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Pairs of rings sharing their units

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Abstract. We are working in the category of commutative unital rings and denote by $\mathrm{U}(R)$ the group of units of a nonzero ring R. An extension of rings $R \subseteq S$, satisfying $\mathrm{U}(R) = R \cap \mathrm{U}(S)$ is usually called local. This paper is devoted to the study of ring extensions such that $\mathrm{U}(R) = \mathrm{U}(S)$, that we call strongly local. P. M. Cohn in a paper, entitled Rings with zero divisors, introduced some strongly local extensions. We generalized under the name Cohn's rings his definition and give a comprehensive study of these extensions. As a consequence, we give a constructive proof of his main result. Now Lequain and Doering studied strongly local extensions, where S is semilocal, so that $S/\mathrm{J}(S)$, where $\mathrm{J}(S)$ is the Jacobson radical of S, is Von Neumann regular. These rings are usually called J-regular. We establish many results on J-regular rings in order to get substantial results on strongly local extensions when S is J-regular. The Picard group of a J-regular ring is trivial, allowing to evaluate the group $\mathrm{U}(S)/\mathrm{U}(R)$ when R is J-regular. We then are able to give a complete characterization of the Doering-Lequain context. A Section is devoted to examples. In particular, when R is a field, the strongly local and weakly strongly inert properties are equivalent.

Key Words: Group of units, local extension, strongly local extension, J-regular ring, integral extension, FCP extension.
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1 Introduction and Notation

This paper deals with commutative unital rings and their morphisms. Any ring R is supposed nonzero. We denote by $\mathrm{U}(R)$ the set of units of a ring R. We will call *strongly local* an extension of rings $R \subseteq S$ such that $\mathrm{U}(R) = \mathrm{U}(S)$, also termed as an SL extension. The reason why is that a ring extension $R \subseteq S$ is classically called local if $\mathrm{U}(R) = \mathrm{U}(S) \cap R$. Naturally these notions do not coincide and have a ring morphism version. If $R \to S$ is a ring morphism and Q is a prime ideal of S lying over P in R, then $R_P \to S_Q$ is usually a called a local morphism of $R \to S$.

Our work takes its origin in the reading of two papers. One of them was written by P.M. Cohn [6]: for any ring R, he exhibits an SL extension $R \subseteq R'$, such that any non unit of R' is a zerodivisor. To prove his main result, Cohn introduces some special rings in a lemma. We have considered rings of the same vein. The idea is as follows: if I is an ideal of a ring R, we define the ring $R/\!\!/I := R[X]/XI[X]$ (where X is an indeterminate over R). This notation may seem weird, but it explains that the ring R/I is shifted. When I is a semiprime ideal, we have an SL extension $R \subseteq R/\!\!/I$ with very nice properties. To have a better understanding, consider a field R and $I = 0$, we recover $R \subseteq R[X]$. We reprove Cohn's result and give a constructive proof, not using a transfinite induction. But Cohn's ring is not necessarily the same as ours, by lack of unicity. All these considerations are developed in Section 6. In Section 7 we consider a ring morphism $R \to R\{X\}$ used by E. Houston in the context of Noetherian rings and their dimensions. We generalize his results and give a link with the rings $R/\!\!/M$, where M is a maximal ideal of R.

The other one was written by Doering and Lequain [9]. This paper deals with pair of semilocal rings sharing their group of units. A first observation is that for a semilocal ring R with Jacosbon radical J, the ring R/J is Von Neumann regular-absolutely flat, in which case the ring is called in the literature J-regular. Note that units are closely linked to Jacobson radicals. In order to generalize Lequain-Doering's results in a substantial way, we were lead to study J-regular rings and their behavior with respect to ring morphisms, a subject not treated in the literature. Actually for an extension $R \subseteq S$, there is an exact sequence of Abelian groups $1 \to \mathrm{U}(R) \to \mathrm{U}(S) \to \mathcal{I}(R,S) \to \mathrm{Pic}(R)$, where $\mathcal{I}(R,S)$ is the group of R-submodules of S that are invertible. Now if R is J-regular, its Picard group is zero, so that $\mathcal{I}(R,S)$ measures the defect of strongly localness of the extension. A first crucial result is that for an SL extension $R \subseteq S$, then S is J-regular if and only if R is J-regular and $R \subseteq S$ is integral seminormal. If these conditions hold, then $\mathrm{J}(R) = \mathrm{J}(S)$. This material is developed in Section 5. The paper culminates in Section 8 with a substantial result in the Doering-Lequain style: an extension $R \subseteq S$, where S is a semilocal ring, is SL if and only if $R \subseteq S$ is a seminormal integral FIP extension, whose residual extensions are isomorphisms and such that $R/M \cong \mathbb{Z}/2\mathbb{Z}$ for each $M \in \mathrm{MSupp}(S/R)$. Now Section 2 examines the behavior of local and SL extensions and many examples are provided. In Section 3 we give a list of extensions that are local. Section 4 explores the properties of SL extensions. We give in Section 9 a series of examples of SL extensions, for example strongly inert extensions. These examples show that it seems impossible to find a general criterion for the SL property, except in the semilocal case.

As an example, we build at the end of the paper a strongly local ring morphism $f : \mathbb{F}_2[X]/(X^3-1) \to \mathbb{F}_2[X]/(X^4-X)$.

If $R \subseteq S$ is a (ring) extension, we denote by $[R,S]$ the set of all R-subalgebras of S.

We will mainly consider ring morphisms that are ring extensions. A property (P) of ring morphisms $f : R \to S$ is called universal if for any base change $R \to R'$, the ring morphism $R' \to R' \otimes_R S$ verifies (P). As usual, $\mathrm{Spec}(R)$ and $\mathrm{Max}(R)$ are the set of prime and maximal ideals of a ring R. We denote by $\kappa_R(P)$ the residual field R_P/PR_P at a prime ideal P of R. If $R \subseteq S$ is a ring extension and $Q \in \mathrm{Spec}(S)$, there exists a residual field extension $\kappa_R(Q \cap R) \to \kappa_S(Q)$.

A *(semi-)local* ring is a ring with (finitely many maximal ideals) one maximal ideal. For an extension $R \subseteq S$ and an ideal I of R, we write $\mathrm{V}_S(I) := \{P \in \mathrm{Spec}(S) \mid I \subseteq P\}$. The support of an R-module E is $\mathrm{Supp}_R(E) := \{P \in \mathrm{Spec}(R) \mid E_P \neq 0\}$, and $\mathrm{MSupp}_R(E) := \mathrm{Supp}_R(E) \cap \mathrm{Max}(R)$. When $R \subseteq S$ is an extension, we will set $\mathrm{Supp}(T/R) := \mathrm{Supp}_R(T/R)$ and $\mathrm{Supp}(S/T) := \mathrm{Supp}_R(S/T)$ for each $T \in [R,S]$, unless otherwise specified.

For a ring R, we denote by $\mathrm{Z}(R)$ the set of all zerodivisors of R, by $\mathrm{Nil}(R)$ the set of nilpotent elements of R and by $\mathrm{J}(R)$ its Jacobson radical. The Picard group of a ring R is denoted by $\mathrm{Pic}(R)$.

Now $(R : S)$ is the conductor of $R \subseteq S$. The integral closure of R in S is denoted by $\overline{R}^S$ (or by $\overline{R}$ if no confusion can occur).

A ring extension $R \subseteq S$ is called an *i-extension* if the natural map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is injective.

An extension $R \subseteq S$ is said to have FIP (for the "finitely many intermediate algebras property") or is an FIP extension if $[R,S]$ is finite. We also say that the extension $R \subseteq S$ has FCP (or is an FCP extension) if each chain in $[R,S]$ is finite, or equivalently, its lattice is Artinian and Noetherian. An FCP extension is finitely generated, and (module) finite if integral.

In the case of an FCP extension $R \subseteq S$ where S is semilocal, we will show in Theorem 8.10 that there is a greatest $T \in [R,S]$ such that $R \subseteq T$ is SL, that is $R \subseteq T$ is a unique MSL-subextension.

Finally, $|X|$ is the cardinality of a set X, $\subset$ denotes proper inclusion and for a positive integer n, we set $\mathbb{N}_n := \{1,\dots,n\}$. If R and S are two isomorphic rings, we will write $R \cong S$. If M and N are R-modules, we write $M \cong_R N$ if M and N are isomorphic as R-modules.

2 First properties of (strongly) local extensions

Generalizing the definition of a local ring morphism between local rings, a ring morphism $f = R \to S$ is called *local* if $f^{-1}(\mathrm{U}(S)) = \mathrm{U}(R)$. We recover the case of a local extension of local rings $f : (R, M) \to (S, N)$ where $f^{-1}(N) = M$. We will mainly be concerned by ring extensions $R \subseteq S$, and in this case the extension is local (reflects units) if $\mathrm{U}(R) = \mathrm{U}(S) \cap R$.

We call a ring extension $R \subseteq S$ *strongly local (SL)* if $\mathrm{U}(R) = \mathrm{U}(S)$. A strongly local extension is obviously local and a ring morphism $f : R \to S$ is called *strongly local (SL)* if $f(\mathrm{U}(R)) = \mathrm{U}(S)$.

Note that if f is a surjective local morphism then f is SL. Indeed we always have $f(\mathrm{U}(R)) \subseteq \mathrm{U}(S)$. Assume that f is surjective and local, and let $y \in \mathrm{U}(S)$. There exists some $a \in R$ such that $y = f(a) \in \mathrm{U}(S)$. It follows that $a \in f^{-1}(\mathrm{U}(S)) = \mathrm{U}(R)$, so that $y \in f(\mathrm{U}(R))$, giving $\mathrm{U}(S) \subseteq f(\mathrm{U}(R))$, and then $f(\mathrm{U}(R)) = \mathrm{U}(S)$, that is f is SL.

A first example of local extension is given by an R-module M and its Nagata extension $f : R \to R \oplus M$, where $f(x) = (x, 0)$. A unit (a, m) of $R \oplus M$ is such that a is a unit of R and (a, m) has an inverse of the form (a', n), where $a' = a^{-1}$ and $n = -a'^2 m$. But this extension is not SL.

The extension $\mathbb{Z} \subseteq \mathbb{Z}[2i]$ is SL.

If R is a ring then the extension $R[X^2] \subseteq R[X]$ is local but not SL in general. For example, if $a \in R$ is nilpotent, then $1 + aX \in \mathrm{U}(R[X])$ but $1 + aX \notin \mathrm{U}(R[X^2])$.

Proposition 2.1. *An extension $R \subseteq S$ such that R and S have the same prime ideals is local and is trivial if it is SL.*

Proof. We know that R and S are local rings with the same maximal ideal [4, Proposition 3.3]. Let M be this common maximal ideal. As usual, we have $\mathrm{U}(R) \subseteq \mathrm{U}(S) \cap R$. Let $x \in \mathrm{U}(S) \cap R$. Then, $x \notin P$, for any $P \in \mathrm{Spec}(S) = \mathrm{Spec}(R)$. Since $x \in R$, it follows that $x \in \mathrm{U}(R)$, so that $\mathrm{U}(R) = \mathrm{U}(S) \cap R$ and $R \subseteq S$ is local. If $R \subseteq S$ is SL, that is $\mathrm{U}(R) = \mathrm{U}(S)$, we have $R = M \cup \mathrm{U}(R) = M \cup \mathrm{U}(S) = S$. □

See [4] for examples and also PVD and $D + M$ construction.

Proposition 2.2. *Let $f : R \to S$ and $g : S \to T$ be two ring morphisms.*

1. *If f and g are (strongly) local, so is $g \circ f$.*
2. *If $g \circ f$ is local, so is f. In case f is surjective, then g is local.*
3. *If $g \circ f$ is SL, so is g. In case g is injective, then f is SL.*

Proof. (1) Assume that f and g are local. Then, $f^{-1}(\mathrm{U}(S)) = \mathrm{U}(R)$ and $g^{-1}(\mathrm{U}(T)) = \mathrm{U}(S)$, so that $(g \circ f)^{-1}(\mathrm{U}(T)) = f^{-1}[g^{-1}(\mathrm{U}(T))] = f^{-1}(\mathrm{U}(S)) = \mathrm{U}(R)$, so that $g \circ f$ is local.

Assume that f and g are SL. Then, $f(\mathrm{U}(R)) = \mathrm{U}(S)$ and $g(\mathrm{U}(S)) = \mathrm{U}(T)$, so that $(g \circ f)(\mathrm{U}(R)) = g[f(\mathrm{U}(R))] = g(\mathrm{U}(S)) = \mathrm{U}(T)$, giving that $g \circ f$ is SL.

(2) Assume that $g \circ f$ is local. Then, $(g \circ f)^{-1}(\mathrm{U}(T)) = \mathrm{U}(R)$, so that $f^{-1}[g^{-1}(\mathrm{U}(T))] = \mathrm{U}(R)$. Obviously, $\mathrm{U}(R) \subseteq f^{-1}(\mathrm{U}(S))$. Let $x \in f^{-1}(\mathrm{U}(S))$. It follows that $f(x) \in \mathrm{U}(S)$, whence $(g \circ f)(x) = g[f(x)] \in g(\mathrm{U}(S)) \subseteq \mathrm{U}(T)$. To end, $x \in (g \circ f)^{-1}(\mathrm{U}(T)) = \mathrm{U}(R)$ and $f^{-1}(\mathrm{U}(S) \subseteq \mathrm{U}(R)$, giving $f^{-1}(\mathrm{U}(S)) = \mathrm{U}(R)$ and f is local.

Assume that, moreover, f is surjective. Obviously, $\mathrm{U}(S) \subseteq g^{-1}(\mathrm{U}(T))$. Let $y \in g^{-1}(\mathrm{U}(T))$, so that $g(y) \in \mathrm{U}(T)$ $(*)$. But, $y \in S$ and f surjective imply that there exists $x \in R$ such that $y = f(x)$. By $(*)$, we get $(g \circ f)(x) = g[f(x)] = g(y) \in \mathrm{U}(T)$, from which it follows that $x \in (g \circ f)^{-1}(\mathrm{U}(T)) = \mathrm{U}(R)$ and $y = f(x) \in f(\mathrm{U}(R)) \subseteq \mathrm{U}(S)$. To end, $\mathrm{U}(S) = g^{-1}(\mathrm{U}(T))$ and g is local.

(3) Assume that $g \circ f$ is SL. Then, $(g \circ f)(\mathrm{U}(R)) = \mathrm{U}(T)$. Obviously, $g[\mathrm{U}(S)] \subseteq \mathrm{U}(T)$. Let $y \in \mathrm{U}(T)$. There exists $x \in \mathrm{U}(R)$ such that $y = (g \circ f)(x) = g[f(x)] \in g[f(\mathrm{U}(R))] \subseteq g[\mathrm{U}(S)]$, which gives $\mathrm{U}(T) \subseteq g[\mathrm{U}(S)]$ and $g[\mathrm{U}(S)] = \mathrm{U}(T)$. Then, g is SL.

Assume that, moreover, g is injective. Obviously, $f[\mathrm{U}(R)] \subseteq \mathrm{U}(S)$. Let $z \in \mathrm{U}(S)$. Then, $g(z) \in \mathrm{U}(T) = (g \circ f)(\mathrm{U}(R))$, so that there exists $x \in \mathrm{U}(R)$ such that $g(z) = (g \circ f)(x) = g[f(x)]$. Since g is injective, it follows that $z = f(x) \in f[\mathrm{U}(R)]$ and $\mathrm{U}(S) \subseteq f[\mathrm{U}(R)]$. To end, $\mathrm{U}(S) = f(\mathrm{U}(R))$ and f is SL. □

Remark 2.3. *Let $f : R \to S$ be a ring morphism. Since the ring morphism $R/\ker(f) \to S$ associated to f is injective and the canonical ring morphism $R \to R/\ker(f)$ is surjective, we may consider the extension $R/\ker(f) \subseteq S$. Then, Proposition 2.2 shows that f is (strongly) local if and only if $R \to R/\ker(f)$ and $R/\ker(f) \to S$ are (strongly) local.*

Corollary 2.4. *Let $R \subseteq S \subseteq T$ be a tower of extensions.*

1. *$R \subseteq T$ is SL if and only if $R \subseteq S$ and $S \subseteq T$ are SL.*
2. *If $R \subseteq S$ and $S \subseteq T$ are local, then $R \subseteq T$ is local.*
3. *If $R \subseteq T$ is local, then $R \subseteq S$ is local.*

Proof. Obvious by Proposition 2.2. □

Proposition 2.5. *If $\{R_i \subseteq S_i\}_{i\in I}$ is a family of extensions, then $\Pi_{i\in I} R_i \subseteq \Pi_{i\in I} S_i$ is a (strongly) local extension if and only if all the elements of the family are (strongly) local.*

Proposition 2.6. *Consider a pullback square in the category of commutative rings:*

$$\begin{array}{ccc} R & \to & S \\ \downarrow & & \downarrow \\ R' & \to & S' \end{array}$$

whose horizontal maps are extensions. Then if $R' \subseteq S'$ is (strongly) local, so is $R \subseteq S$.

Proof. Assume first that $R' \subseteq S'$ is local, so that $\mathrm{U}(R') = \mathrm{U}(S') \cap R'$. As $\mathrm{U}(R) \subseteq \mathrm{U}(S) \cap R$, it is enough to show that $\mathrm{U}(S) \cap R \subseteq \mathrm{U}(R)$. Let $f : S \to S'$ and $a \in \mathrm{U}(S) \cap R$. Then, $b := f(a) \in \mathrm{U}(S') \cap R' = \mathrm{U}(R')$ is a unit in R'. Let $g : R \to R'$ and consider $x = (a, b) = (a, f(a)) \in R$ with $b = g(x) = f(a)$. Set $a' := a^{-1} \in \mathrm{U}(S)$ and $b' := f(a')$. It follows that $b' \in \mathrm{U}(S')$ satisfies $bb' = 1$ in S' and is the (unique) inverse of b, so that $b' \in R'$. Set $x' := (a', b')$, with $b' = g(x') = f(a')$. Then, $xx' = (aa', bb') = 1$, so that $x \in \mathrm{U}(R)$. It follows that for any $a \in \mathrm{U}(S) \cap R$, there exists a unique $x = (a, f(a)) \in \mathrm{U}(R)$, from which we can infer that $\mathrm{U}(S) \cap R \subseteq \mathrm{U}(R)$ and $R \subseteq S$ is local.

Assume now that $R' \subseteq S'$ is SL, so that $\mathrm{U}(R') = \mathrm{U}(S')$. The proof is similar, taking $a \in \mathrm{U}(S)$ instead of $a \in \mathrm{U}(S) \cap R$ and taking in account that $\mathrm{U}(S') = \mathrm{U}(R')$. Then $R \subseteq S$ is SL. □

Corollary 2.7. *Let $R \subseteq S$ be an extension sharing an ideal I such that $R/I \subseteq S/I$ is (strongly) local. Then $R \subseteq S$ is (strongly) local.*

Proof. Obvious by Proposition 2.6. □

Proposition 2.8. *Let $\{R \subseteq S_i\}_{i\in I}$ be an upward directed family of (strongly) local extensions. Then so is $R \subseteq \cup[S_i \mid i \in I]$.*

Proof. Set $\mathcal{F} := \{S_i\}_{i\in I}$ and $T := \cup[S_i \mid i \in I]$. Let $x \in \mathrm{U}(T)$. Since $\mathcal{F}$ is an upward directed family of extensions of R, there exists $y \in \mathrm{U}(T)$ such that $xy = 1$ $(*)$ and there exists some $i \in I$ such that $x, y \in S_i$, with $xy = 1$ in S_i by $(*)$. This shows that $x \in \mathrm{U}(S_i)$ $(**)$.

If $R \subseteq S_i$ is local for each S_i, let $x \in \mathrm{U}(T) \cap R$. Then, $(**)$ shows that $x \in \mathrm{U}(S_i) \cap R = \mathrm{U}(R)$ giving $\mathrm{U}(T) \cap R = \mathrm{U}(R)$ and $R \subseteq T$ is local.

If $R \subseteq S_i$ is SL for each S_i, let $x \in \mathrm{U}(T)$. Then, $(**)$ shows that $x \in \mathrm{U}(S_i) = \mathrm{U}(R)$ giving $\mathrm{U}(T) = \mathrm{U}(R)$ and $R \subseteq T$ is SL. □

Proposition 2.9. *Let $R \subseteq S$ be a ring extension such that $R_M \subseteq S_M$ is (strongly) local for any $M \in \mathrm{MSupp}(S/R)$, then so is $R \subseteq S$.*

Proof. Assume first that $R_M \subseteq S_M$ is local for any $M \in \mathrm{MSupp}(S/R)$. Since we always have $\mathrm{U}(R) \subseteq \mathrm{U}(S) \cap R$, it is enough to prove that any $x \in \mathrm{U}(S) \cap R$ is in $\mathrm{U}(R)$. So, let $x \in \mathrm{U}(S) \cap R$. There exists $y \in \mathrm{U}(S)$ such that $xy = 1$ $(*)$ in S, so that $(x/1)(y/1) = 1$ $(**)$ in S_M for any $M \in \mathrm{MSupp}(S/R)$. Then, $x/1 \in \mathrm{U}(S_M) \cap R_M = \mathrm{U}(R_M)$ for any $M \in \mathrm{MSupp}(S/R)$. It follows that for any $M \in \mathrm{MSupp}(S/R)$, there exists some $y_M/s_M \in \mathrm{U}(R_M)$ such that $(x/1)(y_M/s_M) = 1$ in $R_M \subseteq S_M$. This implies that by $(**)$ we get $y_M/s_M = y/1$ in S_M by the uniqueness of the inverse, so that $y/1 \in R_M$ for any $M \in \mathrm{MSupp}(S/R)$. Moreover, let $M \notin \mathrm{MSupp}(S/R)$. Then, $R_M = S_M$, so that $y/1 \in R_M$ for any $M \in \mathrm{Max}(R)$ and $(*)$ shows that $x \in \mathrm{U}(R)$.

Assume that $R_M \subseteq S_M$ is SL for any $M \in \mathrm{MSupp}(S/R)$. Since we always have $\mathrm{U}(R) \subseteq \mathrm{U}(S)$, it is enough to prove that any $x \in \mathrm{U}(S)$ is in $\mathrm{U}(R)$. So, let $x \in \mathrm{U}(S)$. There exists $y \in \mathrm{U}(S)$ such that $xy = 1$ $(*)$ in S which entails that $(x/1)(y/1) = 1$ $(**)$ in S_M for any $M \in \mathrm{MSupp}(S/R)$. For the rest of the proof, it is enough to copy the same part of the proof of the local case. □

Proposition 2.10. *Let $R \subseteq S$ be a ring extension and Σ a saturated multiplicative closed subset of R which is also a saturated multiplicative closed subset of S. If $R_\Sigma \subseteq S_\Sigma$ is (strongly) local, then so is $R \subseteq S$.*

Proof. Assume first that $R_\Sigma \subseteq S_\Sigma$ is local. Since we always have $\mathrm{U}(R) \subseteq \mathrm{U}(S) \cap R$, it is enough to prove that any $x \in \mathrm{U}(S) \cap R$ is in $\mathrm{U}(R)$. So, let $x \in \mathrm{U}(S) \cap R$. There exists $y \in \mathrm{U}(S)$ such that $xy = 1$ $(*)$ in S which implies that $(x/1)(y/1) = 1$ $(**)$ in S_Σ, which is equivalent to $u = uxy$ for some $u \in \Sigma$. In particular, $y \in \Sigma \subseteq R$ because Σ is closed in S. Using $(*)$, we get that $x \in \mathrm{U}(R)$ and $R \subseteq S$ is local.

Assume now that $R_\Sigma \subseteq S_\Sigma$ is SL. Since we always have $\mathrm{U}(R) \subseteq \mathrm{U}(S)$, it is enough to prove that any $x \in \mathrm{U}(S)$ is in $\mathrm{U}(R)$. So, let $x \in \mathrm{U}(S)$. There exists $y \in \mathrm{U}(S)$ such that $xy = 1$ $(*)$ in S, whence $(x/1)(y/1) = 1$ $(**)$ in S_Σ. For the rest of the proof, it is enough to copy the same part of the proof of the local case. □

We consider the Nagata idealization $R(+)M$ of an R-module M.

Proposition 2.11. *(1) Let M be an R-module and N an R-submodule of M. Then, $R \subseteq R(+)M$ and $R(+)N \subseteq R(+)M$ are local extensions. But $R \subseteq R(+)M$ is SL if and only if $M = 0$ and $R(+)N \subseteq R(+)M$ is SL if and only if $M = N$.*

(2) If $R \subseteq S$ is a ring extension and M an S-module, then M is also an R-module and $R \subseteq S$ is SL if and only if $R(+)M \subseteq S(+)M$ is SL.

Proof. (1) Let $(x,m) \in R(+)M$. Then, $(x,m) \in \mathrm{U}(R(+)M)$ if and only if $x \in \mathrm{U}(R)$. Under this condition, we have $(x,m)^{-1} = (x^{-1}, -x^{-2}m)$. It follows that $\mathrm{U}(R(+)M) \cap R = \mathrm{U}(R)$, so that $R \subseteq R(+)M$ is local but $R \subseteq R(+)M$ is SL if and only if $M = 0$.

If N a submodule of M we get that $\mathrm{U}(R(+)M) \cap (R(+)N) = \mathrm{U}(R(+)N)$, whence $R(+)N \subseteq R(+)M$ is local but $R(+)N \subseteq R(+)M$ is SL if and only if $M = N$.

(2) If M an S-module, then obviously, M is also an R-module. Let $(x,m) \in S(+)M$. By the proof of (1), $(x,m) \in \mathrm{U}(S(+)M)$ if and only if $x \in \mathrm{U}(S)$. A similar equivalence holds for $\mathrm{U}(R(+)M)$ and $\mathrm{U}(R)$. Then the equivalence of (2) is obvious. □

We recall in the next section some results of local extensions we get in [36] and add some new results. We will look more precisely at SL extensions in the other sections.

3 Properties of local extensions

An extension $R \subseteq S$ is called *survival* if for each ideal I of R such that $I \neq R$, then $IS \neq S$ (equivalently $PS \neq S$ for each $P \in \mathrm{Spec}(R)$).

Proposition 3.1. *Lying-over or survival extensions are local.*

In case the ideals of a ring R are linearly ordered, an extension $R \subseteq S$ is survival if and only if it is local.

Proof. [36, Definition before Proposition 8.8] gives the result for a survival extension.

Assume that $R \subseteq S$ has lying-over. Since we always have $\mathrm{U}(R) \subseteq \mathrm{U}(S) \cap R$, it is enough to prove that any $x \in \mathrm{U}(S) \cap R$ is in $\mathrm{U}(R)$. So, let $x \in \mathrm{U}(S) \cap R$ be such that $x \notin \mathrm{U}(R)$. There exists some $P \in \mathrm{Spec}(R)$ such that $x \in P$ and there exists some $Q \in \mathrm{Spec}(S)$ lying over P. Then $x \in P \subseteq PS \subseteq Q$, a contradiction with $x \in \mathrm{U}(S)$.

Assume that the ideals of a ring R are linearly ordered and that the extension $R \subseteq S$ is local. Let I be an ideal of R such that $I \neq R$ and $IS = S$. There exists some $x_1, \ldots, x_n \in I$ and $s_1, \ldots, s_n \in S$ such that $\sum_{i=1}^n s_i x_i = 1$ $(*)$. But the Rx_i are linearly ordered. Let x_k be such that $Rx_i \subseteq Rx_k$ for each $i \in \mathbb{N}_n$. Then $(*)$ implies $sx_k = 1$ for some $s \in S$, so that $x_k \in \mathrm{U}(S) \cap R = \mathrm{U}(R)$, a contradiction. Then, $R \subseteq S$ is survival. □

Proposition 3.2. *[36, Proposition 8.8] An extension $R \subseteq S$ is survival if and only if $R(X) \subseteq S(X)$ is local.*

Recall that an extension $R \subseteq S$ is called *Prüfer* if $R \subseteq T$ is a flat epimorphism for each $T \in [R, S]$ (or equivalently, if $R \subseteq S$ is a normal pair) [18, Theorem 5.2, page 47].

In [31], we defined an extension $R \subseteq S$ to be *quasi-Prüfer* if it can be factored $R \subseteq R' \subseteq S$, where $R \subseteq R'$ is integral and $R' \subseteq S$ is Prüfer. An FCP extension is quasi-Prüfer [31, Corollary 3.4].

Definition 3.3. *A ring extension $R \subseteq S$ has a greatest flat epimorphic subextension $R \subseteq \widehat{R}$ that we call the Morita hull of R in S [22, Corollary 3.4]. We say that the extension is Morita-closed if $\widehat{R} = R$.*

In fact, $\widehat{R}$ coincide with the weakly surjective hull $\mathrm{M}(R, S)$ of [18], since weakly surjective morphisms $f : R \to S$ are characterized by $R/\ker f \to S$ is a flat epimorphism [23, Proposition p.3]. Our terminology is justified by the fact that the Morita construction is earlier. The Morita hull can be computed by using a (transfinite) induction [22] as follows. Let S' be the set of all $s \in S$, such that there is some ideal I of R, such that $IS = S$ and $Is \subseteq R$, or equivalently, $S' = \{s \in S \mid S = (R :_R s)S\}$. Then $R \subseteq S'$ is a subextension of $R \subseteq S$. We set $S_1 := S'$ and $S_{i+1} := (S_i)' \subseteq S_i$, which defines the transfinite induction.

We introduce the condition $(\star)$ on a ring extension $R \subseteq S$: an element $s \in S$ belongs to R if (and only if) $S = (R :_R s)S$. The condition $(\star)$ is clearly equivalent to the Morita-closedness of the extension.

An integral extension is Morita-closed because an injective flat epimorphism is trivial as soon as it has LO [19, Lemme 1.2, p.109].

An extension $R \subseteq S$ is Morita-closed if R is zero-dimensional because a prime ideal of R can be lifted up to S, since such a prime ideal is minimal and then it is enough to again apply [19, Lemme 1.2, p.109].

Proposition 3.4. *A Morita-closed extension $R \subseteq S$ is local. Therefore the Morita hull $\widehat{R}$ of an extension $R \subseteq S$ is such that $\widehat{R} \subseteq S$ is local.*

Proof. Use the multiplicatively closed subset $\Sigma_S := \{r \in R \mid r \in \mathrm{U}(S)\}$ and the factorization $R \subseteq R_{\Sigma_S} \subseteq S$. □

The condition $(\star)$ defined after Definition 3.3 is stronger than the local property. A flat epimorphic extension does not verify the condition $(\star)$ because of [39, Exercise 8, p.242]. Actually, a flat epimorphic extension does not need to be local: it is enough to consider a localization $R \to R_P$, where P is a prime ideal of an integral domain R.

We use the results of Olivier about pure extensions [24]. Recall that an injective ring morphism $f : R \to S$ is called *pure* if $R' \to R' \otimes_R S$ is injective for each ring morphism $R \to R'$, whence purity is an universal property. A faithfully flat morphism is pure.

A pure ring morphism is a strict monomorphism of the category of commutative rings [24, Corollaire 5.2, page 21]. This last condition can be characterized in the category of commutative unital rings as follows. If $f : R \to S$ is a ring morphism, the dominion of f is $\mathrm{D}(f) := \{s \in S \mid s \otimes 1 = 1 \otimes s \text{ in } S \otimes_R S\}$.

A ring extension $R \subseteq S$ is a *strict* monomorphism if and only if its dominion is R. Now a ring morphism is an epimorphism if and only if its dominion is S [19, Lemme 1.0, page 108]. It follows that a ring extension is strict and an epimorphism if and only if it is trivial.

Denote by D the dominion of an extension $R \subseteq S$ then $D \subseteq S$ is strict, because $S \otimes_D S = S \otimes_R S$.

We will consider minimal (ring) extensions, a concept that was introduced by Ferrand-Olivier [10]. Recall that an extension $R \subset S$ is called *minimal* if $[R,S] = \{R,S\}$. A minimal extension is either a flat epimorphism or a strict monomorphism, in which case it is finite. A minimal integral extension is strict [10, Théorème 2.2(ii)].

Lemma 3.5. *A strict monomorphism $f : R \to S$ is local (e.g. either pure or minimal integral).*

Proof. If $a \in R$ is a unit in S, there is a factorization $R \to R_a \to S$. From the natural map $R_a \otimes_R R_a \to S \otimes_R S$ and $\mathrm{D}(R \to R_a) = R_a$, because $R \to R_a$ is an epimorphism, we get that $R_a \subseteq \mathrm{D}(f) = R$. □

We define the class $\mathcal{TP}$ of rings R such that $\mathrm{Pic}(R) = 0$. It contains semi-local rings and Nagata rings.

Proposition 3.6. *A Prüfer local extension $R \subseteq S$ over a $\mathcal{TP}$ ring is trivial.*

Proof. Assume that $R \neq S$. Since $R \subset S$ is Prüfer, it is a flat epimorphism, so that there exists $M \in \mathrm{Max}(R)$ such that $MS = S$ and $1 = \sum_{i=1}^{n} m_i s_i$, for some positive integer n, $m_i \in M$ and $s_i \in S$ for each $i \in \mathbb{N}_n$. Set $I := \sum_{i=1}^{n} Rm_i$. Then $IS = S$, so that I is an S-regular R-submodule of S finitely generated. According to [18, Theorem 1.13, p.91], I is S-invertible, and then a projective R-module of rank 1 by [18, Lemma 4.1, p.109]. It follows that I is a free R-module which implies that I is a principal ideal of R.

Set $I := Rx$, with $x \in R$. But $IS = S$ gives $xS = S$, and x is a unit in S. But $x \in R$ implies $x \in \mathrm{U}(S) \cap R = \mathrm{U}(R)$, a contradiction with $x \in I \subseteq M$. Then, $R = S$. □

This Proposition generalizes [36, Proposition 8.9] which holds for an extension over an arithmetical ring.

4 Generalities about SL extensions

We first give examples of SL extensions.

Example 4.1. *(1a) If $R \subseteq S$ is an extension we can consider $L(S) := R[\mathrm{U}(S)]$. Then $L(S) \subseteq S$ is SL and $L(S)$ is the smallest element T of $[R,S]$ such that $T \subseteq S$ is SL. Actually $L(S)$ is the intersection of all $T \in [R,S]$ such that $T \subseteq S$ is SL.*

(1b) There exist also subextensions $R \subseteq U$ of $[R,S]$, maximal with respect to the property SL. We will call them MSL-subextensions. The proof uses Zorn's Lemma.

If $R \subseteq S$ is a chained extension, then $U := \cup[V \in [R,S] \mid \mathrm{U}(R) = \mathrm{U}(V)]$ is the MSL-subextension.

In Theorem 8.10, we prove that when $R \subseteq S$ is an FCP extension and S is semilocal, there exists a unique MSL-subextension.

(2) Let $R \subseteq S$ be an extension where S is a Boolean ring, then the extension is SL. This is obvious because the only unit of S and R is 1 since R is also a Boolean ring.

(3) Let R be a reduced ring. Then $R \subset R[X]$ is SL because an invertible polynomial of $R[X]$ is of the form $a + Xf(X)$ where a is invertible in R and the coefficients of $f(X)$ are nilpotent, so that $f(X) = 0$.

On the other hand, $R \subset R[[X]]$ is never SL because any power series with a unit of R as first coefficient is a unit in $R[[X]]$.

(4) Let Σ be a multiplicative closed subset of a ring R such that $R \subseteq R_\Sigma$ is SL. Then $\Sigma = \mathrm{U}(R)$ and $R = R_\Sigma$. Of course, $\mathrm{U}(R) \subseteq \Sigma$. Assume $\mathrm{U}(R) \neq \Sigma$ and let $x \in \Sigma \setminus \mathrm{U}(R)$. Then, $x \in \mathrm{U}(R_\Sigma) = \mathrm{U}(R)$, a contradiction, so that $\Sigma = \mathrm{U}(R)$ and $R = R_\Sigma$.

(5) The ring morphism $j : R \to R/\mathrm{J}(R)$ is SL. Indeed if $\bar{x}$ is the class of an element $x \in R$ that is a unit in $R/\mathrm{J}(R)$, there is some $y \in R$, such that $xy - 1 \in \mathrm{J}(R)$. Then $1 - (1 - xy)$ is a unit in R; so that $x \in \mathrm{U}(R)$.

(6) Let $R \subseteq S$ be an extension such that $1 \neq -1$ and $\mathrm{U}(S)$ is a simple group. Since $\mathrm{U}(R)$ is a subgroup of $\mathrm{U}(S)$, the only possibilities are either $\mathrm{U}(R) = \{1\}$ $(*)$ or $\mathrm{U}(R) = \mathrm{U}(S)$ $(**)$. But $1 \neq -1$ and $-1 \in \mathrm{U}(R)$ show that only case $(**)$ can occur. Then $R \subseteq S$ is SL.

(7) Let $R \subseteq S$ be an extension where $1 \neq -1$ and $\mathrm{U}(S)$ has a finite prime order. Then $\mathrm{U}(S)$ is a simple group and $R \subseteq S$ is SL by (6).

We will see that in the FCP case, the situation $1 \neq -1$ often occurs through the ring $\mathbb{Z}/2\mathbb{Z}$.

(8) An SL extension $R \subseteq S$ is trivial if (S, M) is a local ring. Indeed since $S = \mathrm{U}(S) \cup M = \mathrm{U}(R) \cup M$ and any $x \in M$ is such that $1 + x \in \mathrm{U}(S) = \mathrm{U}(R) \subseteq R$, we get that $M \subseteq R$, giving $R = S$.

Proposition 4.2. *The following holds for an extension $R \subseteq S$ where R is reduced:*

1. *If $s \in S$ is not algebraic over R, then $R \subseteq R[s]$ is SL.*
2. *In case S is also reduced and $V \in [R, S]$ is a MSL-subextension, then $V \subseteq S$ is algebraic.*

Proof. (1) Let $s \in S$ which is not algebraic over R and consider the morphism $\varphi : R[X] \to R[s]$ defined by $\varphi(X) = s$. Then φ is a surjective morphism which is also injective since s is not algebraic over R. Then $R[X] \cong R[s]$ so that $R \subseteq R[s]$ is SL by Example 4.1(3).

(2) Assume, moreover, that S is also reduced. By Example 4.1(1), there exists an *MSL*-subextension $V \in [R, S]$. Then, V is also reduced. Assume that $S \neq V$ and let $s \in S \setminus V$. If s is not algebraic over V, then $V \subseteq V[s]$ is SL by (1) and so is $R \subseteq V[s]$ because $\mathrm{U}(R) = \mathrm{U}(V) = \mathrm{U}(V[s])$, a contradiction with the maximality of V. Then, any $s \in S \setminus V$ is algebraic over V and then $V \subseteq S$ is algebraic. □

Corollary 4.3. *Let $R \subseteq S$ be an extension where S is reduced. There exists $U \in [R, S]$ such that $R \subseteq U$ is SL and $U \subseteq S$ is quasi-Prüfer.*

Proof. By Example 4.1(1), there exists an MSL-subextension $U \in [R, S]$. Moreover, $U \subseteq S$ is algebraic as $U \subseteq V$ for any $V \in [U, S]$ by Proposition 4.2. This shows that $U \subseteq S$ is a residually algebraic pair, and then is quasi-Prüfer by [31, Theorem 2.3]. □

Proposition 4.4. *Let $R \subseteq S$ be a ring extension. The following conditions are equivalent:*

1. *$R \subseteq S$ is SL.*
2. *$R[X] \subseteq S[X]$ is SL.*
3. *$R + XS[X] \subseteq S[X]$ is SL.*

If these conditions hold, then $\mathrm{Nil}(R) = \mathrm{Nil}(S)$.

Proof. We begin to remark that $1 - x$ is a unit if x is nilpotent for any $x \in R$ (resp. $x \in S$). Then, $\mathrm{U}(R) = \mathrm{U}(S) \Rightarrow \mathrm{Nil}(R) = \mathrm{Nil}(S)$.

(1) $\Leftrightarrow$ (2) We know that $\mathrm{U}(R[X]) = \mathrm{U}(R) + X\mathrm{Nil}(R)[X]$ $(*)$ and $\mathrm{U}(S[X]) = \mathrm{U}(S) + X\mathrm{Nil}(S)[X]$ $(**)$. Then, we get the equivalence applying the previous remark to $R \subseteq S$ and $R[X] \subseteq S[X]$ and using $(*)$ and $(**)$.

(2) $\Rightarrow$ (3) since $R[X] \subseteq S[X]$ SL implies by Corollary 2.4 that $R + XS[X] \subseteq S[X]$ is SL because $R + XS[X] \in [R[X], S[X]]$.

(3) $\Rightarrow$ (1) Let $a \in \mathrm{U}(S) \subseteq \mathrm{U}(S[X]) = \mathrm{U}(R + XS[X])$, which gives $a \in S \cap \mathrm{U}(R + XS[X]) \subseteq R + XS[X]$. This shows that $a \in R$. The same property holds for $b := a^{-1} \in S$, so that $a \in \mathrm{U}(R)$ and $R \subseteq S$ is SL. □

Corollary 4.5. *Let $R \subseteq S$ be a ring extension such that S is reduced. Then $R \subseteq R + XS[X]$ is SL.*

Proof. Let $a \in \mathrm{U}(R + XS[X]) \subset \mathrm{U}(S[X]) \cap (R + XS[X])$, so that $a = c + Xf(X)$, where $c \in R$ and $f(X) \in S[X]$. But, as a is an element of $\mathrm{U}(S[X])$, we have $f(X) \in \mathrm{Nil}(S)[X] = 0$ because S is reduced. Then, $a = c \in R \cap \mathrm{U}(S[X]) \subseteq R \cap \mathrm{U}(S)$. The same property holds for $b := a^{-1} \in \mathrm{U}(R + XS[X])$, so that $a \in \mathrm{U}(R)$ and $R \subseteq R + XS[X]$ is SL. □

Proposition 4.6. *Let $R \subseteq S$ be a ring extension. If $R(X) \subseteq S(X)$ (resp. $R[[X]] \subseteq S[[X]]$) is SL, then $R = S$.*

Proof. Assume that $R(X) \subseteq S(X)$ is SL and let $s \in S$. Then, $P(X) := s + X \in S[X]$ is such that $P(X)/1 \in \mathrm{U}(S(X)) = \mathrm{U}(R(X))$. This implies that there exist $f(X), g(X) \in R[X]$ with content equal to R such that $P(X)/1 = f(X)/g(X)$. Set $f(X) := \sum_{i=0}^{n+1} a_i X^i$ and $g(X) := \sum_{i=0}^{n} b_i X^i$, $a_i, b_i \in R$ such that $g(X)P(X) = f(X)$. It follows that $sb_0 = a_0, b_n = a_{n+1}$ and $sb_i + b_{i-1} = a_i$ for any $i \in \mathbb{N}_n$ $(*)$. Since $c(g) = R$, there exist $\lambda_0, \dots, \lambda_n \in R$ such that $\sum_{i=0}^{n} \lambda_i b_i = 1$. Multiplying each equality of rank i of $(*)$ by λ_i and adding each of these equalities for each $i \in \{0, \dots, n\}$, we get $s(\sum_{i=0}^{n} \lambda_i b_i) = s = \lambda_0 a_0 + \sum_{i=1}^{n} \lambda_i (a_i - b_{i-1}) \in R$, so that $S = R$.

Assume now that $R[[X]] \subseteq S[[X]]$ is SL. We know that $\mathrm{U}(R[[X]]) = \mathrm{U}(R) + XR[[X]]$ and $\mathrm{U}(S[[X]]) = \mathrm{U}(S) + XS[[X]]$. It follows that $\mathrm{U}(R) + XR[[X]] = \mathrm{U}(S) + XS[[X]]$. Let $s \in S$. Then $1 + sX \in \mathrm{U}(S) + XS[[X]] = \mathrm{U}(R) + XR[[X]]$, so that $s \in R$ and $R = S$. □

Definition 4.7. *Let R be a ring.*

1. *A polynomial $p(X) \in R[X]$ is called comonic if $p(0) \in \mathrm{U}(R)$.*

2. *A ring extension $R \subseteq S$ is called co-integrally closed if any $x \in S$ which is a zero of a comonic polynomial of $R[X]$ is in R.*

Proposition 4.8. *Let $R \subseteq S$ be a ring extension. The following statements hold:*

1. *If $R \subseteq S$ is SL, then $R \subseteq S$ is co-integrally closed.*

2. *Let $R \subseteq S$ be an integral ring extension. Then $R \subseteq S$ is SL if and only if $R \subseteq S$ is co-integrally closed.*

Proof. (1) Assume that $R \subseteq S$ is SL and let $x \in S$ be a zero of a comonic polynomial $p(X) := \sum_{i=0}^{n} a_i X^i \in R[X]$. Then, $p(0) = a_0 \in \mathrm{U}(R)$ and $\sum_{i=0}^{n} a_i x^i = 0$, so that $x(\sum_{i=1}^{n} a_i x^{i-1}) = -a_0 \in \mathrm{U}(R) = \mathrm{U}(S)$ shows that $x \in \mathrm{U}(S) = \mathrm{U}(R) \subseteq R$. Then, $R \subseteq S$ is co-integrally closed.

(2) One part of the proof is gotten in (1). So, assume that $R \subseteq S$ is co-integrally closed. Obviously, $\mathrm{U}(R) \subseteq \mathrm{U}(S)$. Let $x \in \mathrm{U}(S)$. Since $R \subseteq S$ is an integral ring extension, there exists a monic polynomial $p(X) := \sum_{i=0}^{n} a_i X^i \in R[X]$ with $a_n = 1$, such that $p(x) = 0$, giving $x^n + \sum_{i=0}^{n-1} a_i x^i = 0$ $(*)$. Since $x \in \mathrm{U}(S)$, multiplying $(*)$ by x^{-n}, we get $1 + \sum_{i=0}^{n-1} a_i x^{i-n} = 1 + \sum_{i=0}^{n-1} a_i (x^{-1})^{n-i} = 0$ which shows that x^{-1} is a zero of the comonic polynomial $q(X) := 1 + \sum_{i=0}^{n-1} a_i X^{n-i} = \sum_{i=1}^{n} a_{n-i} X^i + 1 \in R[X]$. Then, $x^{-1} \in R$. To sum up, we have shown that any $x \in \mathrm{U}(S)$ is such that $x^{-1} \in R$. Setting $y := x^{-1}$, which is also in $\mathrm{U}(S)$, the previous proof gives that $y^{-1} = x$ is in R. Moreover, since it also shows that x and x^{-1} are in R, this implies that $x \in \mathrm{U}(R)$. To conclude, $\mathrm{U}(R) = \mathrm{U}(S)$ and $R \subseteq S$ is SL. □

5 J-regular rings

In this section, we look at properties of J-regular rings, which will play an important role in the following study of SL extension. An absolutely flat ring is in this paper called a (Von Neumann) *regular ring*. Actually, many results need only rings R with a Jacobson radical J such that R/J is regular. They are called J-*regular* in the literature.

We recall some results concerning regular rings.

(1) A ring R is regular if for any $x \in R$, there exists $y \in R$ such that $x^2y = x$. Under these conditions, such an y is unique when satisfying $y^2x = y$ [25, Lemme, p.69]. Moreover, setting $e := xy$ and $u := 1 - e + x$, we get that e is an idempotent, u is a unit with $1 - e + y$ as inverse and $x = eu$.

(2) A ring is regular if and only if it is reduced and zero-dimensional.

(3) Let R be a regular ring. For any $P \in \mathrm{Spec}(R)$, there is an isomorphism $R/P \cong R_P$.

Lemma 5.1. *If $f : R \to S$ is a strict monomorphism and S is regular, then R is regular.*

Proof. See [26, diagram page 40 and Proposition 19]. □

For any ring R there is an (Olivier) ring epimorphism $t : R \to \mathcal{O}(R)$, whose spectral map is bijective, such that $\mathcal{O}(R)$ is regular and such that any ring morphism $R \to S$ where S is regular can be factored $R \to \mathcal{O}(R) \to S$.

Note that $\ker(t) = \mathrm{Nil}(R)$.

This property is a consequence of the following facts: $\mathrm{V}(t^{-1}(I)) = \overline{{}^at(\mathrm{V}(I))}$ for any ideal I of $\mathcal{O}(R)$ [13, Proposition 1.2.2.3, p.196] and $\mathcal{O}(R)$ is reduced.

The existence of the ring $\mathcal{O}(R)$, called *the universal regular (absolutely flat) ring associated to R* is due to Olivier [25, Proposition, p.70].

Lemma 5.2. *If $f : R \to S$ is an epimorphism and $M \in \mathrm{Max}(R)$, there exists $N \in \mathrm{Max}(S)$ such that $M = {}^af(N)$ if and only if $N = f(M)S$.*

If in addition ${}^af$ is bijective, then, for any $M \in \mathrm{Max}(R)$, we have $f(M)S \in \mathrm{Max}(S)$ and $M = {}^af[f(M)S]$.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/M & \xrightarrow{\overline{f}} & S/f(M)S \end{array}$$

where $\overline{f}$ is deduced from f. Since f is an epimorphism, so is $\overline{f}$. Assume that $M := {}^af(N)$ for some $N \in \mathrm{Max}(S)$. Since $f(M)S \subseteq N \subset S$, it follows that $M = {}^af[f(M)S]$, so that $\overline{f}$ is injective. Now, R/M being a field, $\overline{f}$ is surjective, and then an isomorphism. Hence, $S/f(M)S$ is a field and $f(M)S \in \mathrm{Max}(S)$, which infers that $N = f(M)S$.

Conversely, assume that $N = f(M)S$ for some $N \in \mathrm{Max}(S)$. Then, $f(M)S \subset S$. It follows that ${}^af[f(M)S] \in \mathrm{Spec}(R)$ with $M \subseteq {}^af[f(M)S] \subset R$. But $M \in \mathrm{Max}(R)$ implies that $M = {}^af[f(M)S] = {}^af(N)$.

Now if ${}^af$ bijective, for any $M \in \mathrm{Max}(R)$, there exists $N \in \mathrm{Max}(S)$ such that $M = {}^af(N)$. Then the first part of the Lemma gives that $N := f(M)S \in \mathrm{Max}(S)$ and $M = {}^af[f(M)S]$. □

Proposition 5.3. *Let $t : R \to \mathcal{O}(R)$ be the Olivier ring epimorphism, $\mathcal{T} := \{N \in \mathrm{Spec}(\mathcal{O}(R)) \mid {}^at(N) \in \mathrm{Max}(R)\}$ and $K := \cap[N \in \mathcal{T}]$. Then, ${}^at(\mathcal{T}) = \mathrm{Max}(R)$ and any $N \in \mathcal{T}$ is of the form $t(M)\mathcal{O}(R)$ where $M \in \mathrm{Max}(R)$.*

If R is J-regular, there is an isomorphism $R/\mathrm{J}(R) \cong \mathcal{O}(R)/K$.

Proof. Since ${}^at(N) \in \mathrm{Max}(R)$ for any $N \in \mathcal{T}$, we have ${}^at(\mathcal{T}) \subseteq \mathrm{Max}(R)$. Now, let $M \in \mathrm{Max}(R)$. There exists $N \in \mathrm{Spec}(\mathcal{O}(R))$ such that $M = {}^at(N)$ because ${}^at$ is bijective, so that $N \in \mathcal{T}$, giving $M \in {}^at(\mathcal{T})$. To end, ${}^at(\mathcal{T}) = \mathrm{Max}(R)$.

Let $N \in \mathcal{T}$ and set $M := {}^at(N) \in \mathrm{Max}(R)$. By Lemma 5.2, $N = t(M)\mathcal{O}(R)$.

Assume, moreover, that R is J-regular. Then, $R/\mathrm{J}(R)$ is regular, that is zero-dimensional. Setting $K := \cap[N \in \mathcal{T}]$, we get $t^{-1}(K) = t^{-1}(\cap[N \in \mathcal{T}]) = \cap[{}^at(N) \mid N \in \mathcal{T}] = \cap[M \in {}^at(\mathcal{T})] = \cap[M \in \mathrm{Max}(R)] =$

$\mathrm{J}(R)$. Consider the following commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{t} & \mathcal{O}(R) \\ \downarrow & & \downarrow \\ R/\mathrm{J}(R) & \xrightarrow{\overline{t}} & \mathcal{O}(R)/K \end{array}$$

where $\overline{t}$ is deduced from t. Since t is an epimorphism, so is $\overline{t}$, which is injective because $t^{-1}(K) = \mathrm{J}(R)$. Now, $R/\mathrm{J}(R)$ being zero-dimensional, $\overline{t}$ is surjective, and then $\overline{t} : R/\mathrm{J}(R) \to \mathcal{O}(R)/K$ is an isomorphism. □

We are now looking at topological characterizations of J-regular rings and add a characterization of J-regular rings given in [18].

Proposition 5.4. *Let R be a ring. The following conditions are equivalent:*

1. *R is* J*-regular.*
2. $\mathrm{Max}(R)$ *is closed.*
3. $\mathrm{Max}(R)$ *is proconstructible.*
4. $\mathrm{Max}(R)$ *is compact for the flat topology.*
5. *[18, Proposition 6.4, page 60] For every $x \in R$, there exists $y \in R$ such that $xy \in \mathrm{J}(R)$ and $x+y \in \mathrm{U}(R)$.*

Proof. (1) ⇔ (4) by [41, Theorem 4.5].

(1) ⇒ (2) Equality $\overline{\mathrm{Max}(R)} = \mathrm{V}(\mathrm{J}(R))$ always holds. Then (2) ⇔ $\overline{\mathrm{Max}(R)} = \mathrm{Max}(R)$. But $\dim(R/\mathrm{J}(R)) = 0$ gives that $\mathrm{Max}(R) = \mathrm{V}(\mathrm{J}(R))$. Moreover, (1) ⇔ $\dim(R/\mathrm{J}(R)) = 0$ and $R/\mathrm{J}(R)$ is reduced, this last condition always holding since $\mathrm{J}(R)$ is semiprime. It follows that (1) ⇒ $\mathrm{Max}(R) = \mathrm{V}(\mathrm{J}(R)) = \overline{\mathrm{Max}(R)}$, so that $\mathrm{Max}(R)$ is closed.

(2) ⇒ (1) If $\mathrm{Max}(R)$ is closed, then $\mathrm{Max}(R) = \overline{\mathrm{Max}(R)} = \mathrm{V}(\mathrm{J}(R))$, so that $\dim(R/\mathrm{J}(R)) = 0$ which implies that R is J-regular.

(2) ⇒ (3) because a closed subset is proconstructible.

(3) ⇒ (2) According to [13, Corollaire 7.3.2, page 339], a proconstructible subset stable by specialization is closed. □

Corollary 5.5. *Let $R \subseteq S$ be an integral i-extension. Then R is* J*-regular if and only if S is* J*-regular.*

Proof. Since $R \subseteq S$ is an integral i-extension, the natural map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is an homeomorphism. Then it is enough to use the equivalence (1)⇔(2) of Proposition 5.4. □

Corollary 5.6. *A ring R whose spectrum is Noetherian for the flat topology, is* J*-regular.*

Proof. By [5, Propositions 8 and 9, page 123], $\mathrm{Max}(R)$ is Noetherian, and then compact for the flat topology. Then, use Proposition 5.4. □

Corollary 5.7. *Let $f : R \to S$ be a ring morphism such that ${}^af(\mathrm{Max}(S)) = \mathrm{Max}(R)$. If S is* J*-regular, so is R.*

Proof. By Proposition 5.4, if S is J-regular, $\mathrm{Max}(S)$ is compact for the flat topology. Since ${}^af$ is continuous for the flat topology, we get that ${}^af(\mathrm{Max}(S)) = \mathrm{Max}(R)$ is compact, so that R is J-regular. □

Proposition 5.8. *A ring R is* J*-regular if and only if so is $R(X)$.*

Proof. We know that $\mathrm{Max}(R(X)) = \{MR(X) \mid M \in \mathrm{Max}(R)\}$, so that $\mathrm{J}(R(X)) = \mathrm{J}(R)(X)$. It follows that $R(X)/\mathrm{J}(R(X)) = R(X)/(\mathrm{J}(R)(X))$
$\cong (R/\mathrm{J}(R))(X)$. As S is regular if and only if so is $S(X)$, for a ring S, we get that $(R/\mathrm{J}(R))(X)$ is regular if and only if so is $R/\mathrm{J}(R)$. Then, R is J-regular if and only if so is $R(X)$. □

According to [27, Definitions 1.1, 1.2 and 1.5, Proposition 1.6], [29] and [40], we set $p_r(X) := X^2 - rX \in R[X]$, where $R \subseteq S$ is a ring extension. $R \subseteq S$ is called *s-elementary* (resp.; *t-elementary*, *u-elementary*) if $S = R[b]$, where $p_0(b), bp_0(b) \in R$ (resp.; $p_r(b), bp_r(b) \in R$ for some $r \in R$, $p_1(b), bp_1(b) \in R$). In the following, the letter x denotes s, t or u. $R \subseteq S$ is called *cx-elementary* if $R \subseteq S$ is a tower of finitely many x-elementary extensions, *x-integral* if there exists a directed set $\{S_i\}_{i\in I} \subseteq [R,S]$ such that $R \subseteq S_i$ is cx-elementary and $S = \cup_{i\in I} S_i$. An integral extension $R \subseteq S$ is called *infra-integral* [29], (resp. *subintegral* [40]) if all its residual extensions are isomorphisms (resp.; and is an *i-extension*).

An extension $R \subseteq S$ is called *s-closed* (or *seminormal*) (resp.; *t-closed*, *u-closed* (or *anodal*)) if an element $b \in S$ is in R whenever $p_0(b), bp_0(b) \in R$ (resp.; $p_r(b), bp_r(b) \in R$ for some $r \in R$, $p_1(b), bp_1(b) \in R$) [40, Theorem 2.5]. A ring R is called *seminormal* by Swan if for $x, y \in R$, such that $x^2 = y^3$, there is some $z \in R$ such that $x = z^3$ and $y = z^2$ [40, Definition, page 210]. We say that a ring R is *t-closed* if for $x, y, r \in R$, such that $x^3 + rxy - y^2 = 0$, there is some $z \in R$ such that $x = z^2 - rz$ and $y = z^3 - rz^2$ [28, Définition 1.1]. A t-closed ring is seminormal.

A seminormal ring is reduced. We proved in [28, Proposition 2.1] that a regular ring is t-closed, whence seminormal.

Let x$\in$ {s,t,u}. The *x-closure* ${}^x_S R$ of R in S is the smallest element $B \in [R,S]$ such that $B \subseteq S$ is x-closed and the greatest element $B' \in [R,S]$ such that $R \subseteq B'$ is x-integral. It follows that ${}^u_S R \subseteq {}^t_S R$. Note that the s-closure is actually the seminormalization ${}^+_S R$ of R in S and is the greatest subintegral extension of R in S. Note also that the t-closure ${}^t_S R$ is the greatest infra-integral extension of R in S.

Example 5.9. *(1) Let R be a ring and d a positive integer. Set* $R_d := \{\sum_{i\in I} \varepsilon_i a_i^d \mid \varepsilon \in \{1,-1\},\ a_i \in R,\ |I| < \infty\}$ *which is a subring of R such that* $f : R_d \subseteq R$ *is an integral extension. We claim that f is an i-extension. Let* $P, Q \in \mathrm{Spec}(R)$ *be such that* $P \cap R_d = Q \cap R_d$. *Let* $x \in Q$, *so that* $x^d \in Q \cap R_d = P \cap R_d \subseteq P$, *which implies* $x \in P$, *and then* $Q \subseteq P$. *A similar proof shows that* $P \subseteq Q$, *so that* $P = Q$. *Then* ${}^a f(\mathrm{Max}(R)) = \mathrm{Max}(R_d)$ *and R is* J*-regular if and only if* R_d *is* J*-regular by Corollary 5.5.*

(2) Let $R \subseteq S$ *be a u-closed FCP integral extension. Then* $R \subseteq S$ *is an i-extension by [35, Proposition 5.2] and R is* J*-regular if and only if S is* J*-regular by Corollary 5.5.*

Corollary 5.10. *Let* $R \subseteq S$ *be a subintegral extension, such that S is* J*-regular. Then R is* J*-regular and there is a ring isomorphism* $R/\mathrm{J}(R) \to S/\mathrm{J}(S)$. *In case* $\mathrm{J}(R) = \mathrm{J}(S)$, *then* $R = S$.

Proof. $R/\mathrm{J}(R)$ is regular because reduced and zero-dimensional since so is $S/\mathrm{J}(S)$, and because the extension is integral. This also implies that $\mathrm{J}(R) = R \cap \mathrm{J}(S)$, so that the map $j : A := R/\mathrm{J}(R) \to S/\mathrm{J}(S) := B$ exists, and its residual extensions are isomorphisms. Such residual extensions are of the form $A_P \to B_Q$ where Q is a prime ideal of B above P. It follows that $A \to B$ is a flat epimorphism since $A \to B$ is an i-extension [31, Scholium A (1)]. Finally, since this extension has the lying-over property for maximal ideals, $A \to B$ is a faithfully flat epimorphism, whence an isomorphism [19, Lemme 1.2, page 109]. □

We recall that a ring R verifies the primitive condition if for any element $p(X)$ of the polynomial ring $R[X]$, whose content $c(p)$ is R there is some $x \in R$ such that $f(x) \in \mathrm{U}(R)$ [21]. The Nagata ring $R(X)$ of a ring R verifies the primitive condition [21, p.457].

Let $R \subseteq S$ be an extension. We denote by $\mathcal{I}(R,S)$ the abelian group of all R-submodules of S that are invertible as in [38, Definition 2.1].

There is an exact sequence [38, Theorem 2.4]:

$$1 \to \mathrm{U}(R) \to \mathrm{U}(S) \to \mathcal{I}(R,S) \to \mathrm{Pic}(R) \to \mathrm{Pic}(S)$$

It follows that there is an injective map $\mathrm{U}(S)/\mathrm{U}(R) \to \mathcal{I}(R,S)$.

Proposition 5.11. *Let $R \subseteq S$ be an extension. Then $R \subseteq S$ is an SL extension if and only if $\mathcal{I}(R,S) = \{R\}$.*

Proof. Use the injective map $\mathrm{U}(S)/\mathrm{U}(R) \to \mathcal{I}(R,S)$. □

Proposition 5.12. *The Picard group of a ring R is 0 if either R is* J*-regular or R verifies the primitive condition. Moreover, over such ring, a finitely generated projective module of finite (local) constant rank is free.*

Proof. It is enough to combine [21, Propositions p. 455 and p. 456] and [21, Theorem and Corollary p. 457]. □

Corollary 5.13. *[38, Remark 2.5] Let $R \subseteq S$ be an extension, where $\mathrm{Pic}(R) = 0$ (for example, if either R is a Nagata ring or R is* J*-regular), then $\mathrm{U}(S)/\mathrm{U}(R)$ is isomorphic to $\mathcal{I}(R,S)$.*

Proposition 5.14. *Let $R \subseteq S$ be an extension and M an R-submodule of S. Then M belongs to $\mathcal{I}(R,S)$ if and only if M is an R-module of finite type, $MS = S$ and M is projective of rank one. When R is either a Nagata ring or R is* J*-regular, the last condition can be replaced with M is free of dimension one and then the elements of $\mathcal{I}(R,S)$ are of the form Ru where u is a unit of S.*

Proof. This a consequence of [38, Lemma 2.2 and Lemma 2.3]. □

Remark 5.15. *We now consider an SL extension $R \subseteq S$.*

(1) Since $x \in R$ belongs to $\mathrm{J}(R)$ if and only if $1 - ax$ is a unit for any $a \in R$, let $x \in \mathrm{J}(S)$. In particular, $1-x \in \mathrm{U}(S) = \mathrm{U}(R)$, which implies $x \in R$. It follows easily that $\mathrm{J}(S) \subseteq \mathrm{J}(R)$. Therefore $\mathrm{J}(S)$ is an ideal shared by R and S. We infer from this fact that if P is a prime ideal of R, that does not contains $\mathrm{J}(S)$, we have $R_P = S_P$.

(2) If $2 \in \mathrm{U}(S)$, we claim that R and S have the same idempotents: Let e be an idempotent of S. Then, $(1-2e)^2 = 1 - 4e + 4e^2 = 1$, so that $1 - 2e \in \mathrm{U}(S) = \mathrm{U}(R)$, which implies $2e \in R$. Since $2 \in \mathrm{U}(S) = \mathrm{U}(R)$, we get that $e \in R$.

(3) If the class of an element $s \in S$ is a unit of $S/\mathrm{J}(S)$, there is some $t \in S$, such that $st - 1 \in \mathrm{J}(S)$. It follows that s cannot belong to any maximal ideal of S and is therefore a unit of S and then of R. We deduce from these facts that $\mathrm{U}(R/\mathrm{J}(S)) = \mathrm{U}(S/\mathrm{J}(S))$ and $R/\mathrm{J}(S) \subseteq S/\mathrm{J}(S)$ is SL.

Proposition 5.16. *A ring extension $R \subseteq S$ is SL if and only if $\mathrm{Nil}(R) = \mathrm{Nil}(S)$ and $R/\mathrm{Nil}(R) \subseteq S/\mathrm{Nil}(S)$ is SL.*

Proof. Assume that $R \subseteq S$ is SL. Then $\mathrm{Nil}(R) = \mathrm{Nil}(S)$ by Proposition 4.4. Set $R' := R/\mathrm{Nil}(R)$ and $S' := S/\mathrm{Nil}(S)$. We claim that $R' \subseteq S'$ is SL. Let $\overline{x} \in \mathrm{U}(S')$, where $\overline{x}$ is the class of $x \in S$. There exists $y \in S$ such that $\overline{xy} = \overline{1}$ $(*)$ in S', so that $1 - xy \in \mathrm{Nil}(S) = \mathrm{Nil}(R) \subseteq \mathrm{J}(S)$. Then $1 - xy \in \mathrm{J}(S)$ implies $x, y \in \mathrm{U}(S) = \mathrm{U}(R) \subseteq R$. As $1 - xy \in \mathrm{Nil}(R)$ with $x, y \in R$, we deduce from $(*)$ that $\overline{x}, \overline{y} \in \mathrm{U}(R')$, giving that $\mathrm{U}(S') = \mathrm{U}(R')$ and $R/\mathrm{Nil}(R) \subseteq S/\mathrm{Nil}(S)$ is SL.

Conversely, assume that $\mathrm{Nil}(R) = \mathrm{Nil}(S)$ and $R/\mathrm{Nil}(R) \subseteq S/\mathrm{Nil}(S)$ is SL. By Corollary 2.7, $R \subseteq S$ is SL. □

Theorem 5.17. Let $R \subseteq S$ be an SL extension. Then S is J-regular if and only if R is J-regular and $R \subseteq S$ is integral seminormal. If these conditions hold, then $\mathrm{J}(R) = \mathrm{J}(S)$.

Proof. (1) According to Remark 5.15(3), $R/\mathrm{J}(S) \subseteq S/\mathrm{J}(S)$ is SL. Assume that S is J-regular. We want to show that R is J-regular with $R \subseteq S$ integral seminormal. Since $S/\mathrm{J}(S)$ is regular, it is enough to suppose that the extension $R \subseteq S$ is SL with S regular and to show that R is regular.

Let r be in R. As an element of S, there is some $r' \in S$ such that $r^2r' = r$; so that $e = rr'$ is idempotent and $1 - e + r \in \mathrm{U}(S) = \mathrm{U}(R) \subseteq R$, with $(1 - e + r)^{-1} = 1 - e + r'$ (see (1) at the beginning of Section 5). We have clearly $e \in R$ and, since $1 - e + r' \in \mathrm{U}(S) = \mathrm{U}(R) \subseteq R$, then $r' \in R$. To conclude, R is regular.

Let $x \in S$. Then, $x = ue$, $u \in \mathrm{U}(S) = \mathrm{U}(R)$ and e an idempotent of S, that is $e^2 - e = 0$, so that e is integral over R, and so is x. Then, $R \subseteq S$ is integral. According to the absolute flatness of R and S, we deduce that these rings are seminormal. It follows that $R \subseteq S$ is seminormal by [40, Corollary 3.4].

Coming back to the first case, it follows that if S is J-regular, so is R. Since $R/\mathrm{J}(S) \subseteq S/\mathrm{J}(S)$ is integral seminormal, so is $R \subseteq S$.

(2) We intend to show that $\mathrm{J}(R) = \mathrm{J}(S)$. Returning to the Jacobson ideals, we see that $\mathrm{J}(S)$ is an intersection of maximal ideals of R because $R/\mathrm{J}(S)$ is regular and it follows that $\mathrm{J}(R) \subseteq \mathrm{J}(S)$, the lacking inclusion. By the way, we have shown that a maximal ideal of S contracts to a maximal ideal of R. Actually the natural map $\mathrm{Max}(S) \to \mathrm{Max}(R)$ is surjective because a minimal prime ideal can be lifted up in the extension $R/\mathrm{J}(R) \subseteq S/\mathrm{J}(S)$.

(3) Conversely, assume that R is J-regular and $R \subseteq S$ is integral seminormal. By Remark 5.15(1), we know that $\mathrm{J}(S) \subseteq \mathrm{J}(R)$ and $\mathrm{J}(S)$ is an ideal of R, an intersection of the maximal ideals of S. Then, $\mathrm{J}(S) = \mathrm{J}(S) \cap R$ is also an intersection of maximal ideals of R since $R \subseteq S$ is integral. Hence, $\mathrm{J}(R) \subseteq \mathrm{J}(S)$ giving $\mathrm{J}(R) = \mathrm{J}(S)$. Setting $R' := R/\mathrm{J}(R)$ and $S' := S/\mathrm{J}(S)$, we get that $R' \subseteq S'$ is integral seminormal and SL by Remark 5.15(3). Since R' is regular, it is reduced and we have $\dim(R') = 0 = \dim(S')$ because $R' \subseteq S'$ is integral. We claim that S' is reduced. Assume there exists some $x \in S'$, $x \neq 0$ which is nilpotent, and let n be the least integer > 1 such that $x^n = 0$ with $y := x^{n-1} \neq 0$. We get that $y^2 = x^{n+(n-2)} = 0 = y^3 \in R'$ $(*)$. Since $R' \subseteq S'$ is seminormal, it follows that $y \in R'$. But $(*)$ implies that $y = 0$ since R' is reduced, a contradiction. We have that S' is reduced, and then regular, so that S is J-regular. □

Lequain and Doering have considered in [9] SL extensions $R \subseteq S$ such that S is semilocal. Then Theorem 5.17 implies one of their results, because S is J-regular by the Chinese remainder theorem. We will improve [9, Theorem 1] by using our methods.

Proposition 5.18. *An SL extension $R \subseteq S$, such that S is regular, is u-integral, infra-integral, seminormal, quadratic and R is regular.*

Proof. Since S is regular, any $x \in S$ can be written $x = eu$, where e is an idempotent and $u \in \mathrm{U}(S)$. Set $T := {}^u_S R$. Then, $T \subseteq S$ is u-closed. Since $e^2 - e = e^3 - e^2 = 0 \in T$, we get that $e \in T$. Moreover, $\mathrm{U}(R) \subseteq \mathrm{U}(T) \subseteq \mathrm{U}(S) = \mathrm{U}(R)$ gives $\mathrm{U}(T) = \mathrm{U}(S)$, so that $u \in T$, and then $x \in T$, whence $T = S$ and $R \subseteq S$ is u-integral. Since $S = {}^u_S R \subseteq {}^t_S R \subseteq S$, we get $S = {}^t_S R$ and $R \subseteq S$ is infra-integral. Since S is regular, so is $S/\mathrm{J}(S)$ and $R \subseteq S$ is seminormal by Theorem 5.17.

Let $x \in S$ and $y \in S$ be such that $x^2 y = x$, since S is regular. Keeping the previous notation and properties of the beginning of the Section, $e := xy$ is the idempotent such that $x = eu$, with $u \in \mathrm{U}(S) = \mathrm{U}(R)$. It follows that $e = xu^{-1}$, so that $x^2u^{-1} = xe = x^2y = x$ $(*)$. Then, $x^2 = xu$ gives that any $x \in S$ is quadratic and $R \subseteq S$ is a quadratic extension. Now, if $x \in R$, $(*)$ is still valid with $u \in \mathrm{U}(R)$ as we have just seen. Then, $x^2u^{-1} = x$ shows that R is regular. □

Corollary 5.19. *Let $R \subseteq S$ be an SL extension such that S is* J*-regular. Then $R \subseteq S$ is u-integral, infra-integral and quadratic.*

Proof. Since $R \subseteq S$ is SL, so is $R/\mathrm{J}(S) \subseteq S/\mathrm{J}(S)$ by Remark 5.15 (3). Then, we may apply the results of Proposition 5.18 to the extension $R/\mathrm{J}(R) \subseteq S/\mathrm{J}(S)$ because $\mathrm{J}(R) = \mathrm{J}(S)$ by Theorem 5.17, which gives that $R/\mathrm{J}(R) \subseteq S/\mathrm{J}(S)$ is u-integral, infra-integral and quadratic. It follows that $R \subseteq S$ also verifies these properties. □

Recall that a ring extension $R \subseteq S$ is called a Δ_0-extension if $T \in [R,S]$ for each R-submodule T of S containing R [16].

Corollary 5.20. *Let $R \subseteq S$ be an SL extension such that S is regular. The following properties hold:*

1. *For any $Q \in \mathrm{Spec}(S)$ and $P := Q \cap R$, we have $R_P \cong S_Q$. If, moreover, $R \subseteq S$ is an i-extension, then $R = S$.*

2. *If $R \subseteq S$ is u-closed, then $R = S$.*

3. *If $R \subseteq S$ is simple, then $R \subseteq S$ is a Δ_0-extension.*

4. *If $2 \in \mathrm{U}(S)$, then $R = S$.*

Proof. (1) By Proposition 5.18, $R \subseteq S$ is infra-integral and R is regular. It follows that for any $Q \in \mathrm{Spec}(S)$ and $P := Q \cap R$, we have an isomorphism $R/P \cong S/Q$. But, R and S being regular, they are zero-dimensional, so that $R/P \cong R_P$ and $S/Q \cong S_Q$ giving $R_P \cong S_Q$.

Assume, moreover, that $R \subseteq S$ is an i-extension, then $R \subseteq S$ is a flat epimorphism [31, Scholium A (1)] or [19, Ch. IV]. Then, $R = S$ by [31, Scholium A (2)].

(2) According to Proposition 5.18, $R \subseteq S$ is u-integral. Assume, moreover, that $R \subseteq S$ is u-closed, then $R = {}^u_S R = S$.

(3) According to Proposition 5.18, $R \subseteq S$ is quadratic. Assume, moreover, that $R \subseteq S$ is a simple extension. Set $S := R[y]$, where y is a quadratic element over R. Then, $S = R + Ry$ and it follows from [11, Proposition 4.12] that $R \subset S$ is a Δ_0-extension.

(4) Since S is regular, according to [3, Theorem 2.10], any $x \in S$ can be written $x = u + v$, where $u, v \in \mathrm{U}(S) = \mathrm{U}(R)$ which implies $x \in R$ and $R = S$. □

Proposition 5.21. *Let M be an R-module. The following conditions hold:*

1. $\mathrm{J}(R(+)M) = \mathrm{J}(R)(+)M$.

2. *$R(+)M$ is J-regular if and only if R is J-regular.*

Proof. (1) comes from [2, Theorem 3.2] because any maximal ideal of $R(+)M$ is of the form $P(+)M$ where $P \in \mathrm{Max}(R)$.

(2) $R(+)M$ is J-regular if and only if $(R(+)M)/(\mathrm{J}(R)(+)M)$ is regular. But $(R(+)M)/(\mathrm{J}(R)(+)M) \cong (R/\mathrm{J}(R))(+)(M/M) \cong R/\mathrm{J}(R)$. By [2, Theorem 3.1], we get the result. □

A ring R is called a Max-*ring* if any nonzero R-module has a maximal submodule. An ideal I of R is T-*nilpotent* if for each sequence $\{r_i\}_{i=1}^{\infty} \subseteq I$, there is some positive integer k with $r_1 \cdots r_k = 0$.

Proposition 5.22. *Let $R \subseteq S$ be an SL extension where S is a* Max*-ring. Then R is also a* Max*-ring and R and S are* J*-regular.*

Proof. S is regular and $\mathrm{J}(S)$ is T-nilpotent since S is a Max-ring by [14, Theorem, p.1135]. Then, S is also J-regular, and so is R, with $\mathrm{J}(R) = \mathrm{J}(S)$ by Theorem 5.17. It follows that $\mathrm{J}(R)$ is T-nilpotent. The same reference gives that R is a Max-ring since R is regular by Proposition 5.18. □

6 Cohn's rings

In [6, Theorem 1], P. M. Cohn shows that for each ring R there is an SL extension $R \subseteq R'$ such that $\mathrm{Z}(R') = R' \setminus \mathrm{U}(R')$. We generalize some results.

Lemma 6.1. *Let I be a semiprime ideal of a ring R and $f(X), g(X) \in R[X]$, $a, b \in \mathrm{U}(R)$ such that $af(X) + bg(X) + Xf(X)g(X) \in I[X]$ $(*)$. Then $f(X), g(X) \in I[X]$.*

Proof. We first prove the Lemma for a prime ideal P. We begin to remark that if $f(X) = f_1(X) + f_2(X)$, with $f_2(X) \in P[X]$, then $(*)$ is equivalent to $af_1(X) + bg(X) + Xf_1(X)g(X) \in P[X]$. The same property holds for $g(X)$. Then, the conclusion of the Lemma will result from the fact that $f_1(X), g(X) \in P[X]$. Moreover, once we have proved that $f(X) \in P[X]$, condition $(*)$ shows that $bg(X) \in P[X]$, which gives $g(X) \in P[X]$ because $b \in \mathrm{U}(R)$. So, assume that $f(X)$ and $g(X) \notin P[X]$. Hence, we can set $f(X) := \sum_{i=0}^{n} \alpha_i X^i$ and $g(X) := \sum_{j=0}^{p} \beta_j X^j$ with α_n and $\beta_p \notin P$, according to the precedent discussion. The only term of highest degree in $(*)$ is $\alpha_n\beta_p X^{n+p+1}$ which is in $P[X]$, so that $\alpha_n\beta_p \in P$, a contradiction with α_n and $\beta_p \notin P$. To conclude, $f(X), g(X) \in P[X]$.

Now, assume that $af(X) + bg(X) + Xf(X)g(X) \in I[X]$ $(*)$ for a semiprime ideal I. Then, $I = \cap_{\lambda\in\Lambda} P_\lambda$, for a family of prime ideals $\{P_\lambda\}_{\lambda\in\Lambda}$. Condition $(*)$ shows that $af(X) + bg(X) + Xf(X)g(X) \in P_\lambda[X]$ for each $\lambda \in \Lambda$. By the first part of the proof, it follows that $f(X), g(X) \in P_\lambda[X]$ for each $\lambda \in \Lambda$, which implies that $f(X), g(X) \in \cap_{\lambda\in\Lambda} P_\lambda[X] = I[X]$. □

Proposition 6.2. *For any ring R there exists an SL extension $R \subseteq R'$ such that any nonunit of R is a zerodivisor in R'. The extension $R \subseteq R'$ is pure and t-closed and $(R : R') = \mathrm{J}(R)$. If R is reduced, so is R'.*

Proof. For each $M \in \mathrm{Max}(R)$, let X_M be an indeterminate attached to M. Set $\mathcal{X} := \{X_M \mid M \in \mathrm{Max}(R)\}$. We consider $R[\mathcal{X}]$ as the ring of polynomials in the indeterminates $X_M \in \mathcal{X}$. At last, define $I := \sum_{M\in\mathrm{Max}(R)} X_M M R[\mathcal{X}]$, which is an ideal of $R[\mathcal{X}]$ and $R' := R[\mathcal{X}]/I$. For each $M \in \mathrm{Max}(R)$, we define x_M as the class of X_M in R'. We prove the Proposition in several steps.

(1) Let $M, N \in \mathrm{Max}(R)$, $M \neq N$. We claim that $x_M x_N = 0$ in R'.

Since $M \neq N$, they are comaximal, and there exist $m \in M$, $n \in N$ such that $m + n = 1$. It follows that $X_M X_N = (m+n)X_M X_N = mX_M X_N + nX_N X_M \in I$, giving $x_M x_N = 0$. Then, the class of a polynomial of $R[\mathcal{X}]$ is written as the class of a finite sum of polynomials in one indeterminate, that is in finitely many X_M.

(2) R is a subring of R' because the composite ring morphism $R \subseteq R[\mathcal{X}] \to R[\mathcal{X}]/I$ is injective. Indeed, $R \cap I = 0$ since the constant term of any polynomial in I is 0. Then, we can identify the class of the constant term of any polynomial of $R[\mathcal{X}]$ to this constant term.

(3) Any $y \in R'$ can be written in an unique way

$$y = a + \sum_{M\in\Lambda} x_M f_M(x_M)$$

where $a \in R$, f_M is a polynomial of $R[X]$ and the set Λ is a finite subset of $\mathrm{Max}(R)$.

The existence of this writing results from (1) and (2). Assume that there exist $b \in R$, $g_M \in R[X]$ such that $y = b + \sum_{M\in\Lambda} x_M g_M(x_M)$. There is no harm to choose the same Λ for the two writings. Then, $a + \sum_{M\in\Lambda} x_M f_M(x_M) = b + \sum_{M\in\Lambda} x_M g_M(x_M)$ in R' gives that $(a-b) + \sum_{M\in\Lambda} X_M[f_M(X_M) - g_M(X_M)] \in I = \sum_{M\in\mathrm{Max}(R)} X_M M R[\mathcal{X}]$. First, we get $a = b$ by the substitution $X_M \mapsto 0$ for each M. Now, we get that $X_M[f_M(X_M) - g_M(X_M)] \in X_M M R[X_M]$ for each $M \in \Lambda$, so that $x_M f_M(x_M) = x_M g_M(x_M)$ for each $M \in \Lambda$ showing the uniqueness of the writing.

By the way we proved that $R' = R \bigoplus_{M\in\Lambda} x_M R[x_M]$.

(4) We prove that $(R : R') = \mathrm{J}(R)$.

Let $t \in R$. Then $t \in (R : R')$ if and only if $tx_M \in R$ for any $M \in \mathrm{Max}(R)$ if and only if $tx_M = 0$ by (3), which is equivalent to $tX_M \in I$ for any $M \in \mathrm{Max}(R)$, that is $t \in \cap_{M\in\mathrm{Max}(R)} M = \mathrm{J}(R)$. It follows that $(R : R') = \mathrm{J}(R)$.

(5) $R \subseteq R'$ is SL.

Let $y \in \mathrm{U}(R')$. There exists $z \in \mathrm{U}(R')$ such that $yz = 1$ $(*)$. Write $y = a + \sum_{M\in\Lambda} x_M f_M(x_M)$ and $z = b + \sum_{M\in\Lambda} x_M g_M(x_M)$, where $a, b \in R$ and f_M, g_M are polynomials of $R[X]$. We can choose the same Λ for y and z. Then $(*)$ becomes $1 = (a + \sum_{M\in\Lambda} x_M f_M(x_M))(b + \sum_{M\in\Lambda} x_M g_M(x_M)) = ab + \sum_{M\in\Lambda} x_M[ag_M(x_M) + bf_M(x_M) + x_M f_M(x_M)$

$g_M(x_M)]$ from which it follows that $ab=1$ (**) and $ag_M(X_M)+bf_M(X_M)+X_Mf_M(X_M)g_M(X_M)\in MR[X_M]$ (** *). By (**), we get that $a,b\in \mathrm{U}(R)$. Then, we may apply Lemma 6.1 to (* * *), and we get that $f_M(X_M)$ and $g_M(X_M)$ are in $MR[X_M]$, so that $x_Mf_M(x_M)=x_Mg_M(x_M)=0$ and $y=a\in \mathrm{U}(R)$. Then, $\mathrm{U}(R)=\mathrm{U}(R')$ and $R\subseteq R'$ is SL.

(6) Any nonunit of R is a zerodivisor in R'.

Obvious: If $y\in R$ is not a unit in R, there exists some $M\in \mathrm{Max}(R)$ such that $y\in M$. Then $X_My\in X_MMR[X_M]$ giving $x_My=0$ in R' with $x_M\neq 0$, so that y is a zerodivisor in R'.

(7) $R\to R'$ is pure.

Consider the composite ring morphism $R\subseteq R[\mathcal{X}]\to R[\mathcal{X}]/I$ and let $\varphi: R[\mathcal{X}]\to R$ be the ring morphism defined by the substitution $X_M\mapsto 0$ for each M. Since I is contained in the kernel of φ, there is a ring morphism $R[\mathcal{X}]/I\to R$. It follows that $R[\mathcal{X}]/I$ is a retract of R and the extension $R\to R[\mathcal{X}]/I$ is pure.

(8) $R\subseteq R'$ is t-closed.

Let $y\in R'$ be such that $y^2-ry, y^3-ry^2\in R$ for some $r\in R$. As in the beginning of the proof, set $y=a+\sum_{M\in\Lambda}x_Mf_M(x_M)$ where $a\in R$, $f_M(X_M):=\sum_{i=0}^n\alpha_iX_M^i$ is a polynomial of $R[X_M]$, $n:=\deg(f_M)$ and the set Λ is a finite subset of $\mathrm{Max}(R)$. Then, $y^2-ry=(a^2-ra)+\sum_{M\in\Lambda}[(2a-r)x_Mf_M(x_M)+x_M^2f_M(x_M)^2]$. Assume that $y\notin R$ so that there exists some $M\in\mathrm{Max}(R)$ such that $x_Mf_M(x_M)\neq 0$. As in the proof of Lemma 6.1, we may assume that $\alpha_n\notin M$. But $y^2-ry\in R$ implies that $\alpha_n^2X_M^{2n+2}\in X_MMR[X_M]$, whence $\alpha_n^2\in M$ and $\alpha_n\in M$, a contradiction. Then, any $y\in R'$ such that $y^2-ry, y^3-ry^2\in R$ is in R and $R\subseteq R'$ is t-closed, and, in particular, seminormal. This implies that R' is reduced when R is reduced. □

Corollary 6.3. *Let $R\subseteq S$ be a ring extension such that the spectral map* $\mathrm{Max}(S)\to\mathrm{Max}(R)$ *is bijective and let R' (resp. S') be the ring associated to R (resp. S) gotten in Proposition 6.2. Then, $R\subseteq S$ is SL if and only if $R'\subseteq S'$ is SL.*

Proof. For each $M\in\mathrm{Max}(R)$, there exists a unique $N\in\mathrm{Max}(S)$ lying above M, and for each $N\in\mathrm{Max}(S)$, then, $N\cap R\in\mathrm{Max}(R)$. We use the notation of Proposition 6.2 and consider for each $M\in\mathrm{Max}(R)$, an indeterminate X_M attached to M. Then, we may say that X_M is also attached to any $N\in\mathrm{Max}(S)$ such that $N\cap R=M\in\mathrm{Max}(R)$. Then, the X_M's are attached in a unique way to all maximal ideals of S. Set $\mathcal{X}:=\{X_M\mid M\in\mathrm{Max}(R)\}$. We consider $R[\mathcal{X}]$ as the ring of polynomials in the indeterminates $X_M\in\mathcal{X}$ over R and $S[\mathcal{X}]$ as the ring of polynomials in the indeterminates $X_M\in\mathcal{X}$ over S. Setting $I:=\sum_{M\in\mathrm{Max}(R)}X_MMR[\mathcal{X}]$, we define $J:=\sum_{M\in\mathrm{Max}(R)}[X_MNS[\mathcal{X}]\mid M=N\cap R]$ which is an ideal of $S[\mathcal{X}]$. Set $R':=R[\mathcal{X}]/I$ and $S':=S[\mathcal{X}]/J$. We get the following commutative diagram:

$$\begin{array}{ccc} R & \to & S \\ \downarrow & & \downarrow \\ R[\mathcal{X}] & \to & S[\mathcal{X}] \\ \downarrow & & \downarrow \\ R'=R[\mathcal{X}]/I & \to & S'=S[\mathcal{X}]/J \end{array}$$

where all the maps are extensions since $I=J\cap R[\mathcal{X}]$. By Proposition 6.2, $R\subseteq R'$ and $S\subseteq S'$ are both SL extensions. Now, applying Proposition 2.2, if $R\subseteq S$ is SL, so is $R\subseteq S'$ and then also $R'\subseteq S'$. By the same Proposition, if $R'\subseteq S'$ is SL, so is $R\subseteq S'$ and then also $R\subseteq S$ because $S\subseteq S'$ is injective. □

Theorem 6.4. Let R be a ring. There exists an SL extension $R\subseteq S$ such that any nonunit of S is a zerodivisor in S. The extension $R\subseteq S$ is pure and t-closed. If, moreover, R is reduced, so is S.

Proof. According to Proposition 6.2, there exists an SL extension $R\subseteq R'$ such that any nonunit of R is a zerodivisor in R'. Set $R_0:=R$ and $R_1=R'$. We build by induction, with the same Proposition, a chain $\{R_i\}_{i\in I}$ defined by $R_{i+1}:=R_i'$ verifying $\mathrm{U}(R_{i+1})=\mathrm{U}(R_i)$. The induction shows that $\mathrm{U}(R_i)=\mathrm{U}(R)$. Now any nonunit of R_i is a zerodivisor in R_{i+1}. Setting $S:=\cup_{i\in I}R_i$, we get an extension $R\subseteq S$.

Let $a \in \mathrm{U}(S)$ and set $b := a^{-1} \in \mathrm{U}(S)$, so that $ab = 1$ $(*)$ in S. There exists some $i \in I$ such that $a, b \in R_i$ (we can take the same i for a and b). Then $(*)$ still holds in R_i, so that $a \in \mathrm{U}(R_i) = \mathrm{U}(R)$, giving $\mathrm{U}(S) = \mathrm{U}(R)$ and $R \subseteq S$ is SL.

Let x be a nonunit of S. There exists some $i \in I$ such that $x \in R_i$. Obviously, x is a nonunit of R_i, and then is a zerodivisor in R_{i+1}, and also a zerodivisor in S. So, any nonunit of S is a zerodivisor in S.

Since $R \subseteq R'$ is pure and t-closed by Proposition 6.2, so is $R_i \subseteq R_{i+1}$ for any i. Now, $R \subseteq S$ is pure (resp. t-closed) since $R \subseteq S$ is the union of pure (resp. t-closed) morphisms $R_i \subseteq R_{i+1}$.

If, moreover, R is reduced, so is R' by Proposition 6.2, and so are any R_i, and to end, so is S. □

Remark 6.5. *The ring gotten in Theorem 6.4 is not the only one verifying conditions of Theorem 6.4. For example any regular ring R satisfies these conditions as the ring S built from R in Theorem 6.4.*

We now build SL extensions from a new type of rings whose construction is close to Cohn's rings. Many proofs are similar. If I is an ideal of a ring R, we set $R/\!\!/I := R[X]/XI[X]$. Since $XI[X] \cap R = 0$, we can consider that we have an extension $R \subseteq R/\!\!/I$, with $R \subset R/\!\!/I$ when $I \neq R$ because $X \in XR[X] \setminus XI[X]$ shows that $X - a \notin XI[X]$ for any $a \in R$.

Lemma 6.6. *If P is a prime ideal of a ring R, the extension $R \subseteq R/\!\!/P$ is SL, pure, t-closed and $R/\!\!/P$ is reduced if so is R. Moreover, $P = (R : R/\!\!/P)$, $R/\!\!/P \cong R \bigoplus (X(R/P)[X])$ and if $P \not\subseteq Q$, then, $\kappa(Q) \otimes_R (R[X]/XP[X]) \cong \kappa(Q)$ and if $P \subseteq Q$, then, $\kappa(Q) \otimes_R (R[X]/XP[X]) \cong \kappa(Q)[X]$ for any $Q \in \mathrm{Spec}(R)$.*

Any $a \in R$ is regular in $R/\!\!/P$ if and only if a is regular in R and $a \notin P$.

Proof. Set $P' := XP[X]$. Any $y \in P'$ can be written $y = \sum_{i=1}^{n} p_i X^i$, $p_i \in P$ $(*)$. Then, looking at the terms of degrees 0 and 1 in $(*)$, we get $P' \cap R = 0$ and $X \notin P'$.

An element of $R/\!\!/P = R[X]/XP[X]$ is of the form $y = a + xf(x)$, where x is the class of X in $R/\!\!/P$ and $a \in R$. As $R \cap XR[X] = 0$, we also have $R \cap xR[x] = 0$, so that $R/\!\!/P = R \bigoplus (xR[x])$, with $xP = 0$, a direct sum of R-modules. Obviously, $XR[X]/XP[X] \cong X(R/P)[X]$ as R-modules.

Now, $y \in \mathrm{U}(R/\!\!/P)$ if and only if there exists $z \in R/\!\!/P$ such that $yz = 1$ $(**)$. Set $y = a + xf(x)$ and $z = b + xg(x)$ with $f(X), g(X) \in R[X]$. Then $(**) \Leftrightarrow (a + xf(x))(b + xg(x)) = 1 \Leftrightarrow ab + x[ag(x) + bf(x) + xf(x)g(x)] = 1 \Leftrightarrow ab = 1$ (i) and $ag(X) + bf(X) + Xf(X)g(X) \in P[X]$ (ii). Since $(i) \Leftrightarrow a, b \in \mathrm{U}(R)$, then (i) and (ii) implies $f(X), g(X) \in P[X]$, according to Lemma 6.1, so that $y = a$, $z = b \in \mathrm{U}(R)$ which give $\mathrm{U}(R/\!\!/P) = \mathrm{U}(R)$ and $R \subseteq R/\!\!/P$ is SL.

We now prove the second statement. Let $\varphi : R[X] \to R$ be the ring morphism defined by $X \mapsto 0$. Since $P' \subseteq \ker(\varphi)$, there is a ring morphism $R/\!\!/P \to R$. It follows that $R/\!\!/P$ is a retract of R and the extension $R \to R/\!\!/P$ is pure.

We claim that $R \subseteq R/\!\!/P$ is t-closed. Let $y \in R/\!\!/P$ be such that $y^2 - ry, y^3 - ry^2 \in R$ for some $r \in R$. Assume that $y \notin R$. According to the beginning of the proof, we can write $y = a + xf(x)$ where $a \in R$, $f(X) := \sum_{i=0}^{n} \alpha_i X^i$ is a polynomial of $R[X]$, $n := \deg(f)$ and either $\alpha_i = 0$ or $\alpha_i \notin P$, with in particular, $\alpha_n \notin P$. Then, $y^2 - ry = (a^2 - ra) + (2a - r)xf(x) + x^2 f(x)^2$. But $y^2 - ry \in R$ implies that $\alpha_n^2 X^{2n+2} \in XPR[X]$, so that $\alpha_n^2 \in P$ and $\alpha_n \in P$, a contradiction. Then, any $y \in R/\!\!/P$ such that $y^2 - ry, y^3 - ry^2 \in R$ is in R and $R \subseteq R/\!\!/P$ is t-closed, and, in particular, is seminormal, whence $R/\!\!/P$ is reduced when R is reduced.

Now, let $a \in R$. Then, $a \in (R : R/\!\!/P) \Leftrightarrow aX^k \in R + XP[X]$ for each integer $k \geq 1 \Leftrightarrow a \in P$, so that $P = (R : R/\!\!/P)$ since $P(R : R/\!\!/P) \subseteq R$.

Set $S := R[X]$, so that $R/\!\!/P = S/P'$. Let $Q \in \mathrm{Spec}(R)$. Then, $\kappa(Q) \otimes_R S/P' = (S_Q/P'_Q)/Q(S_Q/P'_Q) \cong (S_Q/P'_Q)/((QS_Q + P'_Q)/P'_Q) \cong S_Q/(QS_Q + P'_Q) \cong R_Q[X]/(QR_Q[X] + XPR_Q[X])$.

If $P \not\subseteq Q$, then, $XPR_Q[X] = XR_Q[X]$ and $\kappa(Q) \otimes_R S/P' \cong R_Q[X]/(QR_Q[X] + XR_Q[X]) \cong R_Q/QR_Q = \kappa(Q)$.

If $P \subseteq Q$, then, $QR_Q[X] + XPR_Q[X] = QR_Q[X]$, so that $\kappa(Q) \otimes_R S/P' \cong R_Q[X]/QR_Q[X] \cong \kappa(Q)[X]$.

Let $a \in R$. If a is a regular element of $R/\!\!/P$, it is regular in R. If $a \in P$, then $aX \in P'$, so that $ax = 0$, with $x \neq 0$, a contradiction. Conversely, if a is regular in R and $a \notin P$, a is a regular element of $R/\!\!/P$. Otherwise, there exists $y \in R/\!\!/P$, $y \neq 0$ such that $ay = 0$. We may assume that $y \notin R$ since a is a regular element of R. As in the beginning of the proof, we can write $y = b + xf(x)$ where $b \in R$, $f(X) := \sum_{i=0}^{n} \alpha_i X^i$ is a polynomial of $R[X]$, $n := \deg(f)$ and either $\alpha_i = 0$ or $\alpha_i \notin P$, with $\alpha_n \notin P$. Then $ay = 0 = a(b + xf(x)) = ab + axf(x)$ implies $ab + aXf(X) \in P' = XPR[X]$, so that $ab = 0$ (iii) and $af(X) \in PR[X]$ (iv). Since a is a regular element of R, it follows that $b = 0$ by (iii) and (iv) gives $f(X) \in PR[X]$ because $a \notin P$. Then, $y = 0$, a contradiction, and a is a regular element of $R/\!\!/P$. □

Proposition 6.7. *If I is a semiprime ideal of a ring R, the extension $R \subseteq R/\!\!/I$ is SL, pure and t-closed. Moreover, $I = (R : R/\!\!/I)$ and is also semiprime in $R/\!\!/I$. If R is reduced, then $R/\!\!/I$ is reduced.*

Proof. Since I is semiprime, set $I := \cap_{\lambda \in \Lambda} P_\lambda$. Let $P \in \mathrm{Spec}(R)$ be such that $I \subseteq P$, so that $XI[X] \subseteq XP[X]$, and there is a surjective morphism $f : R[X]/XI[X] \to R[X]/XP[X]$ giving the following commutative diagram:

$$\begin{array}{ccc} R & \stackrel{i}{\rightarrow} & R[X]/XI[X] = R/\!\!/I \\ & j \searrow & \downarrow f \\ & & R[X]/XP[X] = R/\!\!/P \end{array}$$

with two injective morphisms i and j. This holds for any P_λ. According to Lemma 6.6, j is SL, which implies that f is SL by Proposition 2.2. It follows that $j(\mathrm{U}(R)) = \mathrm{U}(R/\!\!/P_\lambda) = f(\mathrm{U}(R/\!\!/I))$ $(*)$. Let $y \in \mathrm{U}(R/\!\!/I)$. Then, we may write $y = i(a_0) + \sum_{k=1}^{r} \overline{a_k}\overline{X}^k$, where $\overline{a_k}$ and $\overline{X}$ are respectively the classes of $a_k \in R$ and X in $R/\!\!/I$. It follows that $f(y) = j(a_0) + \sum_{k=1}^{r} \tilde{a_k}\tilde{X}^k$, where $\tilde{a_k}$ and $\tilde{X}$ are respectively the classes of $a_k \in R$ and X in $R/\!\!/P_\lambda$. We deduce from $(*)$ that $\sum_{k=1}^{r} \tilde{a_k}\tilde{X}^k = \tilde{0}$, giving $\sum_{k=1}^{r} a_k X^k \in XP_\lambda R[X]$, and then $a_k \in P_\lambda$ $(**)$ for each $k \geq 1$ and for each $\lambda \in \Lambda$. Moreover, $a_0 \in \mathrm{U}(R)$. Because $(**)$ holds for each P_λ, we get $a_k \in I$ for each $k \geq 1$ and then $\overline{a_k}\overline{X}^k = \overline{0}$ for each $k \geq 1$. To conclude, $y = i(a_0) \in i(\mathrm{U}(R))$ and $i(\mathrm{U}(R)) = \mathrm{U}(R/\!\!/I)$, that is $R \subseteq R/\!\!/I$ is SL.

It is enough to mimic the proofs of Lemma 6.6 for each P_λ to get that $R \subseteq R/\!\!/I$ is pure and t-closed. Moreover, $I = (R : R/\!\!/I)$. Since $R \subseteq R/\!\!/I$ is t-closed, it is also seminormal, so that $I = (R : R/\!\!/I)$ is semiprime in $R/\!\!/I$ by [7, Lemma 4.8].

If R is reduced, then $R/\!\!/I$ is reduced as in Lemma 6.6. □

Corollary 6.8. *Let $R \subseteq S$ be an extension with I an ideal of R and K a semiprime ideal of S such that $I = R \cap K$. Then, $R \subseteq S$ is SL if and only if $R/\!\!/I \subseteq S/\!\!/K$ is SL.*

Proof. Since $I = R \cap K$, we get that I a semiprime ideal of R and $I[X] = R[X] \cap K[X]$, so that the following commutative diagram holds:

$$\begin{array}{ccc} R & \subseteq & S \\ \downarrow & & \downarrow \\ R/\!\!/I & \subseteq & S/\!\!/K \end{array}$$

where the vertical maps are injective and SL morphisms by Proposition 6.7. If either $R \subseteq S$ (1) or $R/\!\!/I \subseteq S/\!\!/K$ (2) is SL, so is $R \to S/\!\!/K$ by Proposition 2.2 (1). If (1) holds, then (2) holds by Proposition 2.2 (3). If (2) holds, then (1) holds by the same Proposition because $S \to S/\!\!/K$ is injective. □

Theorem 6.9. Let R be a ring and $N \in \mathrm{Max}(R[X])$. The following conditions are equivalent:

1. $N \cap R \in \mathrm{Max}(R)$.

2. $M[X] \subseteq N$ for some $M \in \mathrm{Max}(R)$.

3. There exists a monic polynomial $f(X) \in N$.

If these conditions hold, then $N = M[X] + f(X)R[X]$ and $M = N \cap R$.

Proof. (1)⇒(2) Assume that $M := N \cap R \in \mathrm{Max}(R)$. It follows that $M \subseteq N$ which implies $M[X] \subseteq N$ for some $M \in \mathrm{Max}(R)$.

(2)⇒(3) Let $M \in \mathrm{Max}(R)$ be such that $M[X] \subseteq N$. Of course, $M[X]+R[X]f(X) \subseteq N$ for any $f(X) \in N$. Consider the following commutative diagram:

$$\begin{array}{ccc} R & \rightarrow & R[X] \\ \downarrow & & \downarrow \\ R/M & \rightarrow & R[X]/M[X] = (R/M)[X] \end{array}$$

Since $M \in \mathrm{Max}(R)$, it follows that R/M is a field and $(R/M)[X]$ is a PID. Then, there exists a monic polynomial $f(X) \in N$ such that $N/M[X] = \overline{f(X)}(R[X]/M[X])$, where $\overline{f(X)}$ is the class of $f(X)$ in $R[X]/M[X]$.

Then any element of N can be written $f(X)g(X)+h(X)$, where $g(X) \in R[X]$ and $h(X) \in M[X]$, giving $N = M[X] + R[X]f(X)$.

(3)⇒(1) Assume that there exists a monic polynomial $f(X) \in N$ and set $f(X) := X^n + \sum_{i=0}^{n-1} a_i X^i \in N$. Consider the following commutative diagram:

$$\begin{array}{ccc} R & \rightarrow & R[X] \\ \downarrow & & \downarrow \\ R/(N \cap R) & \rightarrow & R[X]/N = [R/(N \cap R)][x] \end{array}$$

where x is the class of X in $R[X]/N$. Since $N \in \mathrm{Max}(R[X])$, it follows that $[R/(N \cap R)][x]$ is a field. Moreover, $R/(N \cap R) \rightarrow R[X]/N$ is injective and $x^n + \sum_{i=0}^{n-1} \overline{a_i} x^i = 0$, where $\overline{a_i}$ is the class of a_i in $R/(N \cap R)$. This implies that $R[X]/N$ is integral over $R/(N \cap R)$, which gives that $R/(N \cap R)$ is a field by [17, Theorem 16] because $R/(N \cap R)$ is an integral domain. To conclude, $N \cap R \in \mathrm{Max}(R)$. If the equivalent conditions (1), (2) and (3) hold then $N = M[X] + f(X)R[X]$ by (2) and $M = N \cap R$. □

Proposition 6.10. *Let R be a ring. Then, $R \subseteq R /\!\!/ \mathrm{J}(R)$ is SL, $\mathrm{J}(R) = (R : R /\!\!/ \mathrm{J}(R)) = \mathrm{J}(R /\!\!/ \mathrm{J}(R))$ and $R /\!\!/ \mathrm{J}(R)$ is not* J*-regular.*

Proof. The two first assertions come from Proposition 6.7 since $\mathrm{J}(R)$ is semiprime.

Let $Q \in \mathrm{Max}(R /\!\!/ \mathrm{J}(R))$. Then, there exists $N \in \mathrm{Max}(R[X])$ such that $X\mathrm{J}(R)[X] \subseteq N$, with $Q = N/(X\mathrm{J}(R)[X])$. It follows that either $X \in N$ $(*)$ or $\mathrm{J}(R)R[X] \subseteq N$ $(**)$.

In case $(*)$, Theorem 6.9 shows that $M := N \cap R \in \mathrm{Max}(R)$ and $N = M[X] + XR[X]$. But $\mathrm{J}(R) \subseteq M$ implies that $\mathrm{J}(R)[X] \subseteq N$. Since $\mathrm{J}(R)[X] \subseteq N$ in case $(**)$, we get that $\mathrm{J}(R)[X] \subseteq N$ in both cases. It follows that $N/\mathrm{J}(R)[X] \in \mathrm{Max}(R[X]/\mathrm{J}(R)[X])$.

Conversely, if N is a maximal ideal of $R[X]$ containing $\mathrm{J}(R)[X]$, it follows that $N/(\mathrm{J}(R)[X]) \in \mathrm{Max}(R[X]/(\mathrm{J}(R)[X]))$ and $N/(X\mathrm{J}(R)[X]) \in \mathrm{Max}(R[X]/(X\mathrm{J}(R)[X]))$.

Recall that for a ring T, we have $\mathrm{J}(T[X]) = \mathrm{Nil}(T[X])$. Moreover, $R[X]/\mathrm{J}(R)[X] \cong (R/\mathrm{J}(R))[X]$ which is reduced. It follows that $\cap[N/\mathrm{J}(R)[X] \mid \mathrm{J}(R)[X] \subseteq N$ and $N \in \mathrm{Max}(R[X])] = \mathrm{J}(R[X]/\mathrm{J}(R)[X])$ $= \mathrm{J}((R/\mathrm{J}(R))[X]) = \overline{0}$ $(*)$ where $\overline{0}$ the class of 0 in $R[X]/\mathrm{J}(R)[X]$.

To make the reading easier, we set $I := \mathrm{J}(R)[X]$, $K := \cap[N \in \mathrm{Max}(R[X]) \mid I \subseteq N]$ and $S := R[X]$. Of course, $I \subseteq K$. We have proved that $\cap[N/I \mid I \subseteq N$ and $N \in \mathrm{Max}(S)] = \overline{0}$. Assume that $I \subset K$, so that there exists some $y \in K \setminus I$. This means that y is in any $N \in \mathrm{Max}(S)$ which contains I with $y \notin I$. If $\overline{y}$ the class of y in S/I, we get that $\overline{y} \in \cap[N/I \mid I \subseteq N$ and $N \in \mathrm{Max}(S)] = \overline{0}$ by $(*)$, giving $y \in I$, a contradiction. Therefore, $\cap[N \in \mathrm{Max}(S) \mid XI \subseteq N] = I$ which gives $\mathrm{J}(R/\!\!/\mathrm{J}(R)) = \cap[Q \in \mathrm{Max}(R/\!\!/\mathrm{J}(R))] = \cap[N/XI \in \mathrm{Max}(R/\!\!/\mathrm{J}(R))] = I/IX = \mathrm{J}(R)[X]/X\mathrm{J}(R)[X] = \mathrm{J}(R)$.

At last, $(R/\!\!/\mathrm{J}(R))/\mathrm{J}(R) = (R[X]/X\mathrm{J}(R)[X])/(\mathrm{J}(R)[X]/X\mathrm{J}(R)[X])$ $\cong R[X]/\mathrm{J}(R)[X] \cong (R/\mathrm{J}(R))[X]$, which is not zero-dimensional, giving that $R/\!\!/\mathrm{J}(R)$ is not J-regular. □

7 The ring $R\{X\}$

We now consider a ring used by Houston and some other authors for results concerning the dimension of rings [15]. When R is a semilocal (Noetherian) domain, he introduces the ring $R\langle X\rangle$.

This notation is in conflict with the notation of the ring used to solve the Serre conjecture. Therefore, we will denote it by $R\{X\}$. Let Σ be the multiplicatively closed subset of $R[X]$ defined as follows: let T_1 be the set of all maximal ideals N of $R[X]$ such that $N \cap R \in \mathrm{Max}(R)$, then Σ is the complementary set in $R[X]$ of $\cup[N \mid N \in T_1]$ and $R\{X\} := R[X]_\Sigma$. As we consider arbitrary rings, a more precise characterization of T_1 is given in Theorem 6.9, completing [15, Lemma 1.2].

Proposition 7.1. *For any ring R, there is a factorization $R \to R\{X\} \to R(X)$, where $R(X)$ is the Nagata ring of R.*

Proof. Let $p(X) \in \Sigma$. We claim that $c(p) = R$. Otherwise, there exists $M \in \mathrm{Max}(R)$ such that $c(p) \subseteq M$. Then $p(X) \in M[X] \subseteq N$ for some $N \in \mathrm{Max}(R[X])$. It follows that $M = N \cap R$, so that $N \in T_1$ by Theorem 6.9 and $p(X) \in N$, a contradiction with $p(X) \in \Sigma$. Since $c(p) = R$, we get that $p(X)/1 \in \mathrm{U}(R(X))$ giving the factorization $R \to R\{X\} \to R(X)$. □

We recall that R is a *Jacobson ring* if and only if maximal ideals of $R[X]$ contract to maximal ideals of R. Then we have the following:

Proposition 7.2. *Let R be a Jacobson ring. Then $R\{X\} = R[X]$.*

Proof. Since R is a Jacobson ring, it follows that $T_1 = \mathrm{Max}(R[X])$ where $T_1 = \{N \in \mathrm{Max}(R[X]) \mid N \cap R \in \mathrm{Max}(R)\}$. This implies that $\Sigma = R[X] \setminus \cup[N \mid N \in T_1] = R[X] \setminus \cup[N \mid N \in \mathrm{Max}(R[X])] = \mathrm{U}(R[X])$ and $R\{X\} := R[X]_\Sigma = R[X]$. □

Mimicking [15] and using also [12, Lemma 3], the following proposition characterizes in a more general setting, and when R is J-regular, the maximal ideals of $R\{X\}$.

Proposition 7.3. *If R is* J*-regular,* $\mathrm{Max}(R\{X\}) = \{NR\{X\} \mid N \in T_1\}$.

Proof. Since R is J-regular, we get that $\mathrm{V}(\mathrm{J}(R)) = \mathrm{Max}(R)$. Let $N \in \mathrm{Max}(R[X])$. Then, $N \in T_1$ if and only if $N \cap R \in \mathrm{Max}(R)$ if and only if $\mathrm{J}(R) \subseteq N$ if and only if $\mathrm{J}(R)[X] \subseteq N$. Then, $T_1 = \mathrm{Max}(R[X]) \cap \mathrm{V}(\mathrm{J}(R)[X])$. According to [12, Lemma 3], and since $R\{X\} = R[X]_\Sigma$, it follows that $\mathrm{Max}(R\{X\}) = \{NR\{X\} \mid N \in T_1\}$. □

Proposition 7.4. *If M is a maximal ideal of R, then there is a factorization $R \to R\{X\} \to R/\!\!/M$ and $R \to R\{X\}$ is faithfully flat.*

Proof. Since $R\{X\} = R[X]_\Sigma$, we have $\mathrm{U}(R\{X\}) = \{p(X)/q(X) \mid p(X), q(X) \in \Sigma\}$. Let $p(X) \in \Sigma$, so that $p(X) \notin Q$ for any $Q \in T_1$. Let $M \in \mathrm{Max}(R)$. We claim that $p(X)R[X] + XM[X] = R[X]$. Otherwise, there exists some $N \in \mathrm{Max}(R[X])$ such that $p(X) \in N$ and $XM[X] \subseteq N$ (*). But (*) holds if and only if either $X \in N$ (1) or $M[X] \subseteq N$ (2). In case (1), N contains the monic polynomial X, so that $N \in T_1$ by Theorem 6.9, and in case (2), the same reference shows that $N \in T_1$. In both cases, we infer that we get a contradiction with $N \in T_1$ and $p(X) \in N$ since $p(X) \notin Q$ for any $Q \in T_1$, so that $p(X)R[X] + XM[X] = R[X]$ holds. This implies that there exist $f(X) \in R[X]$ and $g(X) \in M[X]$ such that $p(X)f(X) + Xg(X) = 1$.

Working in $R/\!\!/M = R[X]/XM[X]$, this gives $\overline{p(X)}\ \overline{f(X)} = \overline{1}$, so that $\overline{p(X)} \in \mathrm{U}(R/\!\!/M)$. Then, we have the wanted factorization thanks to the following commutative diagram:

$$\begin{array}{ccccc} R & \to & R[X] & \to & R/\!\!/M = R[X]/XM[X] \\ & \searrow & \downarrow & \nearrow & \\ & & R\{X\} = R[X]_\Sigma & & \end{array}$$

Consider now the following factorization $R \to R[X] \to R\{X\} \to R/\!\!/M$. By Lemma 6.6, the extension $R \subseteq R/\!\!/M$ is pure. Then, so is $R \subset R\{X\}$ by [24, Lemme 2.3, p.19]. Moreover, $R \subseteq R[X]$ is flat as well as $R[X] \subseteq R\{X\}$. Then, so is $R \subset R\{X\}$. Since the maximal ideals of R can be lifted on in $R\{X\}$, then $R \to R\{X\}$ is faithfully flat. □

Proposition 7.5. *Let $R \subseteq S$ be an integral ring extension. Then, there is an integral ring extension $R\{X\} \subseteq S\{X\}$. In case S is* J*-regular, then $R\{X\} \subseteq S\{X\}$ is SL if and only if $R = S$.*

Proof. Set $T_1 := \{N \in \mathrm{Max}(R[X]) \mid N \cap R \in \mathrm{Max}(R)\}$, $T_1' := \{N' \in \mathrm{Max}(S[X]) \mid N' \cap S \in \mathrm{Max}(S)\}$, $\Sigma := R[X] \setminus \cup[N \mid N \in T_1]$ and $\Sigma' := S[X] \setminus \cup[N' \mid N' \in T_1']$. We have the following diagram:

$$\begin{array}{ccccc} R & \to & R[X] & \to & R\{X\} \\ \downarrow & & \downarrow & & \\ S & \to & S[X] & \to & S\{X\} \end{array}$$

Since $R \subseteq S$ is integral, so is $f : R[X] \subseteq S[X]$. Let $N \in T_1$, so that $M := N \cap R \in \mathrm{Max}(R)$, and there exists $N' \in \mathrm{Max}(S[X])$ lying above N. Set $M' := N' \cap S \in \mathrm{Spec}(S)$. Then, $M' \cap R = N' \cap S \cap R = N' \cap R[X] \cap S \cap R = N \cap R = M \in \mathrm{Max}(R)$. It follows that $M' \in \mathrm{Max}(S)$ since it is a prime ideal of S lying above a maximal ideal of R. Then, $N' \in T_1'$. Conversely, let $N' \in T_1'$ and set $N := N' \cap R[X]$. Then, $M' := N' \cap S \in \mathrm{Max}(S)$ and $N \cap R = N' \cap R[X] \cap R = N' \cap S \cap R[X] \cap R = M' \cap R \in \mathrm{Max}(R)$ since $M' \in \mathrm{Max}(S)$. It follows that there is a surjective map $T_1' \to T_1$ which is the restriction of ${}^af : \mathrm{Spec}(S[X]) \to \mathrm{Spec}(R[X])$.

Now, let $p(X) \in \Sigma$, so that $p(X) \notin N$ for any $N \in T_1$. We claim that $p(X) \in \Sigma'$. Otherwise, there exists $N' \in T_1'$ such that $p(X) \in N'$. But $p(X) \in R[X]$, so that $p(X) \in N' \cap R[X] \in T_1$ by the beginning of the proof, a contradiction. Then, $\Sigma \subseteq \Sigma'$ and $\mathrm{U}(R\{X\}) \subseteq \mathrm{U}(S\{X\})$.

We show that f defines an injective morphism $R\{X\} \to S\{X\}$. Let $p(X)/q(X) \in R\{X\}$, $p(X), q(X) \in R[X]$, $q(X) \in \Sigma$ be such that $p(X)/q(X) = 0$ in $S\{X\}$. There exist $g(X) \in \Sigma'$ such that $p(X)g(X) = 0$ $(*)$ in $S[X]$. Then, $g(X) \notin N'$, for any $N' \in T_1'$. We claim that $c(g) = S$. Otherwise, there exists $N \in \mathrm{Max}(S)$ such that $c(g) \subseteq N$. Then $g(X) \in N[X] \subseteq N'$ for some $N' \in \mathrm{Max}(S[X])$. It follows that $N' \in T_1'$ by Theorem 6.9 and $g(X) \in N'$, a contradiction with $g(X) \notin N'$, for any $N' \in T_1'$. Since, $c(g) = S$, we obtain that g is a regular element of $S[X]$ and it results from $(*)$ that $p(X) = 0$. Then there is an integral ring extension $R\{X\} \subseteq S\{X\}$.

If $R = S$, obviously $R\{X\} \subseteq S\{X\}$ is SL.

Conversely, assume that S is J-regular and $R\{X\} \subseteq S\{X\}$ is SL. We mimic the proof of Proposition 4.6. Let $s \in S$ and set $p(X) := s + X \in S\{X\}$. Since S is J-regular, then $\mathrm{Max}(S\{X\}) = \{NS\{X\} \mid N \in T_1'\}$ by Proposition 7.3, so that $p(X) \in \mathrm{U}(S\{X\}) = \mathrm{U}(R\{X\})$. It follows that there exists $h(X), k(X) \in \Sigma$ such that $p(X)/1 = h(X)/k(X)$ in $S\{X\}$, and then there exists $q(X) \in \Sigma'$ such that $q(X)p(X)k(X) = q(X)h(X)$. Because $q(X) \in \Sigma'$, we get that $c(q) = S$ (see the proof we give in the previous paragraph), which implies that $q(X)$ is a regular element of $S[X]$. Then $p(X)k(X) = h(X) = (s+X)k(X)$. As $Xk(X)$ and $h(X)$ are in $R[X]$, it follows that $sk(X) \in R[X]$. Now, R is also J-regular by Corollary 5.7. The same proof as for $q(X) \in S[X]$ before shows that $c(k) = R$. Set $k(X) := \sum_{i=0}^n a_i X^i$, $a_i \in R$ and $b_i := sa_i \in R$ $(*)$ for each $i \in \{0, \ldots, n\}$. There exist $\lambda_0, \ldots, \lambda_n \in R$ such that $\sum_{i=0}^n \lambda_i a_i = 1$. Multiplying each equality of $(*)$ by λ_i and adding each of these equalities for each $i \in \{0, \ldots, n\}$, we get $s(\sum_{i=0}^n \lambda_i a_i) = s = \sum_{i=0}^n \lambda_i b_i \in R$, so that $S = R$. □

Corollary 7.6. *Let $R \subseteq S$ be an integral ring extension. If $R\{X\} \subseteq S\{X\}$ is SL, so is $R \subseteq S$.*

Proof. Let $s \in \mathrm{U}(S)$. In particular, $s/1 \in \mathrm{U}(S\{X\}) = \mathrm{U}(R\{X\})$. Then $s/1 = f(X)/g(X)$ in $S\{X\}$ for some $f(X)/g(X) \in \mathrm{U}(R\{X\})$, that is $f(X), g(X) \in \Sigma$. The same proof as in Proposition 7.5 shows that $sg(X) = f(X)$ $(*)$ in $S[X]$ and $c(f) = c(g) = R$. Set $f(X) := \sum_{i=0}^{n} a_i X^i, a_i \in R$ and $g(X) := \sum_{i=0}^{n} b_i X^i, b_i \in R$, so that $sb_i = a_i$ for each $i \in \{0,\dots,n\}$ $(**)$. There exist $\lambda_0,\dots,\lambda_n \in R$ such that $\sum_{i=0}^{n} \lambda_i b_i = 1$. Multiplying each equality of $(**)$ by λ_i and adding each of these equalities for each $i \in \{0,\dots,n\}$, we get $s(\sum_{i=0}^{n} \lambda_i b_i) = s = \sum_{i=0}^{n} \lambda_i a_i \in R$, so that $s \in R$. Then, $(*)$ shows that $sc(g) = c(f) = R$ and gives that $Rs = R$, that is $s \in \mathrm{U}(R)$ and $\mathrm{U}(R) = \mathrm{U}(S)$. To conclude, $R \subseteq S$ is SL. □

Corollary 7.7. *Let R be a* J*-regular ring. Then there is a factorization $R \to R\{X\} \to R /\!\!/ \mathrm{J}(R)$.*

Proof. We use the beginning of the proof of Proposition 7.4. Let $p(X) \in \Sigma$, so that $p(X) \notin Q$ for any $Q \in T_1$. We claim that $\overline{p(X)} \in \mathrm{U}(R /\!\!/ \mathrm{J}(R))$. Otherwise, there exists some $N \in \mathrm{Max}(R[X])$ with $X\mathrm{J}(R)[X] \subseteq N$ such that $p(X) \in N$. As in the quoted proof, $X \notin N$ because $N \notin T_1$. It follows that $\mathrm{J}(R)[X] \subseteq N$. But R being J-regular, any prime ideal of R containing $J(R)$ is maximal. As $\mathrm{J}(R)[X] \subseteq N$, we get $\mathrm{J}(R) \subseteq N \cap R$, so that $N \cap R \in \mathrm{Max}(R)$, giving $N \in T_1$, a contradiction. Then, $\overline{p(X)} \in \mathrm{U}(R /\!\!/ \mathrm{J}(R))$, and we get the wanted factorization thanks to the following commutative diagram:

$$\begin{array}{ccccc} R & \to & R[X] & \to & R /\!\!/ \mathrm{J}(R) \\ & \searrow & \downarrow & \nearrow & \\ & & R\{X\} = R[X]_{\Sigma} & & \end{array}$$

□

Corollary 7.8. *Let R be a ring and I an ideal intersection of finitely many maximal ideals M_i. Then, $R \to R /\!\!/ I$ is SL and there is a factorization $R \to R\{X\} \to R /\!\!/ I \to R /\!\!/ M_i$ for any M_i.*

If, moreover, R is J*-regular, there is a factorization $R \to R\{X\} \to R /\!\!/ \mathrm{J}(R) \to R /\!\!/ I \to R /\!\!/ M_i$ for any M_i.*

Proof. By Proposition 6.7, $R \to R /\!\!/ I$ is SL. For the other assertions, as we have the inclusions $\mathrm{J}(R) \subseteq I \subseteq M_i$ for any M_i, we get the wanted factorization, mimicking the proof of Corollary 7.7, and using the fact that R/I is regular as a product of finitely many fields. □

8 FCP SL extensions

Proposition 8.1. *Let $R \subseteq S$ be an SL extension where S is a semilocal ring. Then R is semilocal,* $\mathrm{J}(R) = \mathrm{J}(S)$ *and $R \subseteq S$ is an FIP seminormal and infra-integral extension.*

Proof. Since S is semilocal, S is J-regular. According to Remark 5.15 (3), Theorem 5.17 and Proposition 5.18, we have $\mathrm{J}(R) = \mathrm{J}(S)$ and we can assume that $R \subseteq S$ is seminormal and infra-integral, R and S are regular and semi-local and that S is a finite product of fields each of them being a residual field of R. Finally the extension can be viewed as a product of extensions $R/M \to (R/M)^n$ where $M \in \mathrm{Max}(R)$. By using suitable localizations and [8, Proposition 4.15], we get that $R \subseteq S$ is an FIP seminormal and infra-integral extension. □

Corollary 8.2. *Let $R \subseteq S$ be an SL extension where S is a semilocal ring. If $R \subseteq S$ is a flat epimorphism, then $R = S$.*

Proof. By Proposition 8.1, $R \subseteq S$ is integral and a flat epimorphism, then $R = S$. □

Theorem 8.3. An extension $R \subseteq S$, where S is a semilocal ring, is SL if and only if $R \subseteq S$ is a seminormal infra-integral FIP extension such that $R/M \cong \mathbb{Z}/2\mathbb{Z}$ for each $M \in \mathrm{MSupp}(S/R)$.

Proof. Assume that $R \subseteq S$ is SL. According to Proposition 8.1, we get that $\mathrm{J}(R) = \mathrm{J}(S)$ and $R \subseteq S$ is an FIP seminormal and infra-integral extension. Set $I := J(S) = J(R) = \cap_{i=1}^n M_i = \cap[M_{i,j} \mid M_{i,j} \in \mathrm{Max}(S)]$, where $\mathrm{Max}(R) = \{M_i\}_{i=1}^n$ and $\mathrm{Max}(S) = \{M_{i,j} \mid i = 1,\ldots,n,\ M_i = M_{i,j} \cap R$ for each $j \in \mathbb{N}_{n_i}\}$. By Remark 5.15 (3), $R/I \subseteq S/I$ is SL. But $R/I \cong \prod_{i=1}^n R/M_i$ and $S/I \cong \prod[S/M_{i,j} \mid M_{i,j} \in \mathrm{Max}(S)]$. Set $X_1 := \mathrm{MSupp}(S/R)$, $X_2 := \mathrm{Max}(R) \setminus X_1$, $I_1 := \cap_{M \in X_1} M$, $I_2 := \cap_{M \in X_2} M$, $R_1 := R/I_1 = \prod_{M \in X_1} R/M$ and $R_2 := R/I_2 = \prod_{M \in X_2} R/M$. Then, $R/I = R_1 \times R_2$ because I_1 and I_2 are comaximal. It follows that $\mathrm{U}(R/I) = \mathrm{U}(R_1) \times \mathrm{U}(R_2)$. For $M_i \in X_2$, there is a unique $M_{i,j}$ lying above M_i and we have $R/M_i \cong S/M_{i,j}$, so that $R_2 \cong S_2 := \prod_{M_i \in X_2} S/M_{i,j}$ and $\mathrm{U}(R_2) \cong \mathrm{U}(S_2)$ (∗). Set $S_1 := \prod_{M_i \in X_1} S/M_{i,j} = \prod_{M_i \in X_1}[\prod_{j \in \mathbb{N}_i} S/M_{i,j}]$. Since $R \subseteq S$ is infra-integral and $M_i = M_{i,j} \cap R$ for each $j \in \mathbb{N}_{n_i}$, we get that $R/M_i \cong S/M_{i,j}$, so that $S_1 \cong \prod_{M_i \in X_1}(R/M_i)^{n_i}$. Whence $\mathrm{U}(S_1) \cong \prod_{M_i \in X_1}[\mathrm{U}(R/M_i)]^{n_i}$ (∗∗). Since $R \subseteq S$ is SL and because of (∗), we get that $\mathrm{U}(R_1) \cong \mathrm{U}(S_1) \cong \prod_{M_i \in X_1}[\mathrm{U}(R/M_i)]^{n_i}$. But, for any $M_i \in X_1$, we have $R_{M_i} \subset S_{M_i}$ seminormal and infra-integral, which gives that $n_i > 1$ for any $M_i \in X_1$. Then, $\mathrm{U}(R_1) \cong \mathrm{U}(S_1)$ if and only if $|\mathrm{U}(R/M_i)| = 1$ for any $M_i \in X_1$. Since R/M_i is a field for any $M_i \in X_1$, it follows that $R/M_i \cong \mathbb{Z}/2\mathbb{Z}$ for each $M_i \in \mathrm{MSupp}(S/R)$.

Conversely, assume that S is a semilocal ring and $R \subseteq S$ is a seminormal infra-integral FIP extension such that $R/M \cong \mathbb{Z}/2\mathbb{Z}$ for each $M \in \mathrm{MSupp}(S/R)$. We can use the results and notation of the first part of the proof related to the sets I_1, I_2, R_1, R_2 and S_1, S_2. Since $R \subseteq S$ is seminormal FIP with S semilocal, we get that $(R : S) = \cap_{M_i \in X_1} M_i = \cap_{M_i \in X_1} M_{i,j}$ by [34, Proposition 2.4]. In addition, $\mathrm{J}(S) = \cap_{i=1}^n M_{i,j} = (R : S) \cap_{M_i \in X_2} M_{i,j} = (R : S) \cap_{M_i \in X_2} M_i = \mathrm{J}(R)$. Set $I := J(S) = J(R)$. We still have $\mathrm{U}(R/I) = \mathrm{U}(R_1) \times \mathrm{U}(R_2)$, $\mathrm{U}(R_2) \cong \mathrm{U}(S_2)$, $\mathrm{U}(S/I) = \mathrm{U}(S_1) \times \mathrm{U}(S_2)$ and $U(S_1) \cong \prod_{M_i \in X_1}[\mathrm{U}(R/M_i)]^{n_i}$. Since $R/M_i \cong \mathbb{Z}/2\mathbb{Z}$ and $R/M_i \cong S/M_{i,j}$ for any $M_i \in X_1$ and $M_{i,j}$ lying over M_i, it follows that $|\mathrm{U}(R/M_i)| = 1 = |\mathrm{U}(S/M_{i,j})|$ for any $M_i \in X_1$, so that $\mathrm{U}(R_1) \cong \mathrm{U}(S_1)$, which yields that $R/I \subseteq S/I$ is SL and $R \subseteq S$ is SL by Corollary 2.7. □

Corollary 8.4. *A seminormal infra-integral FCP extension $R \subseteq S$ is SL if $R/M \cong \mathbb{Z}/2\mathbb{Z}$ for each $M \in \mathrm{MSupp}(S/R)$.*

Proof. Since $R \subseteq S$ is an FCP extension, it follows that $\mathrm{MSupp}(S/R)$ has finitely many elements. Set $I := (R : S)$. Moreover, $R \subseteq S$ being seminormal infra-integral, I is an intersection of finitely many maximal ideals of S by [34, Proposition 2.4]. It follows that the extension $R/I \subseteq S/I$ satisfies the assumptions of Theorem 8.3 because $R/I \subseteq S/I$ has FIP by [8, Proposition 4.15], and is then SL. Now, it is enough to use Corollary 2.7 to get that $R \subseteq S$ is SL. □

The following result did not seem to appear in earlier papers: we show that there is a seminormal infra-integral closure for FCP extensions. This will be useful to build a closure for SL extensions in an FCP extension.

Lemma 8.5. *Let $R \subseteq S$ be an FCP extension and $T, V \in [R, S]$ be such that $R \subseteq T$ and $R \subseteq V$ are both seminormal infra-integral. Then, $R \subseteq TV$ is seminormal infra-integral.*

Proof. Since $R \subseteq T$ and $R \subseteq V$ are both infra-integral, we have $T, V \in [R, {}_S^tR]$, giving $TV \in [R, {}_S^tR]$, so that $R \subseteq TV$ is infra-integral.

We claim that $R \subseteq TV$ is seminormal. Let $M \in \mathrm{MSupp}(TV/R)$, so that $(TV)_M = T_M V_M \neq R_M$. In particular, either $T_M \neq R_M$ or $V_M \neq R_M$. In both cases, $R_M \subseteq T_M$ and $R_M \subseteq V_M$ are either seminormal or equality. Set $R' := R_M$, $T' := T_M$, $V' := V_M$ and $M' := MR_M$. According to [34, Proposition 2.4], we get that $M' \subseteq (R' : T')$ and $M' \subseteq (R' : V')$, giving $M'T' = M'V' = M' \subseteq R'$, so that $M'T'V' = M' \subseteq R'$. It follows that $M' = (R' : T'V')$. Set $M' = \cap_{i=1}^n M'_i = \cap_{j=1}^m N'_j$ (∗), where the M'_i are in $\mathrm{Max}(T')$ (resp. N'_j are in $\mathrm{Max}(V')$) because $R' \subseteq T'$ and $R' \subseteq V'$ are either seminormal or equality. It follows that $T'/M'_i \cong R'/M' \cong V'/N'_j$ for each i and j since $R' \subseteq T'$ and $R' \subseteq V'$ are both infra-integral. In addition, $T'/M' \cong (R'/M')^n$ and $V'/M' \cong (R'/M')^m$ by (∗). This implies that $(T'V')/M' \cong (T'/M')(V'/M') \cong (R'/M')^{n+m}$ where R'/M' is a field. According [8, Proposition 4.15], we get that $R'/M' \subseteq (T'V')/M'$ is seminormal infra-integral as $R' = R_M \subseteq T'V' = T_M V_M = (TV)_M$. Since this holds for any $M \in \mathrm{MSupp}(TV/R)$, we get that $R \subseteq TV$ is seminormal infra-integral. □

Proposition 8.6. *Let $R \subseteq S$ be an FCP extension. There exists a greatest $T \in [R,S]$ such that $R \subseteq T$ is seminormal infra-integral. It satisfies the following properties:*

1. $T = \Pi[V \in [R,S] \mid R \subseteq V$ *seminormal infra-integral*$]$.
2. $T = \sup[V \in [R,S] \mid R \subseteq V$ *seminormal infra-integral*$]$.
3. $T = \cup[V \in [R,S] \mid R \subseteq V$ *seminormal infra-integral*$]$.

Proof. Set $\mathcal{F} := \{V \in [R,S] \mid R \subseteq V$ seminormal infra-integral$\}$. The result is obvious if $\mathcal{F} = \{R\}$. So, assume that $\mathcal{F} \neq \{R\}$. Since $R \subseteq V$ is infra-integral for any $V \in \mathcal{F}$, it follows that $\mathcal{F} \subseteq [R, {}_S^tR]$. In particular, any $V \in \mathcal{F}$ is integral over R. Then, we may assume that $R \subseteq S$ is infra-integral. Since $R \subseteq S$ has FCP, $\mathcal{F}$ has maximal elements. We claim that $\mathcal{F}$ has only one maximal element. Otherwise, there exist V, V', $V \neq V'$ which are maximal elements of $\mathcal{F}$. Then, $VV' \notin \mathcal{F}$. According to Lemma 8.5, we get that $R \subseteq VV'$ is seminormal infra-integral, a contradiction. Then, $\mathcal{F}$ has only one maximal element. Let T be this maximal element. Equalities (1), (2) and (3) follow obviously because $V \subseteq T$ for any $V \in \mathcal{F}$ and $T \in \mathcal{F}$. □

There are four types of minimal extensions, but only two types are used in the paper, characterized in [7, Theorems 2.1 and 2.2] and [32, Proposition 4.5]. We recall some results about minimal extensions:

Proposition 8.7. *(1) [7, Theorem 2.1] Let $R \subseteq T$ be a minimal integral extension. Then, $M := (R:T) \in \mathrm{Max}(R)$ and there is a bijection $\mathrm{Spec}(T) \setminus \mathrm{V}_T(M) \to \mathrm{Spec}(R) \setminus \{M\}$, with at most two maximal ideals of T lying above M.*

(2) Let $R \subseteq S$ be an FCP extension. Then, any maximal chain of $[R,S]$ results from juxtaposing finitely many minimal extensions.

Proof. Obvious. □

Definition 8.8. *(1) Let $R \subset T$ be an extension and $M := (R:T)$. Then $R \subset T$ is minimal decomposed if and only if $M \in \mathrm{Max}(R)$ and there exist $M_1, M_2 \in \mathrm{Max}(T)$ such that $M = M_1 \cap M_2$ and the natural maps $R/M \to T/M_i$ for $i \in \{1,2\}$ are both isomorphisms. A minimal decomposed extension is seminormal infra-integral.*

(2) A minimal extension $R \subset T$ is minimal Prüfer if and only if $R \subset T$ is a flat epimorphism and there exists $M \in \mathrm{Max}(R)$ such that $MT = T$ with $R_P \cong T_P$ for any $P \in \mathrm{Spec}(R)$, $P \neq M$ [10, Theorem 2.2]. In particular, there is a bijection $\mathrm{Spec}(T) \to \mathrm{Spec}(R) \setminus \{M\}$.

Lemma 8.9. *Let $R \subseteq S$ be an FCP integral extension where S is semilocal.*

1. *Let $T \in]R,S]$ be such that $R \subset T$ is minimal. Then, $R \subset T$ is SL if and only if $R \subset T$ is minimal decomposed such that $M := (R:T)$ satisfies $R/M \cong \mathbb{Z}/2\mathbb{Z}$.*
2. *Let $T \in]R,S]$. Then $R \subset T$ is SL if and only if $T \in [R, {}_S^uR]$ and $T/(R:T) \cong (\mathbb{Z}/2\mathbb{Z})^m$ for some positive integer m.*
3. *Let $T, V \in [R,S]$ be such that $R \subset T$ and $R \subset V$ are SL. Then, $R \subset TV$ is SL.*

Proof. Since $R \subseteq S$ is an FCP integral extension such that S is semilocal, any element of $[R,S]$ is semilocal.

(1) Let $T \in [R,S]$ be such that $R \subset T$ is minimal SL. Then, T is semilocal and $R \subset T$ satisfies conditions of Theorem 8.3, that $R \subset T$ is a seminormal infra-integral extension such that $R/M \cong \mathbb{Z}/2\mathbb{Z}$ for $M = (R : T)$ since $\mathrm{MSupp}(T/R) = \{M\}$. It follows that $R \subset T$ is minimal decomposed by [32, Proposition 4.5].

Conversely, if $R \subset T$ is minimal decomposed such that $M := (R : T)$ satisfies $R/M \cong \mathbb{Z}/2\mathbb{Z}$, then $R \subset T$ is seminormal infra-integral with $R/M \cong \mathbb{Z}/2\mathbb{Z}$ for $M \in \mathrm{MSupp}(T/R)$ by the previous reference. Then Theorem 8.3 shows that $R \subset T$ is SL.

(2) According to Theorem 8.3, $R \subseteq T$ is SL if and only if (i) and (ii) hold where :

(i) $R \subseteq T$ is a seminormal infra-integral FIP extension.

(ii) $R/M \cong \mathbb{Z}/2\mathbb{Z}$ for each $M \in \mathrm{MSupp}(T/R)$.

Assume first that $R \subseteq T$ is SL. By (i) and [35, Theorem 5.9], we get that $R \subseteq T$ is u-integral, so that $T \in [R, {}_S^uR]$. Moreover, since $R \subseteq T$ is seminormal, it follows that $(R : T) = \cap[N_{i,j} \mid N_{i,j} \in \mathrm{Max}(T),\ N_{i,j} \cap R = M_i]$, where $\mathrm{MSupp}(T/R) := \{M_i\}_{i=1}^n$. Then, $T/(R : T) \cong T/(\cap N_{i,j}) \cong \Pi(T/N_{i,j}) \cong \Pi(R/M_i)^{n_i}$, where $n_i := |\{N_{i,j} \in \mathrm{Max}(T) \mid N_{i,j} \cap R = M_i\}|$ and because of (i). Then, (ii) gives that $T/(R : T) \cong (\mathbb{Z}/2\mathbb{Z})^m$ for some positive integer m.

Conversely, assume that $T \in [R, {}_S^uR]$ and $T/(R : T) \cong (\mathbb{Z}/2\mathbb{Z})^m$ for some positive integer m. Since $T \in [R, {}_S^uR]$, we get that $R \subset T$ is u-integral, and in particular infra-integral. Moreover, since $T/(R : T) \cong (\mathbb{Z}/2\mathbb{Z})^m$, a product of finitely many finite fields, $T/(R : T)$ is reduced, so that $(R : T)$ is an intersection of finitely many maximal ideals of T (and R), and $R \subset T$ is seminormal by [34, Proposition 2.4]. Then, (i) holds because $T/(R : T)$ has finitely many elements, giving that $R \subseteq T$ has FIP. At last, $(R : T)$ is semi-prime, it is an intersection of the maximal ideals $N_{i,j}$ of T lying above the maximal ideals M_i of R of $\mathrm{MSupp}(T/R)$. It follows that $R/M \cong \mathbb{Z}/2\mathbb{Z}$ for each $M \in \mathrm{MSupp}(T/R)$ and (ii) holds. To conclude, $R \subseteq T$ is SL (we also can use Corollary 8.4).

(3) Let $T, V \in [R,S]$ be such that $R \subset T$ and $R \subset V$ are SL. Then $R \subset T$ and $R \subset V$ are both seminormal infra-integral, and so is $R \subset TV$ by Lemma 8.5. Moreover, $\mathrm{MSupp}(TV/R) = \mathrm{MSupp}(T/R) \cup \mathrm{MSupp}(V/R)$, so that any $M \in \mathrm{MSupp}(TV/R)$ is either in $\mathrm{MSupp}(T/R)$ or in $\mathrm{MSupp}(V/R)$, which implies that $R/M \cong \mathbb{Z}/2\mathbb{Z}$, whence $R \subseteq TV$ is SL by Corollary 8.4. □

Theorem 8.10. Let $R \subseteq S$ be an FCP extension where S is semilocal. There exists a greatest $T \in [R,S]$ such that $R \subseteq T$ is SL. It satisfies the following properties:

1. $T = \Pi[V \in [R,S] \mid R \subseteq V$ seminormal infra-integral, $R/M \cong \mathbb{Z}/2\mathbb{Z}$ for any $M \in \mathrm{MSupp}(V/R)]$.
2. $T = \Pi[V \in [R,S] \mid R \subseteq V$ SL$]$.
3. $T = \sup[V \in [R,S] \mid R \subseteq V$ SL$]$.
4. $T = \cup[V \in [R,S] \mid R \subseteq V$ SL$]$.

Proof. Let $V \in [R,S]$ be such that $R \subseteq V$ is SL. According to Lemma 8.9, we have $V \in [R, {}_S^uR]$ and $R \subset V$ is u-integral, and in particular $V \in [R, \overline{R}]$, where $\overline{R}$ is the integral closure of $R \subseteq S$. As $\overline{R} \subseteq S$ has FCP, any element of $[\overline{R}, S]$ is semilocal. It follows that we can assume that $R \subseteq S$ is an FCP integral extension such that S is semilocal.

Setting $\mathcal{F}' := \{V \in [R,S] \mid R \subseteq V$ seminormal infra-integral, $R/M \cong \mathbb{Z}/2\mathbb{Z}$ for any $M \in \mathrm{MSupp}(V/R)]\}$ and using notation of Proposition 8.6, we get that $\mathcal{F}' \subseteq \mathcal{F}$ and $V \in \mathcal{F}'$ if and only if $R \subseteq V$ is SL by Theorem 8.3, because $R \subseteq V$ has FIP (see the proof of Lemma 8.9(2). Since $R \subseteq S$ has FCP, $\mathcal{F}'$ has maximal elements. Applying Lemmas 8.5 and 8.9, we get that $\mathcal{F}'$ has only one maximal element T which is the greatest $V \in [R,S]$ such that $R \subseteq V$ is SL. Equations (1), (2), (3) and (4) follow obviously because $V \subseteq T$ for any $V \in \mathcal{F}'$ and since $T \in \mathcal{F}'$. □

9 Some special cases of SL extensions

In this last section, we characterize SL extensions satisfying another property at the same time. We recall that an extension $R \subseteq S$ is called *Boolean* if $[R,S]$ is a Boolean lattice, that is a distributive lattice such that each $T \in [R,S]$ has a complement T' in $[R,S]$ (such that $T \cap T' = R$ and $TT' = S$) [33].

Corollary 9.1. *Let $R \subseteq S$ be an FCP SL extension such that S is semilocal and $|\mathrm{V}_S(MS)| = 2$ for each $M \in \mathrm{MSupp}(S/R)$. Then $R \subseteq S$ is a Boolean extension.*

Proof. Since $R \subseteq S$ is an SL extension such that S is semilocal, Proposition 8.1 gives that $R \subseteq S$ is a seminormal infra-integral FIP extension, and so is $R_M \subseteq S_M$ for each $M \in \mathrm{MSupp}(S/R)$. But $|\mathrm{V}_S(MS)| = 2$ for each $M \in \mathrm{MSupp}(S/R)$ implies that $R_M \subseteq S_M$ is minimal decomposed by [7, Lemma 5.4] and then Boolean by [33, Lemma 3.27], from which we can infer that $R \subseteq S$ is a Boolean extension by [33, Proposition 3.5]. □

Remark 9.2. *The SL criteria of Theorem 8.3 (resp. Corollary 8.4) may hold even if S (resp. $S/(R:S)$) is not a semilocal ring.*

A weaker condition is gotten with the following example. Take for instance $R := \prod_{i\in\mathbb{N}} R_i$ where $R_i \cong \mathbb{Z}/2\mathbb{Z}$ for each $i \in \mathbb{N}$ and set $S := R^2$. According to [30, Proposition 1.4], $R \subseteq S$ is seminormal infra-integral but not FCP with $(R:S) = 0$, so that $S/(R:S)$ is not semilocal. But since $|\mathrm{U}(R)| = |\mathrm{U}(S)| = 1$, $R \subseteq S$ is SL.

Example 9.3. *(1) In Number Theory, we can find a lot of SL extensions $R \subseteq S$ that are not FCP and S is neither semilocal nor regular. Let $K := \mathbb{Q}(\sqrt{-d})$, where d is a square-free positive integer such that $d \neq 1,3$. By [37, 10.2,* **D**, *p.169], $\mathrm{U}(A) = \{1,-1\}$, where A is the ring of integers of K. As $\mathrm{U}(\mathbb{Z}) = \{1,-1\}$, we get that $\mathrm{U}(\mathbb{Z}) = \mathrm{U}(R) = \mathrm{U}(A)$, for any $R \in [\mathbb{Z},A]$, so that $\mathbb{Z} \subseteq R$ and $R \subseteq A$ are SL for any $R \in [\mathbb{Z},A]$. Of course, $\mathbb{Z} \subseteq A$ is an integral extension which has not FCP since $(\mathbb{Z}:A) = 0$ [7, Theorem 4.2] and A is neither semilocal nor regular. This also holds for a ring of integers R with integral closure A. By Dirichlet's Theorem [37, Theorem 1, page 179], these are the only cases where the ring of algebraic integers A of an algebraic number field is such that $\mathbb{Z} \subseteq A$ is SL.*

(2) We can say more considering a particular situation of (1). Let $d := 5$. Then, $A = \mathbb{Z} + \mathbb{Z}\sqrt{-5}$ by [37, 5.4, **V**, *p.97] and 5 is ramified in K [37, 11.2,* **K**, *p.199]. This means that $5A = M^2$, where $M \in \mathrm{Max}(A)$. Set $P := 5A$ and $R := \mathbb{Z} + 5A$. Then, $R \in [\mathbb{Z},A]$ with $R \subseteq A$ SL by (1). We have $P = (R:A) \in \mathrm{Max}(R)$ because $R/P = (\mathbb{Z}+5A)/5A \cong \mathbb{Z}/(\mathbb{Z}\cap 5A) = \mathbb{Z}/5\mathbb{Z}$ which is a field. Since $P = M^2$, it follows that M is the only maximal ideal of A lying above P. Then, A_P is a local ring. Assume that $R_P \subseteq A_P$ is SL. Then, Example 4.1(8) says that $R_P = A_P$, a contradiction with $P = (R:A) \in \mathrm{MSupp}(A/R)$.*

This example shows that Proposition 2.9 has no converse: an extension $R \subseteq S$ may be SL with $R_M \subseteq S_M$ not SL for some $M \in \mathrm{MSupp}(S/R)$.

Let R be a ring and F its *prime subring*, that is $F := \mathbb{Z}/\ker(c)$, where $c : \mathbb{Z} \to R$ is the ring morphism defined by $c(x) := x1_R$. Then, either $F = \mathbb{Z}$ or $F = \mathbb{Z}/n\mathbb{Z}$ where n is the least positive integer such that $n1_R = 0_R$.

Proposition 9.4. *Let R be a ring with F its prime subring. Then $F[\mathrm{U}(R)] \to R$ is SL.*

Proof. Use Example 4.1(1a). □

Corollary 9.5. *If R is a local ring with prime subring F, then $F[\mathrm{U}(R)] = R$. If, in addition, $\mathrm{U}(R)$ is a finitely generated group, R is Noetherian.*

Proof. By Proposition 9.4 and Example 4.1(8) $F[\mathrm{U}(R)] = R$.

Now, assume that $\mathrm{U}(R)$ is a finitely generated group. Let $\{x_1,\dots,x_n\}$ be a system of generators of $\mathrm{U}(R)$, so that any $x \in \mathrm{U}(R)$ is of the form $x = \prod_{i=1}^n x_i^{n_i}$, with $n_i \in \mathbb{Z}$ for each $i \in \mathbb{N}_n$. Then, $R = F[\mathrm{U}(R)]$ is an F-algebra generated by $\{x_i, x_i^{-1} \mid i \in \mathbb{N}_n\}$, where either $F = \mathbb{Z}$ or $F = \mathbb{Z}/m\mathbb{Z}$, and so is Noetherian. □

Proposition 9.6. *Let S be a ring with prime subring F.*

1. *Let Σ be a saturated multiplicatively closed subset of S and set $R := F[\Sigma]$. Then, $R_\Sigma \subseteq S_\Sigma$ is SL and so is $R \subseteq S$.*

   If, in addition, S_Σ is regular, then $R_\Sigma \subseteq S_\Sigma$ is u-integral, infra-integral, seminormal, quadratic and R_Σ is regular.

2. *Assume that Σ is the set of regular elements of S.*

   (a) Then $R_\Sigma \subseteq \mathrm{Tot}(S)$ is SL.

   (b) If, in addition, $\mathrm{Tot}(S)$ is regular, so is R_Σ.

   (c) If S has few zerodivisors, then $\mathrm{Tot}(S)$ is semilocal as R_Σ.

Proof. (1) Let $x = a/s \in S_\Sigma$, with $a \in S$ and $s \in \Sigma$. Then, $x \in \mathrm{U}(S_\Sigma) \Leftrightarrow$ there exists $y = b/t \in S_\Sigma$, with $b \in S$ and $t \in \Sigma$ such that $xy = 1$ $(*)$. Now, $(*) \Leftrightarrow ab/st = 1 \Leftrightarrow$ there exists $u \in \Sigma$ such that $uab = ust \in \Sigma$. It follows that $a \in \Sigma$ and $\mathrm{U}(S_\Sigma) = \{a/s \in S_\Sigma \mid a, s \in \Sigma\}$. Then, $\mathrm{U}(S_\Sigma) \subseteq R_\Sigma$, with obviously $\mathrm{U}(S_\Sigma) \subseteq \mathrm{U}(R_\Sigma) \subseteq \mathrm{U}(S_\Sigma)$ giving $\mathrm{U}(R_\Sigma) = \mathrm{U}(S_\Sigma)$. Then $R_\Sigma \subseteq S_\Sigma$ is SL, and so is $R \subseteq S$ by Proposition 2.10.

Assume, in addition, that S_Σ is regular. Then, according to Proposition 5.18, $R_\Sigma \subseteq S_\Sigma$ is u-integral, infra-integral, seminormal, quadratic and R_Σ is regular.

(2) Now, Σ is the set of regular elements of S. Then $\mathrm{Tot}(S) = S_\Sigma$.

(a) and (b) follow from (1).

(c) If S has few zerodivisors, then $\mathrm{Z}(S) = \cup_{i=1}^n P_i$ is a finite union of prime ideals of S and $\Sigma = S \setminus \mathrm{Z}(R)$ gives that $S_\Sigma = \mathrm{Tot}(S)$ has finitely many maximal ideals, so that $\mathrm{Tot}(S)$ is semilocal as R_Σ since $R_\Sigma \subseteq S_\Sigma$ is integral. □

In [1], D. D. Anderson and S. Chun introduced strongly inert extensions and related extensions in the following way: a ring extension $R \subseteq S$ is *strongly inert* (resp. *weakly strongly inert*) if for nonzero $a, b \in S$, then $ab \in R$ (resp. $ab \in R \setminus \{0\}$) implies $a, b \in R$.

The link between strongly inert and SL extensions has been noticed by Anderson and Chun and gives many examples of SL extensions.

Proposition 9.7. *Let $R \subseteq S$ be a strongly inert extension. The following properties hold:*

1. *[1, Proposition 3.1 (2) and Theorem 3.2] $R \subseteq S$ is SL and either $R = S$ or R and S are integral domains.*

2. *If $R \neq S$, then 0 is the only proper ideal shared by R and S.*

3. *If S is J-regular, then $R = S$.*

Proof. (2) Assume that $R \neq S$, so that R and S are integral domains. Let I be a proper ideal shared by R and S. We claim that $I = 0$. Otherwise, there exists some $a \in I$, $a \neq 0$. Since $R \subseteq S$ is SL and $R \neq S$, there exists some $x \in S \setminus R$ with $x \neq 0$. Then, $ax \in I \subseteq R$. It follows that ax is in R. Since $R \subseteq S$ is strongly inert, this implies $x \in R$, a contradiction. Then, 0 is the only proper ideal shared by R and S.

(3) Since $R \subseteq S$ is strongly inert, $R \subseteq S$ is SL. If, moreover, S is J-regular, then, by Theorem 5.17, $\mathrm{J}(R) = \mathrm{J}(S)$ and $R \subseteq S$ is integral. But, [1, Theorem 3.5 (7)] says that any element of $S \setminus R$ is transcendental over R. It follows that $R = S$. □

Remark 9.8. *The strongly local property is weaker than the strongly inert property in the following way: Let $R \subseteq S$ be SL and $a, b \in S$ be such that $ab \in \mathrm{U}(R)$. Then $ab \in \mathrm{U}(S)$ which implies that $a, b \in \mathrm{U}(S) = \mathrm{U}(R)$, so that $a, b \in R$.*

But, in case R is a field, the following Corollary shows that the notions of SL extension and of weakly strongly inert extension are equivalent.

Corollary 9.9. *Let $K \subseteq S$ be an extension where K is a field. The following properties holds:*

1. *$K \subseteq S$ is weakly strongly inert if and only if $K \subseteq S$ is SL.*
2. *If, in addition, S is an integral domain, then $K \subseteq S$ is strongly inert if and only if $K \subseteq S$ is SL. In this case, $K \subseteq S$ is algebraically closed.*
3. *If $K \subseteq S$ is SL, then $K \subseteq S$ is an FCP extension if and only if either $K = S$ or $K \cong \mathbb{Z}/2\mathbb{Z}$ and $S \cong K^n$ for some integer n.*

Proof. Since $\mathrm{U}(K) = K \setminus \{0\}$, we get that $K \subseteq S$ is SL if and only if $\mathrm{U}(S) = K \setminus \{0\}$.

(1) Assume that $K \subseteq S$ is SL and let $a, b \in S \setminus \{0\}$ be such that $ab \in K \setminus \{0\}$. Then, $ab \in \mathrm{U}(S)$, so that $a, b \in \mathrm{U}(S) = K \setminus \{0\} \subseteq K$ and $K \subseteq S$ is weakly strongly inert. Conversely, if $K \subseteq S$ is weakly strongly inert, then $K \subseteq S$ is SL according to [1, Theorem 4.1].

(2) Assume that $K \subseteq S$ is SL and, in addition, that S is an integral domain. It follows that $K \subseteq S$ is weakly strongly inert by (1) and $\mathrm{Z}(S) = \mathrm{Z}(K) = \{0\}$, so that $K \subseteq S$ is strongly inert by [1, Proposition 3.1] and algebraically closed by [1, Theorem 3.5]. Conversely, if $K \subseteq S$ is strongly inert, then $K \subseteq S$ is SL according to [1, Proposition 3.1].

(3) Assume that $K \subseteq S$ is SL.

If $K = S$, then $K \subseteq S$ has FCP.

If $K \cong \mathbb{Z}/2\mathbb{Z}$ and $S \cong K^n$ for some integer n, then $K \subseteq S$ has FCP by [30, Proposition 1.4] or because S has finitely many elements.

Conversely, assume that $K \subseteq S$ has FCP. Then S is semilocal by Proposition 8.7 and Definition 8.8. It follows from Theorem 8.3 that $K \cong \mathbb{Z}/2\mathbb{Z}$ because $\mathrm{MSupp}(S/K) = \{0\}$ and $S \cong K^n$ for some integer n by Lemma 8.9 since $K \subseteq S$ is infra-integral, and then integral. □

We say that an extension $R \subseteq S$ is *semi-inert* if for nonzero $a, b \in S$, $ab \in R$ implies either $a \in R$ or $b \in R$. Such an extension exists by [10, Proposition 3.1], for example a minimal Prüfer extension.

More generally, if $R \subseteq S$ is semi-inert and $S \subseteq T$ is strongly inert, then $R \subseteq T$ is semi-inert.

Proposition 9.10. *Let $R \subseteq S$ be a semi-inert extension. Then $R \subseteq S$ is integrally closed. If, in addition, $R \subseteq S$ is local, then $R \subseteq S$ is SL.*

Proof. Let $s \in S \setminus R$ and assume that s is integral over R. Then $s^n + \sum_{i=0}^{n-1} a_i s^i = 0$ $(*)$ for some positive integer n and $a_i \in R$ for any $i \in \{0, \dots, n-1\}$. We may assume that n is the least integer such that $(*)$ is satisfied. Then, $n > 1$ since $s \notin R$. It follows that $(*)$ implies $s(s^{n-1} + \sum_{i=0}^{n-2} a_{i+1} s^i) = -a_0 \in R$. Since $s \neq 0$ because $s \in S \setminus R$ and $s^{n-1} + \sum_{i=0}^{n-2} a_{i+1} s^i \neq 0$ by the choice of n, it follows that either $s \in R$ or $s^{n-1} + \sum_{i=0}^{n-2} a_{i+1} s^i \in R$ because $R \subseteq S$ is semi-inert. In both cases, we get a contradiction, giving that s is not integral over R. Then, $R \subseteq S$ is integrally closed.

Obviously, $\mathrm{U}(R) \subseteq \mathrm{U}(S)$. Assume, in addition, that $R \subseteq S$ is local. Then, $\mathrm{U}(S) \cap R = \mathrm{U}(R)$. Let $s \in \mathrm{U}(S)$. There exists $t \in \mathrm{U}(S)$ such that $st = 1 \in R$ $(**)$, that is $s = t^{-1}$. Since $R \subseteq S$ is semi-inert, it follows that either s or $t \in R$. Assume first that $s \in R$. Then, $s \in \mathrm{U}(S) \cap R = \mathrm{U}(R)$. Assume now that $s \notin R$. We get that $t \in R \cap \mathrm{U}(S) = \mathrm{U}(R)$. Then, t has an inverse in R, which is unique and is also its inverse in S, so that $s \in \mathrm{U}(R)$, a contradiction since $s \notin R$. Hence, $\mathrm{U}(S) = \mathrm{U}(R)$ and $R \subseteq S$ is SL. □

We can also build SL extensions with cyclotomic extensions.

Proposition 9.11. *Let p be a prime integer different from 2 and such that 2 is a primitive root in $\mathbb{Z}/p\mathbb{Z}$.*

Set $K := \mathbb{Z}/2\mathbb{Z}$, $L := K[X]/(X^{p-1} + \cdots + 1)$, $R := K[X]/(X^p - 1)$ and $S := K[X]/(X^{p+1} - X)$. The following properties hold:

1. *$X^{p-1} + \cdots + 1$ is irreducible over K.*

2. *There exists an injective ring morphism $f : R \to S$.*
3. *f is SL.*

Proof. (1) comes from [20, Theorem 2.47] because 2 is a primitive root in $\mathbb{Z}/p\mathbb{Z}$.

(2) Since $X^p - 1 = (X-1)(X^{p-1}+\cdots+1)$, the Chinese Remainder Theorem shows that $R \cong [K[X]/(X-1)] \times [K[X]/(X^{p-1}+\cdots+1)] \cong K \times L$ with $K \subseteq L$ a field extension of degree $p-1$. Similarly, $S \cong [K[X]/(X)] \times [K[X]/(X-1)] \times [K[X]/(X^{p-1}+\cdots+1)] \cong K^2 \times L$. There is an injective ring morphism $g : K \times L \to K^2 \times L$, given by $(x,y) \mapsto (x,x,y)$. Then, the following commutative diagram

$$\begin{array}{ccc} K \times L & \xrightarrow{g} & K^2 \times L \\ \uparrow & & \downarrow \\ R & \xrightarrow{f} & S \end{array}$$

implies an injective ring morphism $f : R \to S$ gotten by

$$f : R \to K \times L \to K^2 \times L \to S$$

In fact, g is defined as $g := (g_1, g_2)$ where $g_1 : K \to K^2$ is the diagonal map and g_2 is the identity on L. From now, we may identify R with $K \times L$, S with $K^2 \times L$ and f with g.

Let x be the class of X in L, so that $L = K[x]$. Setting $y := (0,x) \in R$, we get that $y^p = (0, 1_L)$ and any element of R is of the form (a,b), where $a \in K = \{0, 1_K\}$ and $b = \sum_{i=0}^{p-2} \alpha_i x^i \in L$ with $\alpha_i \in K$ for each i. Then, $(a,b) = (a,0) + (0, \sum_{i=0}^{p-2} \alpha_i x^i) = (a,a) - (0,a) + (0, \sum_{i=0}^{p-2} \alpha_i x^i)$ can be written as $a.1_R + \sum_{i=0}^{p-2} \beta_i y^i$, with a and $\beta_i \in K$ for each i. It follows that $R = K[y]$.

Setting $z := (0,0,x)$, we get that $z = f(y)$, so that we have the injective ring morphism $f : R \to S$ defined by $f(y) = z$. We may remark that $S \neq K[z]$ because $(0,1,0) \in S \setminus K[z]$.

(3) Since $|\mathrm{U}(K)| = 1$, we get that $|\mathrm{U}(R)| = |\mathrm{U}(K \times L)| = |\mathrm{U}(L)| = |\mathrm{U}(K^2 \times L)| = |\mathrm{U}(S)|$. It follows that $f(\mathrm{U}(R)) = \mathrm{U}(S)$ since $f(\mathrm{U}(R)) \subseteq \mathrm{U}(S)$ and f is SL. □

In the previous proposition the injective ring morphism $f : R \to S$ we built may not be unique as it is shown in the following example:

Example 9.12. *Set $K := \mathbb{Z}/2\mathbb{Z}$, $R := K[X]/(X^3-1)$ and $S := K[X]/(X^4-X)$. We will build two injective ring morphisms $f : R \to S$ which are SL.*

Let y be the class of X in R and t the class of X in S, so that $R = K[y]$ and $S = K[t]$. We may use the proof of Proposition 9.11 with $p = 3$ since 2 is a primitive root in $\mathbb{Z}/3\mathbb{Z}$. Then, we get that $|\mathrm{U}(R)| = 3$ giving $\mathrm{U}(R) = \{1, y, y^2\}$. Since there exists an injective ring morphism $f : R \to S$ which is SL, we also have $|\mathrm{U}(S)| = 3$. As f is also a linear morphism over the K-vector space R, we may define f by the image of the basis $\{1, y, y^2\}$ of R over K. An easy calculation of $(a+bt+ct^2+dt^3)^3 = 1$, with $a,b,c,d \in K$ shows that $\mathrm{U}(S) = \{1, t^3+t+1, t^3+t^2+1\}$. Since we must have $f(1) = 1$ and $f(y) \neq f(y^2)$ both in $\mathrm{U}(S)$, we get that $f(y)$ and $f(y^2)$ are the two different elements of $\{t^3+t+1, t^3+t^2+1\}$. Whatever the value we give to y, we get that $f(y^2) = f(y)^2$.

So, such an f is a ring morphism, which is obviously injective. At last, $f(\mathrm{U}(R)) = \mathrm{U}(S)$ shows that $f : R \to S$ is SL. There are two such morphisms: f_1 and f_2 defined by $f_1(y) = t^3+t+1$ and $f_2(y) = t^3+t^2+1$.

Proposition 9.13. *Let R be a ring. Set $K := \mathbb{Z}/2\mathbb{Z}$, $T := K^n \times R$ and $S = K^m \times R$, where $n < m$ are two positive integers. There is an injective ring morphism $f : T \to S$ given by $f(x,y) := (\varphi(x), y)$ where $\varphi : K^n \to K^m$ is an injective ring morphism. Then, f is SL.*

Proof. Since $n < m$, we can build an injective ring morphism $\varphi : K^n \to K^m$ given, for instance, by $\varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, x_n, \ldots, x_n)$ where the $m-n$ last terms are all equal to x_n. Then, the map $f : T \to S$ given by $f(x,y) := (\varphi(x), y)$ is an injective ring morphism. Since $K = \mathbb{Z}/2\mathbb{Z}$, it follows that $|\mathrm{U}(K)| = 1 = |\mathrm{U}(K^n)| = |\mathrm{U}(K^m)|$. But, $|\mathrm{U}(T)| = |\mathrm{U}(K^n)||\mathrm{U}(R)|$ and $|\mathrm{U}(S)| = |\mathrm{U}(K^m)||\mathrm{U}(R)|$, giving $|\mathrm{U}(T)| = |\mathrm{U}(S)| = |\mathrm{U}(R)|$, so that f is SL. □

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