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On graded almost-Bézout rings

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Abstract. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a commutative ring graded by an arbitrary abelian group Γ. We say that R is graded almost-Bézout ring if for each $a, b \in h(R)$, there exists $n \geq 1$ and $x \in h(R)$ such that $(a^n, b^n) = (x)$. In this paper, we investigate the transfer of this property in different graded commutative ring extensions, namely, in graded trivial ring extensions ($A \ltimes E$) and graded amalgamated algebras along an ideal ($A \bowtie^f J$). Our aim is to provide examples of new classes of Γ-graded rings satisfying the above mentioned property.

Key Words: Graded almost Bézout ring, graded amalgamation of ring, gr-Bézout ring, graded trivial ring extension.
2020 MSC: 13A02, 13C13.

Dedicated to our Professor David E. Dobbs for his 80th Birthday.

1 Introduction

Throughout this paper, all rings are commutative with unity, Γ will denote an abelian group written additively with an identity element denoted by 0 and all the graded rings and modules are graded by Γ. The purpose of this paper is to study the possible transfer of the property of being a gr-AB ring to graded trivial ring extensions and graded amalgamated algebras along an ideal.

In [1], Anderson and Zaffrullah introduced and studied the notion almost Bézout domain (AB-domain for short). An integral domain R is an AB-domain if for $a, b \in R - \{0\}$ there is n such that (a^n, b^n) is principal. The notion of almost Bézout domains runs along lines somewhat similar to those of Bézout domains (i.e., every two generated, equivalently, every finitely generated, ideal is principal). In [3], Anderson, Knopp, and Lewin continued the study of almost Bézout domains, and after observing that each almost Bézout domain is nearly Bézout, they used the construction $K + XL[X]$ to disprove the converse. The same example shows that a Noetherian almost Bézout domain need not be an almost principal ideal domain (API-domain), even though each Noetherian Bézout domain is a principal ideal domain (PID). In [2], Anderson and Zaffrullah continued their study of almost Bézout domains and gave a new characterization of Cohen-Kaplansky domains. In [13], the generalization of the almost Bézout domains to arbitrary commutative rings (with zero-divisors) is considered as follows: R is called an almost Bézout ring (AB-ring for short) if, for any two elements a and b in R, there exists a positive integer n such that the ideal (a^n, b^n) is principal. In the setting of Γ-graded rings, Ahmed and Moh'D, in [5], introduced two graded types of Bézout modules. The first type is graded-Bézout modules (gr-Bézout modules for short) in which every finitely generated submodule with homogeneous generators is cyclic with a homogeneous generator. The second type is weakly graded-Bézout modules (weakly gr-Bézout modules for short) which satisfy the Bézout property for the Bézout property for the graded submodules. They investigated the relationship among the three types of Bézout modules, the ordinary Bézout modules and the two graded types of Bézout modules. Notably, the graded Bézout ring is a gr-Bézout module over itself (see [5]).

Let A be a ring and E be an A-module. Then $A \ltimes E$, which is called the *trivial* (*ring*) *extension of A by E*, is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by $(a,e)(b,f) := (ab, af + be)$ for all $a,b \in A$ and all $e,f \in E$ (this construction is also known by other terms and by other forms of notation, such as the *idealization* $A(+)E$). Trivial ring extensions have been generalized and studied extensively in graded ring theory, often because of their usefulness in constructing new classes of examples of graded rings satisfying various properties (see [4, 9, 12]). Let Γ be an abelian group, $A = \bigoplus_{\alpha\in\Gamma} A_\alpha$ be a Γ-graded ring and $E = \bigoplus_{\alpha\in\Gamma} E_\alpha$ be a Γ-graded A-module. Then $A \ltimes E$ is a Γ-graded ring with $(A \ltimes E)_\alpha = A_\alpha \oplus E_\alpha$ for every $\alpha \in \Gamma$ (cf. [4, Proposition 2]).

Let $A = \bigoplus_{\alpha\in\Gamma} A_\alpha$ and $B = \bigoplus_{\alpha\in\Gamma} B_\alpha$ be two commutative rings graded by an arbitrary commutative monoid Γ, J be a homogeneous ideal of B, and $f : A \longrightarrow B$ be a graded ring homomorphism. Then $R := A \bowtie^f J$, which is called the amalgamation of A with B along J with respect to f (introduced and studied by D'Anna, Finocchiaro and Fontana in [6, 7]), is a graded ring $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ with $h(R) = \bigcup_{\alpha\in\Gamma} R_\alpha$, which is the set of all homogeneous elements of R, where for each $\alpha \in \Gamma$

$$R_\alpha = \left(A \bowtie^f J\right)_\alpha := \{(a_\alpha, f(a_\alpha) + j_\alpha) \mid a_\alpha \in A_\alpha, j_\alpha \in J_\alpha\}.$$

This construction, first introduced and studied in [10], allows us to meet the challenges posed by providing rich examples that enrich the current literature of graded ring theory. One of the key tools for studying $A \bowtie^f J$ is based on the fact that the graded amalgamation can be studied in the frame of the pullback constructions [11, Section 3]. Other classical constructions, such as the graded amalgamated duplication of a ring along an ideal denoted by $A \bowtie I$ and the graded trivial ring extension of A by E [4] ($A \ltimes E$), can be interpreted as a particular cases of the general graded amalgamation construction [10, Examples 3.3 & 3.4].

We can now specify the main purposes of this paper. More precisely, we start our investigation by studying the possible transfer of the gr-AB ring property to the graded trivial ring extension ($A \ltimes E$), combining the main transfert result (Theorem 2.1) and a well-known result on [13] we construct an example of a gr-AB ring which is not AB-ring (Example 2.4). Additionally, we present another special outcome through an example of gr-AB ring which is neither a gr-Bézout ring nor a gr-coherent ring (Examples 2.5 and 2.8). Subsequently, we turn our attention to explore how the graded amalgamation $A \bowtie^f J$ inherits the properties of being a gr-AB ring from the graded ring A for specific classes of homogeneous ideals J and graded homomorphisms f. Among other things, we show that if $J = 0$, then the graded amalgamation $A \bowtie^f J$ being a gr-AB ring is equivalent to A being a gr-AB ring, however the fact that J is nonzero necessitates additional conditions, such as A being a graded integral domain as per in Theorem 2.11, or a further requirements such as (A, M) being a gr-local ring with specific conditions on $f(M)J$ and $J \subseteq \text{gr-Nil}(B)$ (Proposition 2.15).

We pause to review some definitions and preliminary results on graded modules and rings, see for instance [14, 15]. Let Γ be a grading abelian group. By a *graded ring R*, we mean a ring graded by Γ, that is, a direct sum of subgroups R_α of R such that $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$ for every $\alpha, \beta \in \Gamma$. The set $h(R) = \bigcup_{\alpha\in\Gamma} R_\alpha$ is called the set of homogeneous elements of R. An element $x \in R$ is called *homogeneous* if it belongs to one of the R_α, *homogeneous of degree α* if $x \in R_\alpha$. The element 0 is therefore homogeneous of all degrees; but if $x \neq 0$ is homogeneous, it belongs to only one of the R_α; the index α such that $x \in R_\alpha$ is then called the *degree* of x and is sometimes denoted by $deg(x)$. Every $y \in R$ may be written uniquely as a sum $\sum_\alpha y_\alpha$ of homogeneous elements with $y_\alpha \in R_\alpha$ and y_α is the *homogeneous component of degree α* of y. Clearly, R_0 is a subring of R (and in particular $1 \in R_0$).

By a graded R-module E, we mean an R-module graded by Γ, that is, a direct sum of subgroups E_α of E such that $R_\alpha E_\beta \subseteq E_{\alpha+\beta}$ for every $\alpha, \beta \in \Gamma$. Let R and R' be two graded rings. Then a ring homomorphism $f : R \longrightarrow R'$ is called graded if $f(R_\alpha) \subseteq R'_\alpha$ for all $\alpha \in \Gamma$. A graded ring isomorphism is a bijective graded ring homomorphism. Obviously $\mathrm{Im}(f)$ is a graded subring of R', $\mathrm{Ker}(f)$ is a homogeneous ideal of R, and the bijection $R/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ canonically associated with f is an isomorphism of graded rings.

Let I be an ideal of R. Then I is called a homogeneous ideal of R if one of the following equivalent conditions holds: (i) $I = \bigoplus_{\alpha\in\Gamma} I_\alpha$, where $I_\alpha = I \cap R_\alpha$ for all $\alpha \in \Gamma$ and (ii) $x = x_{\alpha_1} + x_{\alpha_2} + \cdots + x_{\alpha_n} \in I$ implies that $x_{\alpha_i} \in I$, where $x_{\alpha_i} \in R_{\alpha_i}$. Similarly, a submodule N of M is called a homogeneous submodule if and only if $N = \bigoplus_{\alpha\in\Gamma} N_\alpha$, where $N_\alpha = N \cap M_\alpha$ for all $\alpha \in \Gamma$ if and only if $m = m_{\alpha_1} + m_{\alpha_2} + \cdots + m_{\alpha_n} \in N$ implies that $m_{\alpha_i} \in N$, where $m_{\alpha_i} \in M_{\alpha_i}$. If I is a homogeneous ideal of a graded ring R, then R/I is a graded ring, where $(R/I)_\alpha := (R_\alpha + I)/I$. A homogeneous ideal P of R is called a prime homogeneous ideal (gr-prime) if P is a proper homogeneous ideal of R with the property that $a, b \in h(R)$ and $ab \in P$ implies either $a \in P$ or $b \in P$. A homogeneous ideal M of R is called a maximal homogeneous ideal (gr-maximal) if it is maximal among proper homogeneous ideals; equivalently, if every nonzero homogeneous element of R/M is invertible and it is not difficult to show that a maximal homogeneous ideal is prime homogeneous.

Let I be a proper homogeneous ideal of a graded ring R. Then the graded radical of I will be designated by $\mathrm{Gr}(I) = \{x = \sum_{g\in\Gamma} x_g \in R :$ for each $g \in \Gamma$, there exists $n_g \in \mathbb{N}$ such that $x_g^{n_g} \in I\}$. Note that, if x is a homogeneous element, then $x \in \mathrm{Gr}(I)$ if and only if $x^n \in I$ for some positive integer n (see [16]). It is shown in [16, Proposition 2.5] that $\mathrm{Gr}(I)$ is the intersection of all prime homogeneous ideals of R containing I.

A graded ring is said to be graded-local (gr-local) if it has a unique maximal homogeneous (gr-maximal) ideal and a graded ring R is called a graded-field (gr-field) if every nonzero homogeneous element of R is invertible. Clearly every field is a graded field, however, the converse is not true in general, see [14, page 46]. Recall from [14] that a Γ-graded ring $R = \bigoplus_{g\in\Gamma} R_g$ is said to be a crossed product if R_g contain a unit for every $g \in \Gamma$ (i.e, $R_g \cap U(R) \neq \emptyset$ for each $g \in \Gamma$). Note that a Γ-crossed product $R = \bigoplus_{g\in\Gamma} R_g$ is a strongly graded ring, that is $R_{g+h} = R_g R_h$ for every $g, h \in \Gamma$.

We will be using the following definition (which agrees with the classical one if R is a graded integral domain). A graded R-module E is said to be a *torsion R-graded module* if, for each homogeneous $e \in E$, there exists $a \in R \setminus \{0\}$ such that $ae = 0$. We will also use the following standard definitions. A *regular homogeneous element* of a graded ring R is a non-zero-divisor homogeneous element; a graded R-module E is *gr-divisible* if, for each homogeneous $e \in E$ and each regular homogeneous element a of R there exists $f \in E$ such that $e = af$; a graded A-module E is a *torsion-free* (graded *A-module*) if whenever $a \in h(A)$ and $e \in E$ with $ae = 0$ implies that either $a = 0$ or $e = 0$. Denoted by H the multiplicative set of regular homogeneous elements of R. Then, by extending some definitions to the case where rings have zero-divisors, R_H, called the homogeneous total ring of quotients of R, is a Γ-graded ring, where $R_H = \bigoplus_{\alpha\in\Gamma} (R_H)_\alpha$ with

$$(R_H)_\alpha = \left\{ \frac{r}{s} \mid r \in R_\beta, s \text{ regular homogeneous in } R_\gamma \text{ and } \beta - \gamma = \alpha \right\}.$$

If R is a graded integral domain (an integral domain graded by Γ), then R_H is called the homogeneous quotient field of R. Obviously, every nonzero homogeneous element of R_H is invertible and $(R_H)_0$ is a field. For any Γ-graded ring R, we denote respectively by h-Z(R), h-Reg(R), h-Nil(R), the set of all homogeneous zero-divisors of R, the set of regular homogeneous elements of R and the set of nilpotent homogeneous element of R.

The notion of gr-AB rings was recently introduced in [12] as follows:

Definition 1.1. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a Γ-graded ring. Then R is said to be a graded almost Bézout ring (gr-AB ring) if for each $a, b \in h(R)$, there exist $n \geq 1$ and $x \in h(R)$ such that $(a^n, b^n) = (x)$.

2 Main results

Our first result studies the possible transfer of the gr-AB ring property between a graded ring A and a graded trivial ring extension $A \ltimes E$.

Theorem 2.1. Let A be a graded ring, E a graded A-module and $R := A \ltimes E$ be the graded trivial ring extension of A by E. Then, the following statements hold:

1. If R is a gr-AB ring, then so is A.
2. Suppose that A is a graded integral domain with $K := A_H$ its homogeneous quotient field, and E is a graded K-vector space. Then R is a gr-AB ring if and only if so is A.
3. Suppose that (A, M) is a gr-local ring and E is a graded A-module such that $ME = 0$. Then R is a gr-AB ring if and only if so is A.
4. Let $A \subseteq B$ be an extension of graded integral domains, and suppose that $\mathbb{Q} \subseteq B$. Further, assume that A is a crossed product. Then $R := A \ltimes B$ is a gr-AB ring if and only if A is a gr-AB ring and $A_H \subseteq B$.

The proof of this Theorem requires the following lemma, which can be viewed as a general indicator to check whether the graded trivial ring extension of graded integral domains is not gr-AB ring.

Lemma 2.2. *Let $A \subseteq B$ be an extension of graded integral domains, and let $R := A \ltimes B$ be the graded trivial ring extension of A by B. If there exists a nonzero element $m \in A_0$ such that $n1_B \notin mB$ for each positive integer n, then R is not a gr-AB ring.*

Proof. Deny, assume that R is a gr-AB ring. Let $0 \neq m \in A_0$ (along with the hypothesis that $n1_B \notin mB$ for each positive integer n), and consider the homogeneous elements $(m, 0)$ and $(m, 1)$ of R. Then, there exists a positive integer n and a homogeneous element $(a, b) \in R$ such that $R(m, 0)^n + R(m, 1)^n = R(a, b)$. Then there exists $(c_1, d_1), (c_2, d_2), (\alpha_1, \beta_1)$, and (α_2, β_2) in R such that

$$\left\{ \begin{array}{c} (m^n, 0) = (m, 0)^n = (c_1, d_1)(a, b) = (ac_1, bc_1 + ad_1) \\ \left(m^n, nm^{n-1}\right) = (m, 1)^n = (c_2, d_2)(a, b) = (ac_2, bc_2 + ad_2) \\ (a, b) = (m, 0)^n (\alpha_1, \beta_1) + (m, 1)^n (\alpha_2, \beta_2). \end{array} \right\}$$

Then we obtain the following equations:
(1) $m^n = ac_1$;
(2) $0 = bc_1 + ad_1$;
(3) $m^n = ac_2$;
(4) $nm^{n-1} = bc_2 + ad_2$;
(5) $a = (\alpha_1 + \alpha_2) m^n$.

From the first and third equation, we obtain $ac_1 = ac_2$. Since $0 \neq m, a \neq 0$, and so $c_1 = c_2$. Set $c = c_1 = c_2$. Also from the fifth equation, we have $a = (\alpha_1 + \alpha_2) m^n = (\alpha_1 + \alpha_2) ac$. Then $(\alpha_1 + \alpha_2) c = 1$, and so c is a unit of A. Finally, from the second and fourth equation, we have $0 = bc + ad_1$ and $nm^{n-1} = bc + ad_2$. Thus $nm^{n-1} = a(d_2 - d_1) = c^{-1} m^n (d_2 - d_1)$. Hence $n = mc^{-1}(d_2 - d_1) \in mB$, a contradiction. It follows that $R = A \ltimes B$ is not a gr-AB ring. □

Proof of Theorem 2.1.
(1) This follows from [12, Theorem 4.28].
(2) Suppose that A is a graded integral domain with A_H its homogeneous quotient field, and E is a graded K-vector space. By (1), it is only required to prove that if A is a gr-AB ring, then R is a gr-AB

ring. Which means that if $\alpha := (a,e)$ and $\beta := (b,f)$ are homogeneous elements of R, then there exists a positive integer n and a homogeneous element $(x,y) \in R$ such that $R\alpha^n + R\beta^n = R(x,y)$. Then four cases are possible. The first case is when $a = 0$ and $b \neq 0$. Then $b^{-1}e \in E$. Thus

$$(a,e) = (0,e) = (b,f)\left(0, b^{-1}e\right) \subseteq R(b,f)$$

and so (b,f) divides (a,e). Hence $R(a,e) + R(b,f) = R(b,f)$. Likewise, in the case where $a \neq 0$ and $b = 0$. We have $(b,f) = (0,f) = (a,e)\left(0, a^{-1}f\right)$. Thus (a,e) divides (b,f); and hence $R(a,e) + R(b,f) = R(a,e)$. In the case where $a = b = 0$, then

$$(a,e)^2 = (b,f)^2 = (0,0) \text{ and so } R(a,e)^2 + R(b,f)^2 = R(0,0).$$

In the remaining case, $a \neq 0$ and $b \neq 0$. Since A is a gr-AB ring, there exists a positive integer n and a homogeneous element x of A such that $(a^n, b^n)A = Aa^n + Ab^n = Ax = (x)$. So,

$$\begin{aligned}
((a,e)^n, (b,f)^n)R &= R(a,e)^n + R(b,f)^n \\
&= (A \ltimes E)(a^n, na^{n-1}e) + (A \ltimes E)(b^n, nb^{n-1}f) \\
&= Aa^n \ltimes E + Ab^n \ltimes E \\
&= (Aa^n + Ab^n) \ltimes E \\
&= Ax \ltimes E \\
&= Ax \ltimes xE = (x,0)R.
\end{aligned}$$

(3) Suppose that (A,M) is a gr-local ring and E is a graded A-module such that $ME = 0$. If R is a gr-AB ring, then by (1) A is a gr-AB ring. Conversely, suppose that A is a gr-AB ring, and let (a,b) and $(t,s) \in h(R)$. We may assume that (a,b) and (t,s) are nonunit elements of R and so $a, t \in M$. Since $ME = 0$, $(a,b)^2 = \left(a^2, 0\right)$ and $(t,s)^2 = \left(t^2, 0\right)$, and A is a gr-AB ring. Then there exists a positive integer n and a nonunit homogeneous element x of A such that $\left(a^{2n}, t^{2n}\right)A = Aa^{2n} + At^{2n} = Ax = (x)$. Note that $x \in M$. Hence, we obtain

$$\begin{aligned}
\left((a,b)^{2n}, (t,s)^{2n}\right)R &= R(a^2,0)^n + R(t^2,0)^n \\
&= Aa^{2n} \ltimes 0 + At^{2n} \ltimes 0 \\
&= \left(Aa^{2n} + At^{2n}\right) \ltimes 0 \\
&= Ax \ltimes xE = (x,0)R.
\end{aligned}$$

(4) Assume that $R = A \ltimes B$ is a gr-AB ring and $\mathbb{Q} \subseteq B$. Then A is a gr-AB ring by (1). Now, let a be a nonzero homogeneous element of A. Assume that $\deg(a) = h$. Since A is a crossed product ring, we can choose a unit homogeneous element $\alpha \in A_{-h}$. Note that αa is a homogeneous element of A with $\deg(\alpha a) = 0$. By Lemma 2.2, there is a positive integer n such that $n1_B \in \alpha a B$. Since $\mathbb{Q} \subseteq B, n^{-1} \in B$ and so $1 \in \alpha a B = aB$. Thus a is a unit of B and therefore $A_H \subseteq B$. The converse follows from (2). □

Before stating the following corollary, we note that the graded trivial ring extension $R := A \ltimes E$ is a crossed product if and only if so is A. Define the support of A as $sup(A) = \left\{\gamma \in \Gamma \mid A_\gamma \neq 0\right\}$.

Corollary 2.3. *Let A be a graded integral domain containing the field of rational numbers. Then the following statements are equivalent:*

1. *$A \ltimes A$ is a gr-AB ring and A is a crossed product ring.*
2. *A is a gr-field and $sup(A) = G$.*

Proof. Assume that A is a crossed product ring and $A \ltimes A$ is a gr-AB ring. Then by Theorem 2.1(4) A is a gr-field. On the other hand, since A is a crossed product ring, for every $g \in G$, $A_g \cap U(A) \neq \emptyset$. Hence $A_g \neq \{0\}$ for each $g \in G$, and so $sup(A) = G$. Conversely, if A is a gr-field and sup(A)=G. Let $g \in G$, then there exist a nonzero element $x \in A_g$. Then x is unit and so for each $g \in G$, $A_g \cap U(A) \neq \emptyset$. This A is a crossed product. Consequently, by Theorem 2.1(4), $A \ltimes A$ is a gr-AB ring. □

Combining Theorem 2.1 and [13, Corollary 3.3], we broaden our ability to construct straightforward examples of gr-AB ring which is not AB-ring.

Example 2.4. Let $A = \mathbb{Q}\left[X, X^{-1}\right]$ and $R := A \ltimes A$. Then R is gr-AB ring which is not AB-ring.

In the next example, first we show how to build a new family of non gr-coherent gr-AB rings that are not gr-Bézout rings. Recall from the introduction that a graded ring R is said to be a graded Bézout ring if every nonzero finitely generated graded ideal of R is gr-cyclic (i.e, with a homogeneous generator). While a Γ-graded ring R is said to be a weakly graded-Bézout (weakly gr-Bézout) if every finitely generated graded ideal is cyclic.

Example 2.5. Let A be a gr-Bézout domain which is not a gr-field, $K = A_H$, $E = K \oplus K$ and $R := A \ltimes E$ be the graded trivial ring extension of A by E. Then:

1. R is a gr-AB ring;
2. R is not a gr-Bézout ring;
3. R is not gr-coherent.

Proof. (1) By Theorem 2.1(2), R is a gr-AB ring since A is a gr-Bézout domain.

(2) Set the element $e = (1,0)$ and $f = (0,1)$ of E. Then the finitely generated homogeneous ideal of R, $R(0,e) + R(0,f)$ is not principal. Therefore R is not weakly gr-Bézout, and so not a gr-Bézout ring.

(3) Let e be a nonzero homogeneous element of E. Then it is easy to see that $(0 : (0,e)) = 0 \ltimes E$ is not finitely generated, since K is not finitely generated A-module, and therefore R is not gr-coherent. □

Lemma 2.6. *Let A be a graded ring, E a nonzero graded A-module, and $R := A \ltimes E$ is a gr-AB ring. Further, assume that A is a crossed product. Then for all $e \in h(E)$ and $a \in h$-$Reg(A)$, there exist $f \in E$ and a positive integer $n \geq 1$ such that $na^{n-1}e = a^n f$.*

Proof. Now let a be a regular homogeneous element of A and $e \in h(E)$. Assume that $\deg(a) = h_1$ and $\deg(e) = h_2$. Since A is crossed product, we can choose a unit homogeneous element $x \in A_{h_2 - h_1}$. Note that (xa, e) is a homogeneous element of R with $\deg(xa, e) = h_2$, and as R is a gr-AB ring, there is a positive integer n and a homogeneous element (c, d) of R such that $R(xa,0)^n + R(xa,e)^n = R(c,d)$. Then there are $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (c_1, d_1)$, and (c_2, d_2) in R such that:

$$\left\{ \begin{array}{c} (xa,0)^n = (x^n a^n, 0) = (\alpha_1, \beta_1)(c,d) \\ (xa,e)^n = \left(x^n a^n, n x^{n-1} a^{n-1} e\right) = (\alpha_2, \beta_2)(c,d) \\ (c,d) = (c_1, d_1)(x^n a^n, 0) + (c_2, d_2)\left(x^n a^n, n x^{n-1} a^{n-1} e\right) \end{array} \right\}$$

Then we obtain the following equations:
(1)$x^n a^n = \alpha_1 c$.
(2) $x^n a^n = \alpha_2 c$.
(3) $0 = \alpha_1 d + c\beta_1$.
(4) $n x^{n-1} a^{n-1} e = \alpha_2 d + c\beta_2$.
(5) $c = (c_1 + c_2) x^n a^n$.

As $x^n a^n = \alpha_1 c$, we get that c is a regular homogeneous element of A since a is regular. From the first and the second equation, we have $\alpha_1 = \alpha_2 = \alpha$. Finally, from the third, the fourth, and the fifth equation, we have $nx^{n-1}a^{n-1}e = c(\beta_2 - \beta_1) = x^n a^n (c_1 + c_2)(\beta_2 - \beta_1)$. Hence there exist $f \in E$ such that $na^{n-1}e = a^n f$ with $f = x(c_1 + c_2)(\beta_2 - \beta_2)$. □

Proposition 2.7. *Let A be a graded integral domain and E a graded torsion-free A-module such that $\mathbb{Q} \cdot E \subseteq E$. Assume further that A is a crossed product. Then $R = A \ltimes E$ is a gr-AB ring if and only if A is a gr-AB domain and E is a graded divisible A-module.*

Proof. By Theorem 2.1(1) and [12, Theorem 4.28(2)], we need only to prove that if R is a gr-AB ring, then E is a graded divisible A-module; that is, if e is a nonzero homogeneous element of E and a is a nonzero homogeneous element of A, then $e \in aE$. Since R is a gr-AB ring and A is a crossed product, by Lemma 2.6, there is $f \in E$ and a positive integer $n \geq 1$ such that $na^{n-1}e = a^n f$. As E is a graded torsion-free A-module, we get $ne = af$. Therefore $e = an^{-1} f \in aE$ since $\mathbb{Q} \cdot E \subseteq E$. □

Using Example 2.5 and [12, Theorem 4.28(2)], we are able to provide another example of non gr-Bézout gr-AB ring.

Example 2.8. Let A be a gr-Bézout domain, $K = A_H$ the homogeneous quotient field of A, and $F = K \oplus K$. Set $B = A \ltimes F$ and $E = K \ltimes F$. Then, $R = B \ltimes E$ is a gr-AB ring which is not a gr-Bézout ring.

Proof. By Example 2.5, B is a gr-AB ring which is not a gr-Bézout ring and so R is not gr-Bézout. We have $h - Z(B) = 0 \ltimes h(F) = h - Nil(B)$ and E is a graded divisible B-module. By [12, Theorem 4.28(2)], R is a gr-AB ring. □

If (A, M) is a gr-local ring and E a graded A-module such that M is the only homogeneous prime ideal of A, then we obtain a new application in the following proposition.

Proposition 2.9. *Let (R, M) be a graded local ring, suppose that M is the only prime homogeneous ideal of R. Then R is a gr-AB ring.*

Proof. Let $a, b \in h(R)$, if a or b is unit, then $Ra + Rb = R$. In the case where a and b are nonunits in R, then both a and b are in M. Hence there exists a positive integer n such that $Ra^n + Rb^n = 0$. □

Example 2.10. Let K be a gr-field and $E = K \oplus K$. Then $R := K \ltimes E$ is a gr-AB ring which is not gr-Bézout.

Now, we turn our attention to the transfer of the gr-AB ring property to graded amalgamation of rings $A \bowtie^f J$. It is easy to see that, if $J = 0$, then $A \bowtie^f J \cong A$, and so $A \bowtie^f J$ is a gr-AB ring if and only if so is A. Thus in the sequel, we assume that J is a nonzero proper homogeneous ideal of B.

Theorem 2.11. Let A and B be a pair of graded rings, where A is graded integral domain, $f : A \longrightarrow B$ be a graded ring homomorphism and J be a regular homogeneous proper ideal of B. Then the following statements are equivalent:

1. $A \bowtie^f J$ is a gr-AB ring.
2. f is injective, $f(A) + J$ is a gr-AB ring and $f(A) \cap J = (0)$.
3. $f^{-1}(J) = 0$ and $f(A) + J$ is a gr-AB ring.

Before proving this theorem, we recall the following lemma which is a direct application of [10, Theorem 3.5(4)], using the fact that the gr-AB ring property is stable under factor graded ring and in view of the graded isomorphisms:

$$\frac{A \bowtie^f J}{\{0\} \times J} \cong A$$

and

$$\frac{A \bowtie^f J}{f^{-1}(J) \times \{0\}} \cong f(A) + J.$$

Lemma 2.12. *Let A and B be graded rings, $f : A \to B$ be a graded ring homomorphism, and J a homogeneous ideal of B. If $A \bowtie^f J$ is a gr-AB ring, then A and $f(A) + J$ are gr-AB rings.*

Proof of Theorem 2.11.

(1) $\Rightarrow$ (2) Assume that $R := A \bowtie^f J$ is a gr-AB ring, and suppose that f is not injective. Let x be a homogeneous regular element of J and $0 \neq a = \sum_{i=0}^{k} a_{\alpha_i} \in \ker f$. Then there exists $i \in \{0, \ldots, k\}$ such that $a_{\alpha_i} \neq 0$. Hence $(a_{\alpha_i}, 0) = (a_{\alpha_i}, f(a_{\alpha_i}))$ and $(0, x)$ are homogeneous element of $A \bowtie^f J$ for each $i, j \in \{0, \ldots, k\}$ (zero is homogeneous of all degrees). Then there exists a positive integer n and $(r, f(r) + s) \in h(A \bowtie^f J)$ such that

$$(a^n_{\alpha_i}, 0)\left(A \bowtie^f J\right) + (0, x^n)\left(A \bowtie^f J\right) = (r, f(r) + s)\left(A \bowtie^f J\right).$$

Hence there exists $(u, f(u) + k), (v, f(v) + h), (\alpha, f(\alpha) + p)$ and $(\beta, f(\beta) + t)$ in R such that

$$\left\{\begin{array}{c} \left(a^n_{\alpha_i}, 0\right) = (a_{\alpha_i}, 0)^n = (u, f(u) + k)(r, f(r) + s) \\ (0, x^n) = (0, x)^n = (v, f(v) + h)(r, f(r) + s) \\ (r, f(r) + s) = \left(a^n_{\alpha_j}, 0\right)(\alpha, f(\alpha) + p) + (0, x^n)(\beta, f(\beta) + t). \end{array}\right\}$$

Hence $vr = 0$ and $a^n_{\alpha_j} = ur$. Since A is a graded integral domain and $a_{\alpha_j} \neq 0, r \neq 0$. Thus $v = 0$ and so $f(v) = 0$. Therefore, $x^n = h(f(r) + s)$ and $f(r) + s = x^n(f(\beta) + t) = h(f(r) + s).(f(\beta) + t)$. As x is a regular element of B, then $f(r) + s$ is regular. Hence $1 = h(f(\beta) + t) \in J$, which is a contradiction. Therefore, it follows that f is injective. By Lemma 2.12, $f(A) + J$ is a gr-AB ring. It remains to prove that $f(A) \cap J = (0)$. Deny, let $0 \neq f(a) \in J$ for some $a = \sum_{i=0}^{n} a_{\alpha_i}$. Then there exists $i \in \{0, \ldots, n\}$ such that $0 \neq f(a_{\alpha_i}) \in J$, then it is straighforward to see that $(a_{\alpha_i}, 0), (0, f(a_{\alpha_i})) \in h(A \bowtie^f J)$ for each $i \in \{0, \ldots, n\}$. Since $A \bowtie^f J$ is a gr-AB ring, there exists a positive integer n and $(r, f(r) + s) \in h(A \bowtie^f J)$ such that

$$(a_{\alpha_i}, 0)^n\left(A \bowtie^f J\right) + (0, f(a_{\alpha_i}))^n\left(A \bowtie^f J\right) = (r, f(r) + s)\left(A \bowtie^f J\right).$$

By similar reasoning as above, we get that $J = B$, which is a contradiction since J is a proper homogeneous ideal of B. Hence $f(A) \cap J = (0)$, as desired.

(2) $\Rightarrow$ (3) if f is injective and $f(A) \cap J = (0)$, then $f^{-1}(J) = 0$.

(3) $\Rightarrow$ (1) If $f^{-1}(J) = 0$, then $A \bowtie^f J \simeq f(A) + J$ is a ring graded isomorphism and the conclusion is straightforward. □

Recall from [10], that in the case where $B = A$, $I = \bigoplus_{\alpha \in \Gamma} I_\alpha$ is a homogeneous ideal of A and $f = id_A$ the identity map of A. Then $A \bowtie I$ the graded (amalgamated) duplication of A along I is a special graded amalgamation given by $A \bowtie I = \bigoplus_{\alpha \in G} (A \bowtie I)_\alpha$, where for each $\alpha \in G$:

$$(A \bowtie I)_\alpha := (A \bowtie^{id_A} I)_\alpha = \{(a_\alpha, a_\alpha + i_\alpha) \mid a_\alpha \in A_\alpha, i_\alpha \in I_\alpha\}.$$

The following corollary is an immediate result of Theorem 2.14, which examines the case of the graded amalgamated duplication.

Corollary 2.13. *Let A be a graded integral domain and I a nonzero proper homogeneous ideal of A. Then graded amalgamation duplication $A \bowtie I$ is never gr-AB ring.*

The following proposition provide a more general concept of transfer for the gr-AB ring property to graded amalgamation of rings.

Proposition 2.14. *Let A and B be a pair of graded rings, $f : A \longrightarrow B$ be a graded ring homomorphism, and J be a regular homogeneous proper ideal of B. If $R = A \bowtie^f J$ is a gr-AB ring, then $h(f^{-1}(J)) \subseteq h - Z(A)$.*

Proof. Deny, if we assume that $f^{-1}(J)$ contains a homogeneous regular element e and let j be a homogeneous regular element of J. Then, there is a positive integer n and $(r, f(r)+k) \in h(R)$ such that $R(e,0)^n + R(0,j)^n = R(r, f(r)+k)$. Therefore, we have

$$\left\{ \begin{array}{c} (e,0)^n = (e^n,0) = (\alpha_1, f(\alpha_1)+k_1)(r, f(r)+k) \\ (0,j)^n = (0,j^n) = (\alpha_2, f(\alpha_2)+k_2)(r, f(r)+k) \\ (r, f(r)+k) = (c_1, f(c_1)+d_1)(e^n,0) + (c_2, f(c_2)+d_2)(0,j^n) \end{array} \right\}$$

for some $(\alpha_1, f(\alpha_1)+k_1), (\alpha_2, f(\alpha_2)+k_2), (c_1, f(c_1)+d_1)$ and $(c_2, f(c_2)+d_2)$ in R. Then, we obtain the following equations:
(1) $e^n = \alpha_1 r$.
(2) $\alpha_2 r = 0$.
(3) $j^n = (f(\alpha_2)+k_2)(f(r)+k)$.
(4) $f(r)+k = (f(c_2)+d_2)j^n$.

From the first equation we get that r is a homogeneous regular element of A since e is regular and so $\alpha_2 = 0$ by the second equation. Also, $f(r)+k$ is a homogeneous regular element of B, since j is regular. Hence, from the third and the fourth equation, we obtain $f(r)+k = k_2(f(c_2)+d_2)(f(r)+k)$. Therefore, $1 = k_2(f(c_2)+d_2) \in J$, a contradiction. □

Recall, from [16], that gr-Nil$(R) = Gr(0)$ is the set of all $x = \sum_{j=0}^{k} x_{\alpha_j} \in R$ such that x_{α_j} is a nilpotent element of R for each $i \in \{0, \dots, k\}$. We then highlight another context in which it is possible to determine if a graded amalgamated algebra along an ideal is a gr AB-ring.

Proposition 2.15. *Let A and B be a pair of graded rings and $f : A \longrightarrow B$ be a graded ring homomorphism. Suppose that A is a gr-local ring with maximal homogeneous ideal M, and J is a homogeneous proper ideal of B such that $f(M)J = 0$ and $J \subseteq$ gr-Nil(B). Set $R := A \bowtie^f J$. Then R is a gr-AB ring if and only if A is a gr-AB ring.*

Proof. Assume that R is a gr-AB ring. By Lemma 2.12, A is a gr-AB ring. Conversely, let $\alpha = (a, f(a)+i)$ and $\beta = (b, f(b)+j)$ be two homogeneous elements of R. Without loss of generality, we may assume that $a, b \in M$. Since $i, j \in J$, there are positives integers p and m such that $i^p = 0$ and $j^m = 0$. As A is a gr-AB ring, there exists a positive integer n and $c \in h(A)$ such that $Aa^{npm} + Ab^{npm} = Ac$. By using the fact that $f(M)J = 0$, we conclude easily that $R\alpha^{npm} + R\beta^{npm} = R(a, f(a))^{npm} + R(b, f(b))^{npm} = R(c, f(c))$. Therefore R is a gr-AB ring. □

Note that in the case where the A is a graded integral domain and I is a proper homogeneous ideal of A, then the graded amalgamation duplication is gr-AB ring if and only if so is A and $I = 0$ (Corollary 2.13). Besides the context of integral domains, the following example shows the faillure of this characterization.

Example 2.16. Let $R = \mathbb{Q}\left[X, X^{-1}\right] \ltimes \mathbb{Q}\left[X, X^{-1}\right]$. Then R is a gr-local ring with a homogeneous maximal ideal $M = 0 \ltimes \mathbb{Q}\left[X, X^{-1}\right] = gr-Nil(R)$. Using Proposition 2.15 and Example 2.4, we conclude that the graded duplication $R \bowtie M$ is a gr-AB ring, even if $M \neq 0$.

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