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Weakly J-filters property in lattices

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Abstract. Let $\mathcal{L}$ be a bounded distributive lattice. Similar to the definition of weakly J-ideals of commutative rings, we introduce and study weakly J-filters of lattices. The main purpose of this paper is devoted to extend the notion of weakly J-ideal property in commutative rings to weakly J-filter property in lattices.

Key Words: lattice; J-filter; weakly J-filter.
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1 Introduction

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. As algebraic structures, lattices are undoubtedly a natural choice of generalizations of rings. Recently, the study of algebraic structures, using the properties of lattices, has become an research topic, leading to many intersting results . There are growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for instance, [5, 7, 8, 9]).

Over the years, several types of ideals have been developed in order to let us fully understand the structures of rings in general. The notion of prime ideals has a significant place in the theory of rings, and it is used to characterize certain classes of rings. Recall from Atiyah and MacDonald in [1], a prime ideal P of R is a proper ideal having the property that $ab \in P$ implies either $a \in P$ or $b \in P$ for each $a, b \in R$ (also see [2, 4]). In [17], Mohamadian defined a proper ideal I of R as an r-ideal if whenever $a, b \in R$ with $ab \in I$ and $\text{ann}(a) = 0$ imply that $b \in I$, where $\text{ann}(a) = \{r \in R : ra = 0\}$. He investigated the behavior of r-ideals and compare them with other classical ideals such as prime and maximal ideals. In [18], Tekir et al., defined and studied some subclass of r-ideals, namely, the class of n-ideals. A proper ideal I of a ring R is called an n-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$, then $b \in I$. For any ring R, By $J(R)$, we denote the Jacobson radical of R. Khashan and Bani-Ata generalized the concept of n-ideals in [15]. A proper ideal I of a ring R is called a J-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin J(R)$, then $b \in I$. Khashan and Celikel [16] introduced the notion of a weakly J-ideals, i.e. a proper ideal I of a ring R is called a weakly J-ideal if whenever $a, b \in R$ with $0 \neq ab \in I$ and $a \notin J(R)$, then $b \in I$. Let $\mathcal{L}$ be a bounded distributive lattice and $J(\mathcal{L})$ denote the Jacobson radical of $\mathcal{L}$ (i.e. to be the intersection of all the maximal filters of $\mathcal{L}$). In [13], the present author, introduced the concept of J-filters. A proper filter F of $\mathcal{L}$ is called a J-filter if whenever $x \vee y \in F$ with $x \notin J(\mathcal{L})$, then $y \in F$ for every $x, y \in \mathcal{L}$. Our objective in this paper is to extend the notion of weakly J-ideal property in commutative rings to weakly J-filter property in the lattices. Among many other results in this paper, the first, Preliminaries section contains elementary observations needed later on.

Section 3 is devoted to the investigate the basic properties of weakly J-filters as a new generalization of J-filters. A proper filter F of $\mathcal{L}$ is called a weakly J-filter if whenever $1 \neq x \vee y \in F$ with $x \notin J(\mathcal{L})$, then $y \in F$ for every $x, y \in \mathcal{L}$. At first, we provide an example of lattices for which their J-filters and weakly J-filters are the same (Theorem 3.3). Many equivalent characterizations of weakly J-filters for any bounded distributive lattice are presented in Proposition 3.4 and Theorem 3.8. In 2003, Anderson and Smith in [1] defined weakly prime ideals which is a generalization of prime ideals. A proper ideal P of a ring R is said to be a weakly prime if $0 \neq xy \in P$ for each $x, y \in R$ implies either $x \in P$ or $y \in P$. A proper filter P of a lattice $\mathcal{L}$ is said to be a weakly prime if $1 \neq x \vee y \in P$ for each $x, y \in \mathcal{L}$ implies either $x \in P$ or $y \in P$. In the Example 3.10, it is shown that, in general, the class of weakly J-filters is not comparable with the classes of weakly prime filters. Then we justify the relationships between these two concepts in Proposition 3.11 and Proposition 3.12. Further, for two weakly J-filters F_1 and F_2 of a lattice $\mathcal{L}$, we show that $F_1 \cap F_2$ is a weakly J-filter (see Proposition 3.13), but the converse is not true (see Example 3.14). Here, we provide some condition under which the converse of Proposition 3.13 is true (see Theorem 3.15).

We continue in this Section by investigation the stability of weakly J-filters in various lattice-theoretic constructions. In particular, we investigate the behavior of weakly J-filters under homomorphism, in factor lattices and in cartesian products of lattices (see Theorem 3.16, Corollary 3.17, Proposition 3.18, Proposition 3.19 and Theorem 3.31). Further, for two weakly J-filters F and G of a lattice $\mathcal{L}$, we show that $F \wedge G$ is a weakly J-filter of $\mathcal{L}$ (see Theorem 3.21). An element x of $\mathcal{L}$ is called *identity join* of a lattice $\mathcal{L}$, if there exists $1 \neq y \in \mathcal{L}$ such that $x \vee y = 1$. The set of all identity joins of a lattice $\mathcal{L}$ is denoted by $\mathrm{I}(\mathcal{L})$. Similar to the definition of *presimplifiable* ring from Bouvier in [6], a lattice $\mathcal{L}$ is called presimplifiable if $\mathrm{I}(\mathcal{L}) \subseteq J(\mathcal{L})$. It is well known from Anderson and Axtell in [3] that presimplifiable property does not pass in general to homomorphic images. However, we show that this holds under a certain condition: If F is a weakly J-filter of a presimplifiable lattice $\mathcal{L}$, then $\mathcal{L}/F$ is a presimplifiable lattice (see Proposition 3.23, Theorem 3.24 and Theorem 3.27). Similar to the idealization of a module over a ring (Huckaba in [14]), the remaining part of this section mainly devoted to investigation the stability of weakly J-filters in the *filterlization* of a lattice and a filter. Let $\mathcal{M}$ be a filter of lattice $\mathcal{L}$. The filterlization $\mathcal{L}(+)\mathcal{M} = \{(a, m) : a \in \mathcal{L}, m \in \mathcal{M}\}$ of $\mathcal{M}$ in $\mathcal{L}$ is a lattice with respect to the following definitions: (1) $(a, m) = (a', m')$ if $a = a'$ and $m = m'$, (2) $(a, m_1) \wedge_F (b, m_2) = (a \wedge b, m_1 \wedge m_2)$ and (3) $(a, m_1) \vee_F (b, m_2) = (a \vee b, (a \vee m_2) \wedge (b \vee m_1))$. Here we clarify the relationships between weakly J-filters in a lattice $\mathcal{L}$ and in a filterlization lattice $\mathcal{L}(+)\mathcal{M}$ in Theorem 3.33.

2 Preliminaries

A *poset* $(\mathcal{L}, \leq)$ is a *lattice* if $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist for all $a, b \in \mathcal{L}$ (and call $\wedge$ the *meet* and $\vee$ the *join*).

Definition 2.1. (1) A lattice $\mathcal{L}$ is called a *distributive lattice* if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in $\mathcal{L}$ (equivalently, $\mathcal{L}$ is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in $\mathcal{L}$).

(2) A non-empty subset F of a lattice $\mathcal{L}$ is called a *filter*, if for $a \in F$, $b \in \mathcal{L}$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\mathcal{L}$ is a lattice with 0 and 1, then $1 \in F$, $\{1\}$ is a filter of £ and $0 \in F$ if and only if $F = \mathcal{L}$).

(3) A proper filter F of $\mathcal{L}$ is called *prime* if $x \vee y \in F$, then $x \in F$ or $y \in F$.

(4) A proper filter F of $\mathcal{L}$ is said to be *maximal* if G is a filter in $\mathcal{L}$ with $F \subsetneq G$, then $G = \mathcal{L}$. We define the *Jacobson radical* of $\mathcal{L}$, denoted by $J(\mathcal{L})$, to be the intersection of all the maximal filters of $\mathcal{L}$. The set of all maximal filters of $\mathcal{L}$ is denoted $\mathrm{Max}(\mathcal{L})$.

(5) Let D be subset of a lattice $\mathcal{L}$. Then the filter generated by D, denoted by $T(D)$, is the intersection of all filters that is containing D. A filter F is called finitely generated if there is a finite subset D of F such that $F = T(D)$.

(6) A lattice $\mathcal{L}$ with 1 is called $\mathcal{L}$-domain if $a \vee b = 1$ $(a, b \in \mathcal{L})$, then $a = 1$ or $b = 1$ (so $\mathcal{L}$ is $\mathcal{L}$-domain if and only if $\{1\}$ is a prime filter of $\mathcal{L}$).

(7) A lattice $\mathcal{L}$ is called *local* if it has exactly one maximal filter that contains all proper filters.

(8) If $x \in \mathcal{L}$, then a *complement* of x in $\mathcal{L}$ is an element $y \in \mathcal{L}$ such that $x \vee y = 1$ and $x \wedge y = 0$. The lattice $\mathcal{L}$ is *complemented* if every element of $\mathcal{L}$ has a complement in $\mathcal{L}$.

(9) If $\mathcal{L}$ and $\mathcal{L}'$ are lattices, then a *lattice homomorphism* $f : \mathcal{L} \to \mathcal{L}'$ is a map from $\mathcal{L}$ to $\mathcal{L}'$ satisfying $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for $x, y \in \mathcal{L}$.

(10) A filter G of $\mathcal{L}$ is called *small* in $\mathcal{L}$, written $G \ll \mathcal{L}$, if for every filter H of $\mathcal{L}$, the equality $G \wedge H = \mathcal{L}$ implies $H = \mathcal{L}$.

For undefined notations or terminologies in lattice theory, we refer the reader to [5, 7]. First we need the following easy observation proved in [5, 7, , 9, 11].

Lemma 2.2. *Let $\mathcal{L}$ be a lattice.*

(1) A non-empty subset F of $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in \mathcal{L}$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in \mathcal{L}$.

(2) If F_1, F_2 are filters of $\mathcal{L}$ and $a \in \mathcal{L}$, then $F_1 \vee F_2 = \{a_1 \vee a_2 : a_1 \in F_1, a_2 \in F_2\}$ and $a \vee F_1 = \{a \vee a_1 : a_1 \in F_1\}$ are filters of $\mathcal{L}$ and $F_1 \vee F_2 = F_1 \cap F_2$.

(3) Let A be an arbitrary non-empty subset of $\mathcal{L}$. Then

$$T(A) = \{x \in \mathcal{L} : \ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x \text{ for some } a_i \in A \ (1 \leq i \leq n)\}.$$

Moreover, if F is a filter and A is a subset of $\mathcal{L}$ with $A \subseteq F$, then $T(A) \subseteq F$, $T(F) = F$ and $T(T(A)) = T(A)$

(4) If $\mathcal{L}$ is distributive, F, G are filters of $\mathcal{L}$ and $y \in \mathcal{L}$, then $(G :_{\mathcal{L}} F) = \{x \in \mathcal{L} : x \vee F \subseteq G\}$, $(F :_{\mathcal{L}} T(\{y\})) = (F :_{\mathcal{L}} y) = \{a \in \mathcal{L} : a \vee y \in F\}$ and $(1 :_{\mathcal{L}} y) = \{x \in \mathcal{L} : x \vee y = 1\}$ are filters of $\mathcal{L}$.

(5) If $\mathcal{L}$ is distributive and F_1, F_2 are filters of $\mathcal{L}$, then $F_1 \wedge F_2 = \{a_1 \wedge a_2 : a_1 \in F_1, a_2 \in F_2\}$ is a filter of $\mathcal{L}$ and $F_1, F_2 \subseteq F_1 \wedge F_2$.

Lemma 2.3. *[12, Lemma 3.13] Let $£_1$ and $£_2$ be lattices and $f : £_1 \to £_2$ be a lattice homomorphism such that $f(1) = 1$. The following hold:*

(1) $\mathrm{Ker}(f) = \{x \in £_1 : f(x) = 1\}$ *is a filter of $£_1$;*

(2) If f is injective, then $\mathrm{Ker}(f) = \{1\}$;

(3) If $£_1$ is a complemented lattice, then f is injective if and only if $\mathrm{Ker}(f) = \{1\}$.

3 Characterization of weakly J-filters

In this section, we collect some basic properties concerning weakly J-filters and remind the reader with the following definition.

Definition 3.1. A proper filter F of a lattice $\mathcal{L}$ is called a weakly J-filter if whenever $x, y \in \mathcal{L}$ with $1 \neq x \vee y \in F$ and $x \notin J(\mathcal{L})$, then $y \in F$.

Example 3.2. (1) It is easy to see that every J-filter is a weakly J-filter.

(2) Let $A = \{1, 2, 3\}$. Then $\mathcal{L} = \{X : X \subseteq A\}$ forms a distributive lattice under set inclusion with greatest element A and least element $\emptyset$ (note that if $x, y \in \mathcal{L}$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). It can be easily seen that the set of all proper filters $\mathcal{L}$ is $\{\{A\}, F_1, F_2, F_3, F_4, F_5, F_6\}$, where $F_1 = \{A, \{1, 2\}\}$, $F_2 = \{A, \{1, 3\}\}$, $F_3 = \{A, \{2, 3\}\}$,

$$F_4 = \{A, \{1, 3\}, \{1, 2\}, \{1\}\},$$

$F_5 = \{A, \{2, 3\}, \{1, 2\}, \{2\}\}$ and $F_6 = \{A, \{1, 3\}, \{3, 2\}, \{3\}\}$ with $J(\mathcal{L}) = F_4 \cap F_5 \cap F_6 = \{A\}$. . Set $F = \{A\}$. Then F is not a J-filter. For example, $\{1, 2\}, \{3\} \in \mathcal{L}$ with $\{1, 2\} \vee \{3\} \in F$ and $\{3\} \notin J(\mathcal{L})$ but $\{1, 2\} \notin F$. However, since F is always weakly J-filter (by definition), a weakly J-filter need not be J-filter.

The next result determines the class of lattices for which their J-filters and weakly J-filters are the same. Compare the next theorem with Theorem 2.3 in [16].

Theorem 3.3. If $\mathcal{L}$ is a lattice, then the following statements are equivalent:
(1) $\mathcal{L}$ is a local lattice;
(2) Every proper filter of $\mathcal{L}$ is a J-filter;
(3) Every proper filter of $\mathcal{L}$ is a weakly J-filter;
(4) Every proper principal filter of $\mathcal{L}$ is a weakly J-filter.

Proof. (1) $\Rightarrow$ (2) Let G be a proper filter of a local lattice $\mathcal{L}$ with unique maximal filter M and let $a, b \in \mathcal{L}$ with $a \vee b \in G$ and $a \notin J(\mathcal{L}) = M$. Then $M \wedge T(\{a\}) = \mathcal{L}$ by maximality of M which implies that $T(\{a\}) = \mathcal{L}$, as M is small in $\mathcal{L}$ by [9, Lemma 3.1]. So $0 = a \vee c$ for some $c \in \mathcal{L}$. Then $a = 0$ gives $b \in G$. Thus, G is a J-filter.

The implication (2) $\Rightarrow$ (3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) Let M be a maximal filter of $\mathcal{L}$. If $M = \{1\}$ and $1 \neq a \in \mathcal{L}$, then $\{1\} \subsetneq T(\{a\}) \subseteq \mathcal{L}$ gives $T(\{a\}) = \mathcal{L}$; hence $a \leq a \vee b = 0$ for some $b \in \mathcal{L}$. Therefore, $\mathcal{L} = \{0, 1\}$ and so the result follows clearly. Otherwise, let $1 \neq x \in M$. Now, $T(\{x\})$ is a weakly J-filter by (4) and $1 \neq x \vee 0 \in T(\{x\})$. If $x \notin J(\mathcal{L})$, then $0 \in T(\{x\})$ which is impossible. Hence, $x \in J(\mathcal{L})$ and $J(\mathcal{L}) = M$, i.e. (1) holds. □

For a filter F of a lattice $\mathcal{L}$, the Jacobson radical of F, denoted by $J(F)$, is defined as the intersection of all maximal filters of $\mathcal{L}$ containing F. The following properties can be easily verified for any filters F and G of $\mathcal{L}$:
(1) $F \subseteq J(F)$.
(2) If $F \subseteq G$, then $J(F) \subseteq J(G)$.
(3) $J(\mathcal{L}) \subseteq J(F)$.
(4) $J(J(F)) = J(F)$.

Proposition 3.4. *If F is a proper filter of a lattice $\mathcal{L}$, then the following statements are equivalent:*
(1) F is a weakly J-filter of $\mathcal{L}$;
(2) $F \subseteq J(\mathcal{L})$ and whenever $x, y \in \mathcal{L}$ with $1 \neq \vee x \vee y \in F$, then $x \in J(F)$ or $y \in F$.

Proof. (1) $\Rightarrow$ (2) Let $1 \neq f \in F$. Since $1 \neq f \vee 0 \in F$ and $0 \notin F$, we conclude that $f \in J(\mathcal{L})$. Therefore, $F \subseteq J(\mathcal{L})$. Now, let $1 \neq x \vee y \in F$ with $x \notin J(F)$. Since $J(\mathcal{L}) \subseteq J(F)$ and F is weakly J-filter, we infer that $y \in F$.

(2) $\Rightarrow$ (1) Let $x, y \in \mathcal{L}$ such that $1 \neq x \vee y \in F$ and $x \notin J(\mathcal{L})$. As $F \subseteq J(\mathcal{L})$, we conclude that $J(F) \subseteq J(J(\mathcal{L})) = J(\mathcal{L})$ and so we have $x \notin J(F)$. Thus, $y \in F$ and F is a weakly J-filter. □

The following example shows that we can find a filter F of a lattice $\mathcal{L}$ with $\{1\} \neq F \subseteq J(\mathcal{L})$ which is not a weakly J-filter.

Example 3.5. Assume that $\mathcal{L} = \{0, a, b, c, 1\}$ is a lattice with the relations $0 \leq a \leq c \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$ and $a \wedge b = 0$. An inspection will show that the nontrivial filters of $\mathcal{L}$ are $F_1 = \{1, a, c\}$, $F_2 = \{1, b, c\}$ and $F_3 = \{1, c\}$ with $F_3 \subseteq J(\mathcal{L}) = F_1 \cap F_2 = F_3$. But F_3 is not a weakly J-filter since $1 \neq a \vee b = c \in F_3$ with $a \notin J(\mathcal{L})$ and $b \notin F_3$.

A lattice $\mathcal{L}$ is called *semiprimitive* if $J(\mathcal{L}) = \{1\}$.

Corollary 3.6. *If $\mathcal{L}$ is a semiprimitive lattice, then $\{1\}$ is the only weakly J-filter of $\mathcal{L}$.*

Proof. This is a direct consequence of Proposition 3.4 (2). □

Compare the next theorem with Theorem 2.5 in [16].

Theorem 3.7. Let F be a weakly J-filter of $\mathcal{L}$. If F is not J-filter, then $F = \{1\}$.

Proof. On the contrary, assume that $F \neq \{1\}$. We show that F is a J-filter. Let $a, b \in \mathcal{L}$ such that $a \vee b \in F$ and $a \notin J(\mathcal{L})$. If $1 \neq a \vee b \in F$, then F is a weakly J-filter gives $b \in F$. Now, suppose that $a \vee b = 1$. Since $F \neq \{1\}$, there exists $f \in F$ such that $f \neq 1$. Clearly, $a \wedge f \notin J(\mathcal{L})$ (otherwise, $a \in J(\mathcal{L})$ by Lemma 2.2 (1)). Then $1 \neq (a \wedge f) \vee (b \wedge f) = f \in F$ gives $f \wedge b \in F$. Therefore, $b \in F$ by Lemma 2.2 (1). This shows that F is a J-filter, as required. □

We next give four other characterizations of weakly J-filters. Compare the next theorem with Theorem 2.7 in [16].

Theorem 3.8. Let P be a proper filter of a lattice $\mathcal{L}$. Then the following statements are equivalent:
(1) P is a weakly J-filter of $\mathcal{L}$;
(2) $(P :_{\mathcal{L}} x) = P \cup (1 :_{\mathcal{L}} x)$ for every $x \notin J(\mathcal{L})$;
(3) $(P :_{\mathcal{L}} x) \subseteq J(\mathcal{L}) \cup (1 :_{\mathcal{L}} x)$ for every $x \notin P$;
(4) If $x \in \mathcal{L}$ and F is a filter of $\mathcal{L}$ with $\{1\} \neq x \vee F \subseteq P$, then $F \subseteq J(\mathcal{L})$ or $x \in P$;
(5) If F and G are filters of $\mathcal{L}$ with $\{1\} \neq F \vee G \subseteq P$, then $F \subseteq J(\mathcal{L})$ or $G \subseteq P$.

Proof. (1) $\Rightarrow$ (2) Let $x \notin J(\mathcal{L})$. Since the inclusion $P \cup (1 :_{\mathcal{L}} x) \subseteq (P :_{\mathcal{L}} x)$ is clear, we will prove the reverse inclusion. Let $y \in (P :_{\mathcal{L}} x)$. If $x \vee y \neq 1$, then by (1), $y \in P$. If $x \vee y = 1$, then $y \in (1 :_{\mathcal{L}} x)$. Therefore, $(P :_{\mathcal{L}} x) \subseteq P \cup (1 :_{\mathcal{L}} x)$ and so we have equality.

(2) $\Rightarrow$ (1) Let $x, y \in \mathcal{L}$ such that $1 \neq x \vee y \in P$ and $x \notin J(\mathcal{L})$. Since $y \notin (1 :_{\mathcal{L}} x)$ and $y \in (P :_{\mathcal{L}} x)$, we conclude that $y \in P$ by (2).

(1) $\Rightarrow$ (3) Let $x \notin P$ and $y \in (P :_{\mathcal{L}} x)$. If $x \vee y \neq 1$, then by (1), $y \in J(\mathcal{L})$. If $x \vee y = 1$, then $y \in (1 :_{\mathcal{L}} x)$. Therefore, $(P :_{\mathcal{L}} x) \subseteq J(\mathcal{L}) \cup (1 :_{\mathcal{L}} x)$.

(3) $\Rightarrow$ (4) Let $\{1\} \neq x \vee F \subseteq P$ and $x \notin P$. Then $(1 :_{\mathcal{L}} x) \subsetneqq (P :_{\mathcal{L}} x)$ and so $F \subseteq (P :_{\mathcal{L}} x) \subseteq J(\mathcal{L})$ by [8, Remark 2.3 (i)], as required.

(4) $\Rightarrow$ (5) On the contrary, assume that there exist filters F and G of $\mathcal{L}$ such that $\{1\} \neq F \vee G \subseteq P$ but $F \nsubseteq J(\mathcal{L}$ and $G \nsubseteq P$. Since $F \vee G \neq \{1\}$, we conclude that there exists $g \in G$ such that $\{1\} \neq g \vee F \subseteq P$ which implies that $g \in P$ by (4), as $F \nsubseteq J(\mathcal{L})$. Consider $g' \in G \setminus P$. If $g' \vee F \neq \{1\}$, then $g' \in P$ by (4), a contradiction. Thus, $g' \vee F = \{1\}$. Since $g \wedge g' \in G$ and $g \vee F \neq \{1\}$, we infer that $\{1\} \neq (g \wedge g') \vee F \subseteq P$ and $F \nsubseteq J(\mathcal{L})$ implies that $g \wedge g' \in P$; so $g' \in P$ by Lemma 2.1 (1) which is a contradiction.

(5) $\Rightarrow$ (1) Let $x, y \in \mathcal{L}$ such that $1 \neq x \vee y \in P$ and $x \notin J(\mathcal{L})$. Set $F = T(\{x\})$ and $G = T(\{y\})$. Then $\{1\} \neq F \vee G \subseteq P$ and $F \nsubseteq J(\mathcal{L})$. Now the assertion follows from (5). □

Proposition 3.9. *Let S be a non-empty subset of $\mathcal{L}$. If F and $(1 :_{\mathcal{L}} S)$ are weakly J-filters with $S \nsubseteq F$, then $(F :_{\mathcal{L}} S)$ is a weakly J-filter.*

Proof. If $(F :_{\mathcal{L}} S) = \mathcal{L}$, then $0 \in (F :_{\mathcal{L}} S)$ and so $S \subseteq F$, a contradiction. Thus, $(F :_{\mathcal{L}} S)$ is a proper filter. Let $x, y \in \mathcal{L}$ such that $1 \neq x \vee y \in (F :_{\mathcal{L}} S)$ with $x \notin J(\mathcal{L})$. If $1 \neq x \vee y \vee s \in F$ for every $s \in S$, then F is a J-filter gives $y \vee s \in F$ and so $y \in (F :_{\mathcal{L}} S)$. If $x \vee y \vee S = \{1\}$, then $1 \neq x \vee y \in (1 :_{\mathcal{L}} S)$ gives $y \in (1 :_{\mathcal{L}} S) \subseteq (F :_{\mathcal{L}} S)$, as $(1 :_{\mathcal{L}} S)$ is a J-filter, as needed. □

In the following example, it is shown that, in general, the class of weakly J-filters is not comparable with the classes of weakly prime filters.

Example 3.10. (1) Let $\mathcal{L}$ be the lattice as in Example 3.2 (2). Then the filter F_5 is a prime filter (so a weakly prime filter). On the other hand, F_5 is not a weakly J-filter of $\mathcal{L}$, as $F_5 \nsubseteq J(\mathcal{L})$.

(2) The collection of ideals of $\mathbb{Z}$, the ring of integers, form a lattice under set inclusion which we shall denote by $\mathcal{L}$ with respect to the following definitions: $m\mathbb{Z} \vee n\mathbb{Z} = (m, n)\mathbb{Z}$ and $m\mathbb{Z} \wedge n\mathbb{Z} = [m, n]\mathbb{Z}$ for all ideals $m\mathbb{Z}$ and $n\mathbb{Z}$ of $\mathbb{Z}$, where (m, n) and $[m, n]$ are greatest common divisor and least common multiple of m, n, respectively. Note that $\mathcal{L}$ is a distributive complete lattice with least element the

zero ideal and the greatest element $\mathbb{Z}$. By [8, Theorem 2.9 (ii)], $\mathcal{L} \setminus \{0\}$ is the only maximal filter of $\mathcal{L}$ and so $\mathcal{L}$ is a local lattice. It follows from Theorem 3.3 that every proper filter of $\mathcal{L}$ is a weakly J-filter. Consider the filter $P = \{\mathbb{Z}, 2\mathbb{Z}\}$. Since $1 \neq 14\mathbb{Z} \vee 18\mathbb{Z} = 2\mathbb{Z} \in P$ with $14\mathbb{Z}, 18\mathbb{Z} \notin P$, we infer that P is not a weakly prime filter.

Proposition 3.11. *Let P be a weakly prime filter such that $P \subseteq J(\mathcal{L})$. Then P is a weakly J-filter of $\mathcal{L}$.*

Proof. The proof is straightforward. □

The following proposition provides some condition under which a weakly J-filter is a weakly prime filter.

Proposition 3.12. *Let P be a filter of $\mathcal{L}$. Assume P is maximal with respect to property: P and $(1 :_{\mathcal{L}} x)$ are weakly J-filters for all $x \notin P$. Then P is a weakly prime filter in $\mathcal{L}$.*

Proof. Let $x, y \in \mathcal{L}$ such that $1 \neq x \vee y \in P$ and $x \notin P$. Set $S = \{x\}$. Then $(P :_{\mathcal{L}} x)$ is a weakly J-filter by Proposition 3.9. Since $P \subsetneq (P :_{\mathcal{L}} x)$, we conclude that $z \notin P$ for all $z \notin (P :_{\mathcal{L}} x)$ and so $(1 :_{\mathcal{L}} z)$ is a weakly J-filters. Therefore, we have $y \in (P :_{\mathcal{L}} x) = P$ by maximality of P, as needed. □

Proposition 3.13. *Let $\mathcal{L}$ be a lattice. If $\{F_i\}_{i\in\Lambda}$ is a nonempty family of weakly J-filters of $\mathcal{L}$, then $\cap_{i\in\Lambda} F_i$ is a weakly J-filter.*

Proof. (1) Let $a, b \in \mathcal{L}$ such that $1 \neq a \vee b \in \cap_{i\in\Lambda} F_i$ and $a \notin J(\mathcal{L})$. By the hypothesis, $1 \neq a \vee b \in F_i$ for all $i \in \Lambda$ gives $b \in F_i$ for all $i \in \Lambda$ and so $b \in \cap_{i\in\Lambda} F_i$, as needed. □

Example 3.14. In general, the converse Proposition 3.13 is not true. For example, while $F_4 \cap F_5 = \{1\}$ is a weakly J-filters of the lattice as in 3.2 (2), non of the filters F_4 and F_5 are (weakly) J-filters.

The following theorem provides some condition under which the converse Proposition 3.13 is true.

Theorem 3.15. Let $F_1, \cdots, F_n$ be weakly prime filters of $\mathcal{L}$ which are not comparable and $(\mathcal{L} \setminus F_i) \cap \mathrm{I}(\mathcal{L}) = \emptyset$ for all $1 \leq i \leq n$. Then $\cap_{i=1}^{n} F_i$ is a weakly J-filter if and only if F_i is a weakly J-filter for $i \in \{1, \cdots, n\}$.

Proof. One side is clear by Proposition 3.13. To see the other side, let $1 \neq a \vee b \in F_i$ with $a \notin J(\mathcal{L})$ and take $c \in (\bigvee_{i\neq j} F_j) \setminus F_i$. Therefore, $a \vee b \vee c \in \bigvee_{i=1}^{n} F_i = \cap_{i=1}^{n} F_i$. If $a \vee b \vee c = 1$, then $c \in (\mathcal{L} \setminus F_i) \cap \mathrm{I}(\mathcal{L})$ which is a contradiction So we may assume that $a \vee b \vee c \neq 1$. Since $\cap_{i=1}^{n} F_i$ is a weakly J-filter and $a \notin J(\mathcal{L})$, we conclude that $b \vee c \in \cap_{i=1}^{n} F_i$ and so $b \vee c \in F_i$. This implies that $b \in F_i$, i.e. F_i is a weakly J-filter of $\mathcal{L}$. □

We continue this section with the investigation of the stability of weakly J-filters in various lattice-theoretic constructions.

Theorem 3.16. Let $\mathcal{L}$ be a complemented lattice. If $f : \mathcal{L} \to \mathcal{L}'$ is a lattice homomorphism such that $f(0) = 0$ and $f(1) = 1$, then the following hold:

(1) If f is a monomorphism and K is a weakly J-filter of $\mathcal{L}'$, then $f^{-1}(K)$ is a weakly J-filter of $\mathcal{L}$.

(2) If f is an epimorphism and G is a weakly J-filter of $\mathcal{L}$ with ker(f) $\subseteq G$, then $f(G)$ is a weakly J-filter of $\mathcal{L}'$.

Proof. (1) Let $a, b \in \mathcal{L}$ with $1 \neq a \vee b \in f^{-1}(K)$ and $a \notin J(\mathcal{L})$. Then $f(a) \vee f(b) = f(a \vee b) \in K$. We show that $f(a) \notin J(\mathcal{L}')$. On the contrary, assume that $f(a) \in J(\mathcal{L}')$. Let M be a maximal filter of $\mathcal{L}$. Then $f(M) \neq \{1\}$, as f is a monomorphism. Let $f(M) \subsetneqq F \subseteq \mathcal{L}'$ for some filter F of $\mathcal{L}'$. The there exists $c \in \mathcal{L} \setminus M$ such that $f(c) \in F \setminus f(M)$. Since $M \subsetneqq M \wedge T(\{c\})$, we conclude that $M \wedge T(\{c\}) = \mathcal{L}$ and so $0 = m \wedge (c \vee b) = (m \wedge c) \vee (m \wedge b)$ for some $m \in M$ and $b \in \mathcal{L}$; hence $m \wedge c = 0$. It follows that

$f(m \wedge c) = f(m) \wedge f(c) = f(0) = 0 \in F$ which implies that $F = \mathcal{L}'$. Thus, $f(M)$ is a maximal filter of $\mathcal{L}'$ which implies that $f(a) \in f(M)$ and so $a \in M$. Hence $a \in J(\mathcal{L})$ which is impossible. Therefore, $f(a) \notin J(\mathcal{L}')$. Since by Lemma 2.3, $\mathrm{Ker}(f) = \{1\}$, we conclude that $\{1\} \neq f(a \vee b) = f(a) \vee f(b) \in K$; hence $f(b) \in K$, as K is a weakly J-filter and so $b \in f^{-1}(K)$. Thus, $f^{-1}(K)$ is a J-filter of $\mathcal{L}$.

(2) Let $x, y \in \mathcal{L}'$ such that $1 \neq x \vee y \in f(G)$ and $x \notin J(\mathcal{L}')$. Since f is an epimorphism, there exist $a, b \in \mathcal{L}$ such that $x = f(a)$ and $y = f(b)$. Then $x \vee y = f(a \vee b) \in f(G)$ (so $a \vee b \neq 1$) and then $f(a \vee b) = f(g)$ for some $g \in G$. By the hypothesis, $g \vee g' = 1$ and $g \wedge g' = 0$ for some $g' \in \mathcal{L}$. Since $f(a \vee b \vee g') = f(a \vee b) \vee f(g') = 1$, we conclude that $a \vee b \vee g' \in \ker(f) \subseteq G$; hence $a \vee b = (a \vee b) \vee (g \wedge g') = (a \vee b \vee g) \wedge (a \vee b \vee g') \in G$, as G is a filter. Also, note that $a \notin J(\mathcal{L})$ since otherwise if $a \in J(\mathcal{L})$, then $x = f(a) \in J(\mathcal{L}')$, as $f(J(\mathcal{L})) = f(\bigcap_{M \in \mathrm{Max}(\mathcal{L})} M) \subseteq \bigcap_{M \in \mathrm{Max}(\mathcal{L})} f(M) \subseteq J(\mathcal{L}')$ which is impossible. Since G is a J-filter, we infer that $b \in G$ and so $y = f(b) \in f(G)$, as needed. □

If F is a filter of a lattice $(\mathcal{L}, \leq)$, we define a relation on $\mathcal{L}$, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on $\mathcal{L}$, and we denote the equivalence class of a by $a \wedge F$ and these collection of all equivalence classes by $\mathcal{L}/F$. We set up a partial order $\leq_Q$ on $\mathcal{L}/F$ as follows: for each $a \wedge F, b \wedge F \in \mathcal{L}/F$, we write $a \wedge F \leq_Q b \wedge F$ if and only if $a \leq b$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}/F, \leq_Q)$ is a lattice with $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$ and $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$ for all elements $a \wedge F, b \wedge F \in \mathcal{L}/F$. Note that $f \wedge F = F$ if and only if $f \in F$ (see [10, Remark 4.2 and Lemma 4.3]).

Corollary 3.17. *Suppose $\mathcal{L}$ is a complemented lattice and let F, G be two proper filters of $\mathcal{L}$ with $F \subseteq G$. If G is a weakly J-filter of $\mathcal{L}$, then G/F is a weakly J-filter of $\mathcal{L}/F$.*

Proof. Let $v : \mathcal{L} \to \mathcal{L}/F$ be the natural epimorphism defined by $v(x) = x \wedge F$. Then $\ker(v) = \{x \in \mathcal{L} : x \wedge F = 1 \wedge F\} = F \subseteq G$ by [10, Lemma 4.3] and so by Theorem 3.16 (2) and [10, Lemma 4.3] that $v(G) = \{x \wedge F : x \in G\} = G/F$ is a weakly J-filter of $\mathcal{L}/F$. □

Proposition 3.18. *Suppose $\mathcal{L}$ is a lattice and let F, G be two proper filters of $\mathcal{L}$ with $F \subseteq G$. Then the followings hold:*

(1) If F is a J-filter of $\mathcal{L}$ and G/F is a weakly J-filter of $\mathcal{L}/F$, then G is a J-filter of $\mathcal{L}$.

(2) If F is a weakly J-filter of $\mathcal{L}$ and G/F is a weakly J-filter of $\mathcal{L}/F$, then G is a weakly J-filter of $\mathcal{L}$.

Proof. (1) Let $x, y \in \mathcal{L}$ such that $x \vee y \in G$ and $x \notin J(\mathcal{L})$. If $x \vee y \in F$, then $y \in F \subseteq G$. So we may assume that $x \vee y \notin F$ (so $(x \vee y) \wedge F \neq F = 1 \wedge F$). By [10, Lemma 4.3], $M/F \in \mathrm{Max}(\mathcal{L}/F)$ if and only if $M \in \mathrm{Max}(\mathcal{L})$ with $F \subseteq M$. Since F is a J-filter, we infer that $F \subseteq J(\mathcal{L})$ and so clearly $x \wedge F \notin J(\mathcal{L}/F)$. Since $F = 1 \wedge F \neq (x \wedge F) \vee_Q (y \wedge F) = (x \vee y) \wedge F \in G/F$ by Lemma [10, Lemma 4.3] and G/F is a weakly J-filter, we include that $y \wedge F \in G/F$; hence $y \in G$, as required.

(2) The proof is similar to that in case (1) and we omit it. □

One can easily show that if G is a filter of a complemented lattice $\mathcal{L}$, then $\mathcal{L}/G$ is a complemented lattice.

Proposition 3.19. *Let F and G be two filters of a complemented lattice $\mathcal{L}$. Then there is a lattice isomorphism $\phi : F/(F \cap G) \to (F \wedge G)/G$ that sends each residue class $x \wedge (F \cap G)$ to $x \wedge G$.*

Proof. At first, note that since $(a \wedge b) \wedge b = a \wedge b$ for some $b \in G$ and $a \in \mathcal{L}$, we conclude that $(a \wedge b) \wedge G = a \wedge G$. If $x \wedge (F \cap G) = y \wedge (F \cap G)$, then $x \wedge a = y \wedge b$ for some $a, b \in F \cap G \subseteq G$; so $x \wedge G = y \wedge G$. This shows that ϕ is well defined. Clearly, ϕ is serjective. If $A = x \wedge (F \cap G)$ and $B = y \wedge (F \cap G)$ are elements of $F/(F \cap G)$, then $\phi(A \vee_Q B) = \phi((x \vee y) \wedge F \cap G) = (x \vee y) \wedge G = (x \wedge G) \vee_Q (y \wedge G) = \phi(A) \vee_Q \phi(B)$. Similarly, $\phi(A \wedge_Q B) = \phi(A) \wedge_Q \phi(B)$. Since $\mathrm{Ker}(\phi) = \{x \wedge (F \cap G) \in F/(F \cap G) : x \wedge G = 1 \wedge G\} = (F \wedge G)/(F \wedge G) = \{\bar{1}\}$, we infer that ϕ is injective by Lemma 2.3, as required. □

Lemma 3.20. *If G is a weakly J-filter of $\mathcal{L}$, then it is small in $\mathcal{L}$.*

Proof. Let K be a filter of $\mathcal{L}$ such that $G \wedge K = \mathcal{L}$. Assume on the contrary, that $K \neq \mathcal{L}$. Since $K \neq \{1\}$, we conclude that $K \subseteq M$ for some maximal filter M of $\mathcal{L}$ by [10, Lemma 2.1]. It follows from Proposition 3.4 that $G \subseteq J(\mathcal{L}) \subseteq M$ and so $\mathcal{L} = K \wedge G \subseteq M$ which is a contradiction. Therefore, $K = \mathcal{L}$. □

Compare the next theorem with Proposition 2.26 in [16].

Theorem 3.21. If G and F are weakly J-filters of a complemented lattice $\mathcal{L}$, then $F \wedge G$ is a weakly J-filter of $\mathcal{L}$.

Proof. Let F and G be weakly J-filters of $\mathcal{L}$. By Lemma 3.20, $F \wedge G \neq \mathcal{L}$. Since $F \cap G$ is a weakly J-filter by Proposition 3.13, then $F/(F \cap G)$ is a weakly J-filter of $\mathcal{L}/(F \cap G)$ by Corollary 3.17. Now, Proposition 3.19 gives $(F \wedge G)/G$ is a weakly J-filter of $\mathcal{L}/G$. Hence, $F \wedge G$ is a weakly J-filter of $\mathcal{L}$ by Proposition 3.18 (2). □

Proposition 3.22. *Let $\mathcal{L}$ be a presimplifiable lattice. Then every weakly J-filter of $\mathcal{L}$ is a J-filter.*

Proof. At first, we show that $F = \{1\}$ is a J-filter of $\mathcal{L}$. Let $a, b \in \mathcal{L}$ such that $a \vee b \in F$ and $a \notin J(\mathcal{L})$. By the hypothesis, $a \notin \mathrm{I}(\mathcal{L})$ and so $b = 1 \in F$. Suppose that G is a weakly J-filter of $\mathcal{L}$. We show that G is a J-filter. Let $x, y \in \mathcal{L}$ such that $x \vee y \in G$ and $x \notin J(\mathcal{L})$. If $x \vee y \neq 1$, then G is a weakly J-filter gives $y \in G$. If $x \vee y = 1$, then $\{1\}$ is a J-filter implies that $y = 1 \in G$. This shows that G is a J-filter. □

Proposition 3.23. *Let F be a proper filter of $\mathcal{L}$. Then F is a J-filter if and only if $F \subseteq J(\mathcal{L})$ and $\mathcal{L}/F$ is a presimplifiable lattice.*

Proof. Suppose F is a J-filter of $\mathcal{L}$. Then $F \subseteq J(\mathcal{L})$ by Proposition 3.4. Now, let $x \wedge F \in \mathrm{I}(\mathcal{L}/F)$. Then there exists $F = 1 \wedge F \neq y \wedge F \in \mathcal{L}/F$ (so $y \notin F$) such that $(x \wedge F) \vee_Q (y \wedge F) = (x \vee y) \wedge F = 1 \wedge F$. So there are elements $p, q \in F$ such that $(x \wedge p) \vee (y \wedge p) = (x \vee y) \wedge p = 1 \wedge q = q \in F$. Since F is a filter and $y \notin F$, we conclude that $y \wedge p \notin F$ by Lemma 2.2 (1). As F is a J-filter and $p \wedge y \notin F$, we infer that $p \wedge x \in J(\mathcal{L})$; so $x \in J(\mathcal{L})$. This shows that $x \wedge F \in J(\mathcal{L})/F = J(\mathcal{L}/F)$ which gives $\mathrm{I}(\mathcal{L}/F) \subseteq J(\mathcal{L}/F)$. Conversely, let $a, b \in \mathcal{L}$ such that $a \vee b \in F$ and $a \notin J(\mathcal{L})$. Then $a \wedge F \notin J(\mathcal{L})/F = J(\mathcal{L}/F)$ and by assumption $a \wedge F \notin \mathrm{I}(\mathcal{L}/F)$. Since $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F = 1 \wedge F$ by [10, Lemma 4.3], we infer that $b \wedge F = 1 \wedge F$. So there exists $c, d \in F$ such that $b \wedge c = 1 \wedge d = d \in F$. This shows that $b \in F$. Thus, F is a J-filter. □

Compare the next theorem with Proposition 2.20 in [16].

Theorem 3.24. If $\mathcal{L}$ is a presimplifiable lattice and F is a weakly J-filter of $\mathcal{L}$, then $\mathcal{L}/F$ is presimplifiable.

Proof. By Proposition 3.22, F is a J-filter. Now the assertion follows from Proposition 3.23. □

Definition 3.25. Let $F \neq \{1\}$ be a filter of $\mathcal{L}$. An element $x \wedge F \in \mathcal{L}/F$ is called *strongly identity join* in $\mathcal{L}/F$ if there exists $1 \wedge F \neq y \wedge F \in \mathcal{L}/F$ such that $(x \wedge F) \vee_Q (y \wedge F) = 1 \wedge F$ and $x \vee y \neq 1$.

Example 3.26. One can easily show that any strongly identity join in $\mathcal{L}/F$ is a identity join. We provide an example of lattices for which The converse is not true. Let $\mathcal{L} = \{0, a, b, c, d, 1\}$ be a lattice with the relations $0 \leq a \leq d \leq 1$, $0 \leq b \leq d \leq 1$, $0 \leq c \leq 1$ and $a \wedge b = a \wedge c = d \wedge c = c \wedge b = 0$. Set $P = \{1, a, d\}$. Since $(b \wedge P) \vee_Q (c \wedge P) = (b \vee c) \wedge P = 1 \wedge P$, we conclude that $b \wedge P$ is an identity join in $\mathcal{L}/P$ which is not a strongly identity join.

Let $F \neq \{1\}$ be a filter of $\mathcal{L}$. The set of all strongly identity joins of a lattice $\mathcal{L}/F$ is denoted by $\mathrm{SI}(\mathcal{L}/F)$. A lattice $\mathcal{L}/F$ is called *S-presimplifiable* if $\mathrm{SI}(\mathcal{L}/F) \subseteq J(\mathcal{L}/F)$. The next theorem gives a more explicit description of weakly J-filters $F \neq \{1\}$ in terms of S-presimplifiable quotient lattices.

Compare the next theorem with Theorem 2.23 in [16].

Theorem 3.27. Let $F \neq \{1\}$ be a filter of $\mathcal{L}$. Then the following statements are equivalent:
(1) F is a weakly J-filter of $\mathcal{L}$;
(2) $F \subseteq J(\mathcal{L})$ and $\mathcal{L}/F$ is S-presimplifiable.

Proof. (1) $\Rightarrow$ (2) Assume F is a weakly J-filter of $\mathcal{L}$ (so $F \subseteq J(\mathcal{L})$ by Proposition 3.4). Let $x \wedge F \in \mathrm{SI}(\mathcal{L}/F)$. Then there exists $1 \wedge F \neq y \wedge F \in \mathcal{L}/F$ (so $y \notin F$) such that $(x \wedge F) \vee_Q (y \wedge F) = (x \vee y) \wedge F = 1 \wedge F$ and $x \vee y \neq 1$ which implies that $(x \vee y) \wedge f = 1 \wedge e = e \in F$ for some $e, f \in F$. This shows that $1 \neq x \vee y \in F$ by Lemma 2.2 (1). Hence, $x \in J(\mathcal{L})$, as F is a weakly J-filter. Therefore, $x \wedge F \in J(\mathcal{L})/F = J(\mathcal{L}/F)$, i.e. (2) holds.

(2) $\Rightarrow$ (1) Let $a, b \in \mathcal{L}$ such that $1 \neq a \vee b \in F$ and $b \notin F$. One can easily show that $a \wedge F$ is a strongly identity join in $\mathcal{L}/F$ and so $a \wedge F \in J(\mathcal{L}/F) = J(\mathcal{L}/F)$ by (2) which gives $a \in J(\mathcal{L})$, i.e. (1) holds. □

Definition 3.28. If S is a nonempty subset of a lattice $\mathcal{L}$ such that $\mathcal{L} \setminus J(\mathcal{L}) \subseteq S$, then S is called a *weakly J-join-subset* of $\mathcal{L}$ if $x \vee y \in S$ or $x \vee y = 1$ for all $x \in \mathcal{L} \setminus J(\mathcal{L})$ and all $y \in S$.

In the following proposition, we describe the relation between weakly J-filters and weakly J-join-subsets of $\mathcal{L}$.

Proposition 3.29. *A filter F is a weakly J-filter of a lattice $\mathcal{L}$ if and only if $\mathcal{L} \setminus F$ is a weakly J-join-subset of $\mathcal{L}$.*

Proof. Let F be a weakly J-filter of $\mathcal{L}$. Since $F \subseteq J(\mathcal{L})$ by Proposition 3.4, we conclude that $\mathcal{L} \setminus J(\mathcal{L}) \subseteq \mathcal{L} \setminus F$. Let $a \in \mathcal{L} \setminus J(\mathcal{L})$ and $b \in \mathcal{L} \setminus F$. If $a \vee b = 1$, then we are done. So we may assume that $a \vee b \neq 1$. Now, we show that $a \vee b \in \mathcal{L} \setminus F$. On the contrary, assume that $a \vee b \in F$. Then F is a weakly J-filter and $x \notin J(\mathcal{L})$ implies that $y \in F$ which is a contradiction. Thus, $x \vee y \in \mathcal{L} \setminus F$ and so $\mathcal{L} \setminus F$ is a weakly J-join-subset of $\mathcal{L}$.

Conversely, let $a, b \in \mathcal{L}$ and $1 \neq a \vee b \in F$ with $a \notin J(\mathcal{L})$. Then we have $b \in F$ since otherwise we would have $a \vee b \in \mathcal{L} \setminus F$ which is impossible. Therefore, F is a weakly J-filter of $\mathcal{L}$. □

Proposition 3.30. *Suppose S is a weakly J-join-subset of a lattice $\mathcal{L}$ with $S \cap \cup_{y \notin J(\mathcal{L})}(1 :_{\mathcal{L}} y) = \emptyset$. If a filter F of $\mathcal{L}$ is maximal with respect to the property $F \cap S = \emptyset$, then F is a weakly J-filter of $\mathcal{L}$.*

Proof. On the contrary, assume that F is not a weakly J-filter of $\mathcal{L}$. Then there are elements $x \notin J(\mathcal{L})$ and $y \notin F$ such that $1 \neq x \vee y \in F$. Since $F \subsetneq (F :_{\mathcal{L}} x)$, we infer that $(F :_{\mathcal{L}} x) \cap S \neq \emptyset$. Consider $s \in (F :_{\mathcal{L}} x) \cap S$. Then $x \vee s \in F$ and S is a weakly J-join-subset gives either $x \vee s \in S$ or $x \vee s = 1$. If $x \vee s \in S$, then $x \vee s \in F \cap S$ which is a contradiction. If $x \vee s = 1$, then $s \in S \cap \cup_{y \notin J(\mathcal{L})}(1 :_{\mathcal{L}} y)$, a contradiction. Thus, F is a weakly J-filter of $\mathcal{L}$. □

Assume that $(\mathcal{L})_1, \leq_1), (\mathcal{L})_2, \leq_2)$ are lattices and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$. We set up a partial order $\leq_c$ on $\mathcal{L}$ as follows: for each $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$, we write $x \leq_c y$ if and only if $x_i \leq_i y_i$ for each $i \in \{1, 2\}$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}, \leq_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2)$ and $x \wedge_c y = (x_1 \wedge y_1, x_2 \wedge y_2)$. In this case, we say that $\mathcal{L}$ is a *decomposable lattice*.

In the next theorem, we characterize weakly J-filters of a decomposable lattice.

Theorem 3.31. Suppose $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice and let $F \neq \{(1,1)\}$ be a filter of $\mathcal{L}$. Then the following statements are equivalent:
(1) F is a weakly J-filter of $\mathcal{L}$;
(2) $F = F_1 \times \mathcal{L}_2$ where F_1 is a J-filter of $\mathcal{L}_1$ or $F = \mathcal{L}_1 \times F_2$ where F_2 is a J-filter of $\mathcal{L}_2$;
(3) F is a J-filter of $\mathcal{L}$.

Proof. (1) $\Rightarrow$ (2) At first, note that if P is a maximal filter of $\mathcal{L}_1$ and Q is a maximal filter of $\mathcal{L}_2$, then $M = P \times \mathcal{L}_2$ and $M' = \mathcal{L}_1 \times Q$ are maximal filters of $\mathcal{L}$ and $(a,0),(0,b) \notin J(\mathcal{L})$ for all $a \in \mathcal{L}_1$ and $b \in \mathcal{L}_2$. Let $F = F_1 \times F_2 \neq \{(1,1)\}$ be a weakly J-filter of $\mathcal{L}$. Suppose $F_1 \neq \mathcal{L}_1$, $F_2 \neq \mathcal{L}$ and consider $(1,1) \neq (x,y) \in F$. Since $(1,1) \neq (x,0) \vee_Q (0,y) = (x,y) \in F$ and $(x,0),(0,y) \notin J(\mathcal{L})$, we conclude that $F = \mathcal{L}$ which is impossible. So we may assume with no loss of generality that $F_1 \neq \mathcal{L}_1$ and $F_2 = \mathcal{L}_2$. Since $F \neq (1,1)$, we infer that F is a J-filter by Theorem 3.7. It remains to show that F_1 is a J-filter of $\mathcal{L}_1$. Let $a,b \in \mathcal{L}_1$ such that $a \vee b \in F_1$ and $a \notin J(\mathcal{L}_1)$. Then $(a,0) \vee_Q (b,0) \in F = F_1 \times \mathcal{L}_2$ and $(a,0) \notin J(\mathcal{L})$ gives $(b,0) \in F$ and so $b \in F_1$, as needed.

(2) $\Rightarrow$ (3) We can assume that $F = F_1 \times \mathcal{L}_2$, where F_1 is a J-filter of $\mathcal{L}_1$. Let $(a,b),(c,d) \in \mathcal{L}$ such that $(a,b) \vee_q (c,d) = (a \vee c, b \vee d) \in F$ and $(a,b) \notin J(\mathcal{L})$. Then $a \notin J(\mathcal{L}_1)$ and $a \vee c \in F_1$ which gives $c \in F_1$. Therefore, $(c,d) \in F$ and so F is a J-filter of $\mathcal{L}$. The implication (3) $\Rightarrow$ (1) is clear. $\square$

Suppose $\mathcal{M}$ is a filter of a lattice $\mathcal{L}$ and let $\mathcal{L}(+)\mathcal{M}$ be the filterlization lattice. Let M be a module over a commutative ring R. Anderson and Smith in [2] determine when $P(+)M$ is a weakly prime ideal in $R(+)M$. Now, we will give a similar result for weakly J-filters.

Lemma 3.32. *Let $\mathcal{L}$ and $\mathcal{M}$ be as above. The following hold:*
(1) If H is a filter of $\mathcal{L}(+)\mathcal{M}$, then $H = G(+)K$, where G is a filter of $\mathcal{L}$ and K is a subfilter of $\mathcal{M}$;
(2) If G is a filter of $\mathcal{L}$ and K is a subfilter of $\mathcal{M}$, then $G(+)K$ is a filter of $\mathcal{L}(+)\mathcal{M}$ if and only if $G \vee \mathcal{M} \subseteq K$.
(3) $J(\mathcal{L}\,(+)\,\mathcal{M}) = J(\mathcal{L})\,(+)\,\mathcal{M}$.

Proof. (1) Set $G = \{x \in \mathcal{L} : (x,m) \in H \text{ for sme } m \in \mathcal{M}\}$ and $K = \{m \in \mathcal{M} : (x,m) \in H \text{ for sme } x \in \mathcal{L}\}$. Let $x,y \in G$ and $z \in \mathcal{L}$. Then $(x,m),(y,m') \in H$ for some $m,m' \in \mathcal{M}$ implies that $(x \wedge y, m \wedge m') = (x,m) \wedge_F (y,m') \in H$ and $(x \vee z, m \vee z) = (x,m) \vee_F (z,1) \in H$, as H is a filter; hence $x \wedge y, x \vee z \in G$. Thus G is a filter of $\mathcal{L}$. Similarly, K is a filter of $\mathcal{M}$. Finally, it is easy to see that $H = G(+)K$.

(2) Suppose $H = G(+)K$ is a filter of $\mathcal{L}(+)\mathcal{M}$ and let $g \vee m \in G \vee \mathcal{M}$ for some $g \in G$ and $m \in \mathcal{M}$. Then $(g,1) \in H$ gives $(g,1) \vee_F (g,m) = (g, g \vee m) \in H$; so $g \vee m \in K$. Thus, $G \vee \mathcal{M} \subseteq K$. Conversely, assume that $G \vee \mathcal{M} \subseteq K$ and let $(a,n),(b,m) \in H$ and $(c,x) \in \mathcal{L}$. Then $(a,n) \wedge_F (b,m) = (a \wedge b, n \wedge m) \in H$ and $(a,n) \vee_F (c,x) = (a \vee c, (a \vee x) \wedge (n \vee c)) \in H$ since $a \vee x \in G \vee \mathcal{M} \subseteq K$ and $n \vee c \in K$, i.e. H is a filter of $\mathcal{L}(+)\mathcal{M}$.

(3) One can easily show that $G(+)K$ is a maximal filter of $\mathcal{L}(+)\mathcal{M}$ if and only if G is a maximal filter of $\mathcal{L}$, i.e. (3) holds. $\square$

Compare the next theorem with Theorem 2.30 in [16].

Theorem 3.33. Let $\mathcal{L}$ and $\mathcal{M}$ be as above. If G is a filter of $\mathcal{L}$ and K is a filter of $\mathcal{M}$, then the following hold:

(1) If $G(+)K$ is a weakly J-filter of $\mathcal{L}(+)\mathcal{M}$, then G is a weakly J-filter of $\mathcal{L}$.

(2) $G(+)\mathcal{M}$ is a weakly J-filter of $\mathcal{L}(+)\mathcal{M}$ if and only if G is a weakly J-filter of $\mathcal{L}$ and for $a,b \in \mathcal{L}$ with $a \vee b = 1$ but $a \notin J(\mathcal{L})$ and $b \notin G$, $a,b \in (1 :_{\mathcal{L}} \mathcal{M})$.

Proof. (1) Clearly, $G \neq \mathcal{L}$. Let $x,y \in \mathcal{L}$ such that $1 \neq x \vee y \in G$ and $x \notin J(\mathcal{L})$. Then $(1,1) \neq (x,1) \vee_F (y,1) = (x \vee y, 1) \in G(+)K$ and $(x,1) \notin J(\mathcal{L})(+)\mathcal{M} = J(\mathcal{L}(+)\mathcal{M})$ gives $(y,1) \in G(+)K$, as $G(+)K$ is a weakly J-filter; hence $y \in G$.

(2) Let $G(+)\mathcal{M}$ be a weakly J-filter of $\mathcal{L}(+)\mathcal{M}$. Then G is a weakly J-filter of $\mathcal{L}$ by (1). Now, for $a,b \in \mathcal{L}$, assume that $a \vee b = 1$ but $a \notin J(\mathcal{L})$ and $b \notin G$. On the contrary, suppose that $a \notin (1 :_{\mathcal{L}} \mathcal{M})$. Then $a \vee m \neq 1$ for some $m \in \mathcal{M}$. It follows that $(1,1) \neq (a,1) \vee_F (b,m) = (1, a \vee m) \in G(+)\mathcal{M}$ but $(a,1) \notin J(\mathcal{L})(+)\mathcal{M}$ and so $(b,m) \in G(+)\mathcal{M}$ which is a contradiction. Therefore, $a \in (1 :_{\mathcal{L}} \mathcal{M})$. Similarly, $b \in (1 :_{\mathcal{L}} \mathcal{M})$. Conversely, let $(a,n),(b,m) \in \mathcal{L}(+)\mathcal{M}$ such that $(1,1) \neq (a,n) \vee_F (b,m) \in G(+)\mathcal{M}$ and $(a,n) \notin J(\mathcal{L}(+)\mathcal{M})$. So $a \vee b \in G$ and $a \notin J(\mathcal{L})$. If $a \vee b \neq 1$, then G is a weakly J-filter gives $b \in G$ and so $(b,m) \in G(+)\mathcal{M}$. So we may assume that $a \vee b = 1$ but $a \notin J(\mathcal{L})$ and $b \notin G$. By the hypothesis, $a,b \in (1 :_{\mathcal{L}} \mathcal{M})$ and then $(a,n) \vee_F (b,m) = (1,1)$ which is impossible. Therefore, $a \vee b \neq 1$ and clearly $G(+)\mathcal{M}$ is a weakly J-filter of $\mathcal{L}(+)\mathcal{M}$, as required. $\square$

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