beta_n)d^2(x_n,q)+\beta_n\left(\frac{\pi-2\eta}{n^2\sqrt{k}}\right)+\beta_n d^2(x_n,q). \end{aligned}$$

Hence we obtain that

$$d^2(y_n,q)\le d^2(x_n,q)+\beta_n\left(\frac{\pi-2\eta}{n^2\sqrt{k}}\right) \tag{26}$$

Now, from CN^*-inequality and using (26), we get

$$\begin{aligned}
d^2(x_{n+1},q) &= d^2(P((1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n),q) \\
&= d^2(P((1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n),P(q)) \\
&\leq d^2((1-\alpha_n)T(PT)^{n-1}x_n \oplus \alpha_n T(PT)^{n-1}y_n,q) \\
&\leq (1-\alpha_n)d^2(T(PT)^{n-1}x_n,q)+\alpha_n d^2(T(PT)^{n-1}y_n,q) \\
&- \frac{R\alpha_n(1-\alpha_n)}{2}d^2(T(PT)^{n-1}y_n,T(PT)^{n-1}x_n) \\
&\leq (1-\alpha_n)\left(\frac{\pi-2\eta}{n^2\sqrt{k}}+d^2(x_n,q)\right)+\alpha_n\left(\frac{\pi-2\eta}{n^2\sqrt{k}}+d^2(y_n,q)\right) \\
&- \frac{R\alpha_n(1-\alpha_n)}{2}d^2(T(PT)^{n-1}y_n,T(PT)^{n-1}x_n)
\end{aligned}$$

Therefore from the last inequality and (26) we have

$$\begin{aligned}
d^2(x_{n+1},q) &\leq \frac{2(1+\alpha_n\beta_n)}{n^2}\frac{\frac{\pi}{2}-\eta}{\sqrt{k}}+d^2(x_n,q) \\
&- \frac{R\alpha_n(1-\alpha_n)}{2}d^2(T(PT)^{n-1}y_n,T(PT)^{n-1}x_n).
\end{aligned} \tag{27}$$

Or

$$d^2(x_{n+1},q) \leq \frac{2(1+\alpha_n\beta_n)}{n^2}\frac{\frac{\pi}{2}-\eta}{\sqrt{k}}+d^2(x_n,q). \tag{28}$$

Hence by (28) and lemma 1.14, we find that $lim_{n\to\infty}d(x_n,q)$ exists for each $q \in F(T)$. Finally, we prove that $lim_{n\to\infty}d(x_n,Tx_n)=0$. From (28) we have

$$\begin{aligned}
\frac{R\alpha_n(1-\alpha_n)d^2(T(PT)^{n-1}y_n,T(PT)^{n-1}x_n)}{2} &\leq \frac{(1+\alpha_n\beta_n)(\pi-2\eta)}{n^2\sqrt{k}} \\
&+ d^2(x_n,q)-d^2(x_{n+1},q).
\end{aligned} \tag{29}$$

Since $\sum_{n=1}^{\infty}\frac{2(1+\alpha_n\beta_n)}{n^2}\frac{\frac{\pi}{2}-\eta}{\sqrt{k}} < \infty$ and $d(x_n,q) < R'$, so we have

$$\textstyle\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)d^2(T(PT)^{n-1}y_n,T(PT)^{n-1}x_n) < \infty,$$

and by the fact that $liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$ we get

$$lim_{n\to\infty}d(T(PT)^{n-1}y_n,T(PT)^{n-1}x_n)=0.$$

On the other hand

$$\begin{aligned}
d^2(y_n,q) &= d^2(P((1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n),q) \\
&= d^2(P((1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n),P(q)) \\
&\leq d^2((1-\beta_n)x_n \oplus \beta_n T(PT)^{n-1}x_n),q) \\
&\leq (1-\beta_n)d^2(x_n,q)+\beta_n d^2(T(PT)^{n-1}x_n,q)-\frac{R\beta_n(1-\beta_n)d^2(T(PT)^{n-1}x_n,x_n)}{2}.
\end{aligned}$$

Also, for the right hand side of the above inequality, we have

$$\frac{R\beta_n(1-\beta_n)d^2(T(PT)^{n-1}x_n,x_n)}{2} \leq (1-\beta_n)d^2(x_n,q)-d^2(y_n,q)+\beta_n\left(\frac{\frac{\pi}{2}-\eta}{n^2\sqrt{k}}\right).$$

Now, since $\sum_{n=1}^{\infty}\frac{\beta_n}{n^2}\frac{\frac{\pi}{2}-\eta}{\sqrt{k}}<\infty$, $d(x_n,q)<R'$ and $d(y_n,q)<R'$, we obtain that

$$\sum_{n=1}^{\infty}\beta_n(1-\beta_n)d^2(x_n,T(PT)^{n-1}x_n)<\infty,$$

and by the fact that $liminf_{n\to\infty}\beta_n(1-\beta_n)>0$, we get

$$lim_{n\to\infty}d(x_n,T(PT)^{n-1}x_n)=0 \tag{30}$$

Now using (30) we get

$$\begin{aligned} d(y_n,x_n) &= d(P((1-\beta_n)x_n\oplus\beta_nT(PT)^{n-1}x_n),x_n)\\ &= d(P((1-\beta_n)x_n\oplus\beta_nT(PT)^{n-1}x_n),P(x_n))\\ &\leq d((1-\beta_n)x_n\oplus\beta_nT(PT)^{n-1}x_n),x_n)\\ &\leq \beta_nd(T(PT)^{n-1}x_n),x_n)\longrightarrow 0\end{aligned}$$

Also we observe that

$$\begin{aligned} d(x_{n+1},x_n) &= d(P((1-\alpha_n)T(PT)^{n-1}x_n\oplus\alpha_nT(PT)^{n-1}y_n),x_n)\\ &= d(P((1-\alpha_n)T(PT)^{n-1}x_n\oplus\alpha_nT(PT)^{n-1}y_n),P(x_n))\\ &\leq d((1-\alpha_n)T(PT)^{n-1}x_n\oplus\alpha_nT(PT)^{n-1}y_n),x_n)\\ &\leq (1-\alpha_n)d(T(PT)^{n-1}x_n),x_n)+\alpha_nd(T(PT)^{n-1}y_n,x_n)\\ &\leq (1-\alpha_n)d(T(PT)^{n-1}x_n),x_n)+\alpha_n[d(T(PT)^{n-1}y_n,T(PT)^{n-1}x_n)\\ &+ d(T(PT)^{n-1}x_n),x_n)]\\ &= d((T(PT)^{n-1}x_n),x_n)+\alpha_nd(T(PT)^{n-1}y_n,T(PT)^{n-1}x_n)\to 0,\end{aligned}$$

as $n\longrightarrow\infty$. Therefore, we obtain that $d(x_{n+1},y_n)\leq d(x_{n+1},x_n)+d(x_n,y_n)\to 0$ as $n\longrightarrow\infty$. Furthermore

$$\begin{aligned} d(x_{n+1},T(PT)^{n-1}y_n) &\leq d(x_{n+1},x_n)+d(x_n,T(PT)^{n-1}x_n)\\ &+ d(T(PT)^{n-1}x_n,T(PT)^{n-1}y_n)\to 0\end{aligned}$$

as $n\longrightarrow\infty$. So we have

$$d(x_{n+1},T(PT)^{n-1}y_n)\longrightarrow 0. \tag{31}$$

Furthermore, we have

$$\begin{aligned} d(x_n,Tx_n) &\leq d(x_n,T(PT)^{n-1}x_n)+d(T(PT)^{n-1}x_n,T(PT)^{n-1}y_{n-1})\\ &+ d(T(PT)^{n-1}y_{n-1},Tx_n)\\ &\leq d(x_n,T(PT)^{n-1}x_n)+d(T(PT)^{n-1}x_n,T(PT)^{n-1}y_{n-1})\\ &+ d(T(PT)^{n-1}y_{n-1},Tx_n,\end{aligned} \tag{32}$$

and since $lim_{n\to\infty}d(T(PT)^{n-2}y_{n-1},x_n)=0$ and TP is uniformly continuous, we have

$$lim_{n\to\infty}d(TPT(PT)^{n-2}y_{n-1},TPx_n)=lim_{n\to\infty}d(T(PT)^{n-1}y_{n-1},Tx_n)=0.$$

Finally, from this, (30), (31) and (32), we have $d(x_n, Tx_n) \to 0$.

Step 3: Now , we prove that $\{x_n\}_{n=1}^{+\infty}$ is $\Delta-$convergent to a fixed point of T.
Let $\omega_W = \cup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}_{n=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$. First of all, we will show that $\omega_W(x_n) \subseteq F(T)$. For this, let $u \in \omega_W(x_n)$ and so, there exists a subsequence $\{u_n\}_{n=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$ such that $A(\{u_n\} = \{u\}$.
On the other hand, by lemma 1.21 there exists a subsequences $\{v_n\}_{n=1}^{+\infty}$ of $\{u_n\}_{n=1}^{+\infty}$ such that $\Delta - lim_{n\to\infty} v_n = v \in K$. Furthermore, by the fact that $d(x_n, Tx_n) \to 0$ we conclude that $d(v_n, Tv_n) \to 0$. Also, it follows from theorem 2.7 that $v \in F(T)$ and then by step1, we obtain that $lim_{n\to\infty} d(x_n, v)$ exists. Thus, $u = v$ by lemma 1.22 and this implies that $\omega_W(x_n) \subseteq F(T)$. Next, we show that $\omega_W(x_n)$ consists of exactly one point. Let $\{u_n\}_{n=1}^{+\infty}$ be a subsequences of $\{x_n\}_{n=1}^{+\infty}$ with $A(\{u_n\}_{n=1}^{+\infty}) = \{u\}$ and $A(\{x_n\}_{n=1}^{+\infty}) = \{x\}$. Since $u \in \omega_W(x_n) \subseteq F(T)$, so from step1 we have $lim_{n\to\infty} d(x_n, u)$ exists. Again, we obtain that $x = u$ and therefore $\omega_W(x_n) = \{x\}$. This means that $\{x_n\}_{n=1}^{+\infty}$ is $\Delta-$convergent to a fixed point of T and the proof is completed. □

Theorem 2.10. Let X, K, P, T and $\{x_n\}_{n=1}^{+\infty}$ be the same as in theorem 2.9. Then $\{x_n\}_{n=1}^{+\infty}$ converges strongly to a fixed point of T if and only if $liminf_{n\to\infty} d(x_n, F(T)) = 0$ where $d(x, F(T)) = inf\{d(x,q) : q \in F(T))\}$.

Proof. If $\{x_n\}_{n=1}^{+\infty}$ converges to $q \in F(T)$ then $lim_{n\to\infty} d(x_n, q) = 0$ and since $0 \le d(x_n, F(T)) \le d(x_n, q)$, we have $liminf_{n\to\infty} d(x_n, F(T)) = 0$.
Conversely, suppose that $lim_{n\to\infty} d(x_n, F(T)) = 0$, then from step1 of the theorem 2.9, we have $lim_{n\to\infty} d(x_n, F(T))$ exists . Thus by hypothesis $lim_{n\to\infty} d(x_n, F(T)) = 0$. Next, we show that $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence. In fact, it follows from (25) that for any $q \in F(T)$

$$d(x_{n+1}, q) \le d(x_n, q) + \frac{\beta_n \alpha_n + 1}{n^2} \le d(x_n, q) + \frac{2}{n^2}.$$

Hence for any positive integers n, m, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\le d(x_{n+m}, q) + d(q, x_n) \\ &\le d(x_{n+m-1}, q) + \frac{2}{(n+m-1)^2} + d(q, x_n) \\ &\le d(x_{n+m-2}, q) + \frac{2}{(n+m-2)^2} + \frac{2}{(n+m-1)^2} + d(q, x_n) \\ \dots &\le d(x_n, q) + \frac{2}{n^2} + \dots + \frac{2}{(n+m-2)^2} + \frac{2}{(n+m-1)^2} + d(q, x_n) \end{aligned}$$

So we get

$$d(x_{n+m}, x_n) \le \sum_{i=n}^{n+m-1} \frac{2}{i^2} + 2d(x_n, q) \le \sum_{i=n}^{n+m-1} \frac{2}{i^2} + 2d(x_n, F(T)) \to 0.$$

as $m, n \to \infty$ and this implies that $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in K. Since, K is a closed subset in a complete $CAT(k)$ space X, it is complete. So, we can assume that $\{x_n\}_{n=1}^{+\infty}$ converges strongly to some point $q^* \in K$. Furthermore, as T is continuous, so $F(T)$ is closed subset in K and since $lim_{n\to\infty} d(x_n, F(T)) = 0$ we obtain $q^* \in F(T)$. This completes the proof. □

Theorem 2.11. Let X, K, P, T and $\{x_n\}_{n=1}^{+\infty}$ be the same as in theorem 2.9.
(i) if T is semi-compact, then $\{x_n\}_{n=1}^{+\infty}$ converges strongly to a fixed point of T.
(ii) if T satisfies condition (I) then $\{x_n\}_{n=1}^{+\infty}$ converges strongly to a fixed point of T .

Proof. It follows from step1 of theorem 2.9 that $\{x_n\}_{n=1}^{+\infty}$ is a bounded sequence in K. Also , since $lim_{n\to\infty} d(x_n, Tx_n) = 0$ by the semi-compactness of T , there exists a subsequence $\{x_{n_k}\}_{k=1}^{+\infty} \subset \{x_n\}_{n=1}^{+\infty}$ such that $\{x_{n_k}\}_{k=1}^{+\infty}$ converges strongly to some point $q \in K$. Moreover , by the uniform continuity of T, we have $d(q, Tq) = lim_{n\to\infty} d(x_n, Tx_n) = 0$ and this implies that $q \in F(T)$. Again by 24 $lim_{n\to\infty} d(x_n, q)$ exists. Hence q is the strong limit of the sequence $\{x_n\}_{n=1}^{+\infty}$. As a result $\{x_n\}_{n=1}^{+\infty}$ converges strongly to a fixed point q of T.

For the proof of (ii), again from step1 of theorem 2.9 we know that $lim_{n\to\infty} d(x_n, F(T))$ exists. Furthermore, by condition (I) and the fact that $lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have

$$lim_{n\to\infty} f(d(x_n, F(T)) \leq lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

That is $lim_{n\to\infty} f(d(x_n, F(T))) = 0$. since f is a non-decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, it follows that $lim_{n\to\infty} d(x_n, F(T)) = 0$. Now theorem 2.8 implies that $\{x_n\}_{n=1}^{+\infty}$ converges strongly to a fixed point q in $F(T)$. □

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