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# On rings over which every flat module is finitely projective

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**Abstract.** The main goal of this paper is to investigate the class of rings for which every flat module is finitely projective (called *FMF*-ring, for short). We examine the stability of this property in several distinguished contexts of commutative ring extensions such as direct product, polynomial ring, power series ring, localization, homomorphic image, trivial ring extensions and amalgamation rings. Our results enrich the current literature with various new and original families of non-coherent, non-perfect, non-arithmetical and non-Noetherian rings that satisfying this property.
**Key Words**: FMF-ring, direct product, localization, homomorphic image, amalgamation of rings, trivial ring extension.
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Dedicated to the memory of Oussama Aymane Es-Safi,
our esteemed co-author, who has recently passed away

## 1 Introduction

All rings considered in this paper are assumed to be commutative with identity elements and all modules are unitary. For a ring $R$, we denote by $Q(R)$, the total ring of quotients of $R$, that is, the localization of $R$ by the set of all its regular elements, $Z(R)$ denotes the set of all zero-divisors of $R$, and $\mathrm{gldim}(R)$ (resp. $\mathrm{wdim}(R)$) denotes the classical global (resp. weak) dimension of $R$. If $R$ is an integral domain, we will usually denote its quotient field by $\mathrm{qf}(R)$.
Recall that an $R$-module $M$ is called finitely projective if, for any finitely generated submodule $N$, the inclusion map $N \to M$ factors through a free module $F$. The notion of finitely projective module is due to Jones [25]. An interesting study of finitely projective modules is also done by Azumaya in [2]. Note that Jones [25] uses the term $f$-projective, Mao [31] and Simson [39] use the term $\aleph_{-1}$-projective. It is well known that every projective module is finitely projective and any finitely generated finitely projective module is projective and also every finitely projective module is flat. The following diagram of implications summarizes the relations between them:

$$M \text{ is projective} \Longrightarrow M \text{ is finitely projective} \Longrightarrow M \text{ is flat.}$$

But these are not generally reversible, for example the rationals are finitely projective as $\mathbb{Z}$-module, though not projective. Let $F$ be any field, $R := \prod_{n\in\mathbb{N}} F$ and $K := \oplus_{n\in\mathbb{N}} F$. $R/K$ is $R$-flat since $R$ is regular. But $R/K$ is not finitely projective (see [25, page 1611]).

Let $A$ and $B$ be two rings, $J$ an ideal of $B$ and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$: called the amalgamation of $A$ and $B$ along $J$ with respect to $f$ is the subring of $A \times B$ defined by:

$$A \bowtie^f J := \{(a, f(a) + j) \ ; \ a \in A, j \in J\}.$$

This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna, Finacchiaro, and Fontana in [12, 13, 14, 15, 16, 18]. Amalgamation rings are a class of rings quite recently introduced and widely studied; the motivation is that this construction is a sort of generalization of many classical constructions and it allows to build rings with prescribed properties, so being a good source of examples.

Let $A$ be a ring and $E$ an $A$-module. The trivial ring extension of $A$ by $E$ (also called the idealization of $E$ over $A$) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a,e)(a',e') := (aa', ae' + a'e)$. For the reader's convenience, recall that if $I$ is an ideal of $A$ and $E'$ is a submodule of $E$ such that $IE \subseteq E'$, then $J := I \propto E'$ is an ideal of $R$. Recall that, prime (resp., maximal) ideals of $R$ have the form $p \propto E$, where $p$ is a prime (resp., maximal) ideal of $A$ [1, Theorem 3.2]. Suitable background on commutative trivial ring extensions is [1, 17, 22, 26, 27, 28, 29, 30].

The aim of this paper is to investigate the class of rings for which every flat module is finitely projective (called $FMF$-ring, for short). We examine the transfer of this property in several distinguished contexts of commutative ring extensions such as direct product, localization, homomorphic image, trivial ring extensions and amalgamation rings. Our results enrich the current literature with new and original families of non-coherent, non-perfect, non-arithmetical and non-Noetherian rings satisfying this property. Examples of $FMF$-rings are perfect rings, Noetherian rings and Prüfer domains(see for instance [2, 6, 8, 25, 38], for more details).

## 2 Main Results

Recall that a ring $R$ is called semihereditary if every finitely generated ideal of $R$ is projective and is said to have weak global dimension $\leq 1$ ( denoted by $wdim(R) \leq 1$) if every finitely generated ideal of $R$ is flat. It is well known that a semi-hereditary ring $R$ have $wdim(R) \leq 1$. In the domain context, all these conditions coincide with the definition of a Prüfer domain. Glaz [21, Example 3.2.1] provides example of non-semihereditary ring of $wdim \leq 1$. See for instance [3, 4, 21].

We start with examples of non-$FMF$-rings.

**Proposition 2.1.** 1. *Any non-semihereditary ring of $wdim \leq 1$ is a non-$FMF$-ring.*

2. *A non-Noetherian von Neumann regular ring is a non-$FMF$-ring.*

*Proof.* (1) Let $R$ be a non-semihereditary ring with $wdimR \leq 1$. Then there exists a finitely generated ideal $I$ of $R$ which is not projective. So, $I$ is not finitely projective. On the other hand, $I$ is flat since $wdimR \leq 1$. Hence, $R$ is a non-$FMF$-ring.
(2) Let $R$ be a non-Noetherian von Neumann regular ring and $I$ be a non finitely generated ideal of $R$. Asssume that $R$ is a $FMF$-ring. Since $R$ is a von neumann regular ring, we get $R/I$ is a flat

$R$-module which is finitely generated. Hence, $R/I$ is a projective $R$-module which implies that $I$ is finitely generated, a contradiction. Finally, $R$ is a non-$FMF$-ring. □

**Remark 2.2.** In [2, Proposition 18], Azumaya proved that a Prüfer domain is a $FMF$-ring. However, [2, Proposition 18] can not be extended beyond the context of integral domains, since a non-semihereditary ring of $wdim \leq 1$ is a Prüfer ring which is not a $FMF$-ring by Proposition 2.1(1).

Recall that it is well known that a ring is semisimple if and only if it is von Neumann regular Noetherian. In the next proposition, we give a similar characterization of semisimple rings by replacing the concept “Noetherian" with “$FMF$-ring".

**Proposition 2.3.** *A ring $R$ is von Neumann regular $FMF$-ring if and only if $R$ is semisimple.*

*Proof.* Assume that $R$ is von Neumann regular $FMF$-ring and let $I$ be an ideal of $R$. Clearly, $R/I$ is a finitely projective module of $R$ and so $R/I$ is projective since it is finitely generated. Then gldim($R$)= 0. Therefore, $R$ is semisimple. The converse is straighforward via the well known fact that a commutative semisimple ring is Von Neumann regular Noetherian and so is Von Neumann regular $FMF$-ring. □

The next proposition establishes some facts of $FMF$-rings. Recall that a ring $R$ is called $S$-ring if every finitely generated flat module is projective (see [38]).

**Proposition 2.4.**

1. *Let $R$ be a ring such that every finitely generated submodule of a flat $R$-module is finitely presented. Then $R$ is an $FMF$-ring.*
2. *Any $FMF$-ring is an $S$-ring.*
3. *Assume that $R$ is an $FMF$-ring. Then $R$ is a perfect ring if and only if every finitely projective $R$-module is projective.*

*Proof.* (1) Let $M$ be a flat $R$-module and $N$ be a finitely generated submodule of $M$. From assumption, $N$ is finitely presented. Hence, by [7, Theorem 1], we have the desired result.
(2) Assume that $R$ is an $FMF$-ring and let $M$ be a finitely generated flat $R$-module. Then, $M$ is finitely projective and so $M$ is projective since it is finitely generated.
(3) Trivial. □

The $FMF$-property descends into an injective (respectively finitely generated flat, faithfully flat ) ring homomorphism.

**Proposition 2.5.** *Let $f$: $R \longrightarrow S$ be a ring homomorphism.*

1. *Assume that $f$ is injective. Then if $S$ is an $FMF$-ring, then so is $R$.*
2. *Assume that $f$ is flat epimorphism. If $R$ is an $FMF$-ring, then so is $S$.*
3. *Assume that $S$ is finitely generated flat $R$-module. Then if $R$ is an $FMF$-ring, then so is $S$.*
4. *Assume that $S$ is a finitely generated faithfully flat $R$-module. Then $R$ is an $FMF$-ring if and only if so is $S$.*

Before proving this proposition, we establish the following Lemmas.

**Lemma 2.6.** *[8, Lemma 5] Let $R$ be a subring of a ring $S$ and let $M$ be a flat left $R$-module. Assume that $S \otimes_R M$ is finitely projective over $S$. Then $M$ is finitely projective.*

**Lemma 2.7.** *Let $R \longrightarrow S$ be a ring homomorphism making $S$ a finitely generated faithfully flat $R$-module. If an $S$-module $M$ is finitely projective as an $R$-module, then $M$ is a finitely projective as an $S$-module.*

*Proof.* Let $N$ be a finitely generated $S$-submodule of $M$. Then $N$ is a finitely generated $R$-submodule of $M$ since $S$ is finitely generated $R$-module. So, there exist a free $R$-module $F$, a morphism $\varphi : N \longrightarrow F$ and a morphism $\psi : F \longrightarrow M$ such that $\psi \circ \varphi := id_N$. Consider the following morphisms: $g$, $\varphi_1$ and $\psi_1$ defined by: $g : F \longrightarrow F \otimes_R S$, $\varphi_1 : N \longrightarrow F \otimes_R S$, where $\varphi_1 := g \circ \varphi$ and $\psi_1 : F \otimes_R S \longrightarrow M$, where $\psi := \psi_1 \circ g$. Then $\psi_1 \circ \varphi_1 = id_N$. Therefore, $M$ is finitely projective $S$-module, which completes the proof. □

*Proof of Proposition 2.5.*
(1) Assume that $S$ is a $FMF$-ring and let $M$ be a flat $R$-module. Then $M \otimes_R S$ is a flat $S$-module and so $M \otimes_R S$ is a finitely projective $S$-module (since $S$ is an $FMF$-ring). Hence, $M$ is a finitely projective $R$-module by Lemma 2.6. It follows that $R$ is an $FMF$-ring.
(2) Assume that $R$ is an $FMF$-ring and let $M$ be a flat $S$-module. Then $M$ is a flat $R$-module (as $S$ is $R$-flat) and so $M$ is a finitely projective $R$-module (since $R$ is an $FMF$-ring). Hence, $M$ is a finitely projective $S$-module by [5, Proposition 3.2]. It follows that $S$ is an $FMF$-ring.
(3) Assume that $R$ is an $FMF$-ring and let $M$ be a flat $S$-module. Then $M$ is a flat $R$-module (since $S$ is $R$-flat) and so $M$ is a finitely projective $R$-module (since $R$ is an $FMF$-ring). Hence, $M$ is a finitely projective $S$-module by Lemma 2.7. It follows that $S$ is an $FMF$-ring.
(4) $f$ is injective since $S$ is faithfully flat $R$-module. We do a similar proof as in assertion (2) above. □

Now, we study the transfer of $FMF$ property to polynomial and power series rings.

**Corollary 2.8.** *Let $R$ be a ring and let $X$ be an indeterminate over $R$. The following statements are equivalent:*

1. *$R$ is an $FMF$-ring.*
2. *$R[X]$ is an $FMF$-ring.*
3. *$R[[X]]$ is an $FMF$-ring.*

*Proof.* This follows from assertion (4) of Proposition 2.5, as $R[X]$ (resp. $R[[X]]$) is a faithfully flat $R$-module. □

The next result establishes the transfer of the $FMF$-property to localization.

**Theorem 2.9.** Let $R$ be a commutative ring and let $S$ be a multiplicative subset of $R$. Then:

1. If $R$ is an $FMF$-ring, then so is $S^{-1}(R)$.
2. If $S$ contains no zero-divisors then, $S^{-1}(R)$ is an $FMF$-ring if and only if so is $R$. (In particular $Q(R)$ is an $FMF$-ring if and only if so is $R$).

Before proving this theorem, we need the following Lemma.

**Lemma 2.10.** *[8, Proposition 6.] Let $R$ be a ring and let $S$ be a multiplicative subset of $R$. Then:*

1. *For each finitely projective $R$-module $M$, $S^{-1}M$ is finitely projective over $S^{-1}R$.*
2. *Let $M$ be a finitely projective $S^{-1}R$-module. If $S$ contains no zero-divisors, then $M$ is finitely projective over $R$.*

*Proof of Theorem 2.9* .
(1) We assume that $R$ is an $FMF$-ring. Let $M$ be a flat $S^{-1}(R)$-module. Then $M$ is flat over $R$, so it is finitely projective over $R$, since $R$ is an $FMF$-ring. It follows that $M = S^{-1}M$ is finitely projective over $S^{-1}R$ by Lemma 2.10. We get that $S^{-1}R$ is an $FMF$-ring.
(2) Notice that if $S$ contains no zero-divisors, then $R$ is a subring of $S^{-1}R$. We conclude by Proposition 2.5 and (1). □

**Remark 2.11.** Notice that in Theorem 2.9(2), the assumption that $S$ contains no zero-divisors is essential. For example, let $R$ be a non-Noetherian von Neumann regular ring. Then $R_M$ is an $FMF$-ring for a maximal ideal $M$ of $R$, since $R_M$ is a field. But, by Proposition 2.1(2) $R$ is a non-$FMF$-ring. Furthermore, $S = R \setminus M$ contains zero-divisors elements since $R$ is not local and $M \subsetneq Z(R)$.

**Corollary 2.12.** *Every domain is an $FMF$-ring.*

The following proposition studies the $FMF$-ring property into a particular homomorphic image.

**Proposition 2.13.** *Let $R$ be a ring and let $I$ be a pure ideal of $R$. If $R$ is an $FMF$-ring, then so is $R/I$.*

*Proof.* $R/I$ is a finitely generated flat $R$-module since $I$ is a pure ideal of $R$. Then $R/I$ is an $FMF$-ring by Proposition 2.5. □

The converse of Proposition 2.13, is not true in general, as shown by the following example.

**Example 2.14.** Let $R$ be any non-$FMF$-ring and let $p$ be a prime ideal of $R$. Then $R/p$ is always an $FMF$-ring.

Next, we study the transfer of the $FMF$-property to direct products.

**Theorem 2.15.** Let $(R_i)_{i:=1,..,n}$ be a family of commutative rings. Then $R = \prod_{i:=1}^{n} R_i$ is an $FMF$-ring if and only if so is $R_i$ for each $i = 1, \ldots, n$.

*Proof.* We do as in the proof of [9, Theorem 2.10]. □

As an application of Proposition 2.5, we have the following result that examines the transfer of $FMF$-property between a ring $A$ and the trivial ring extension of $A$ by $E$, where $E$ be an $A$-module.

**Proposition 2.16.** *Let $A$ be a ring, let $E$ be an $A$-module and let $R := A \propto E$ be a trivial ring extension of $A$ by $E$. Then:*

1. *If $R$ is an $FMF$-ring, then so is $A$.*
2. *Assume that $E$ is a flat $A$-module. Then $R$ is an $FMF$-ring if and only if so is $A$.*

*Proof.* (1) By Proposition 2.5(1) since $A$ is a subring of $R$.
(2) Notice that, if $E$ is a flat $A$-module. Then $R := A \propto E$ is faithfully flat over $A$. Hence, Proposition 2.5(3) completes the proof of (2).
□

**Corollary 2.17.** *Let $A$ be a ring. Then $A \propto A$ is an $FMF$-ring if and only if so is $A$.*

The aforementioned result enriches the current literature with new examples of $FMF$-rings with zero-divisors which are neither coherent nor arithmetical.

**Example 2.18.** Let $A$ be a domain which is not Prüfer, $K =: qf(A)$, and let $R := A \propto K$ be the trivial ring extension of $A$ by $K$. Then:

1. $R$ is an $FMF$-ring by Proposition 2.16(2).

2. $R$ is not coherent by [26, Theorem 2.8(1)].

3. $R$ is not an arithmetical ring by [3, Corollary 2.4.].

**Example 2.19.** Let $K$ be a field and $E$ be a $K$-vector space with $dim_K(E) \geq 2$. Then $R = K \propto E$ is a non-arithmetical $FMF$-ring by Proposition 2.16(2) and [3, Theorem 3.1].

We combine Theorem 2.15 with Proposition 2.5 to get the transfer of the $FMF$-property to the amalgamation $A \bowtie^f J$.

**Proposition 2.20.** *Let $A$ and $B$ be two rings, $f : A \longrightarrow B$ be a ring homomorphism and let $J$ be an ideal of $B$. Then the following assertions hold:*

1. *Assume that $f^{-1}(J)$ is a pure ideal of $A$. Then the following assertions are equivalent:*
   *$(a)$ $A \bowtie^f J$ is an $FMF$-ring.*
   *$(b)$ $A \times f(A) + J$ is an $FMF$-ring.*
   *$(c)$ $A$ and $f(A) + J$ are $FMF$-rings.*

2. *Assume that $J$ and $f^{-1}(J)$ are regular ideals of $B$ and $A$, respectively. Then $A \bowtie^f J$ is an $FMF$-ring if and only if so are $A$ and $B$.*

*Proof.* $(a) \Rightarrow (c)$ Assume that $A \bowtie^f J$ is an $FMF$-ring. By Proposition 2.5(1), it follows that $A$ is an $FMF$-ring. Next, by [15, Proposition 5.1(3)], $f(A) + J \simeq \frac{A \bowtie^f J}{f^{-1}(J) \times \{0\}}$. Using the fact that $f^{-1}(J)$ is a pure ideal of $A$, it follows that $f^{-1}(J) \times \{0\}$ is a pure ideal of $A \bowtie^f J$. By Proposition 2.13, $f(A) + J$ is an $FMF$-ring.
$(c) \Leftrightarrow (b)$ This follows from Theorem 2.15.
$(b) \Rightarrow (a)$ Assume that $A \times f(A) + J$ is an $FMF$-ring. Since $A \bowtie^f J$ is a subring of $A \times (f(A) + J)$. Then by Proposition 2.5(1), it follows that $A \bowtie^f J$ is an $FMF$-ring.
(2) By [16, Proposition 3.1], we have $Q(A \bowtie^f J) := Q(A) \times Q(B)$. Then $A \bowtie^f J$ is an $FMF$-ring if and only if $Q(A \bowtie^f J)$ is an $FMF$-ring if and only if so are $Q(A)$ and $Q(B)$ if and only if so are $A$ and $B$, by Theorem 2.15 and Theorem 2.9. ☐

Let $I$ be a *proper* ideal of $A$. The (amalgamated) duplication of $A$ along $I$ is a special amalgamation given by

$$A \bowtie I := A \bowtie^{id_A} I = \Big\{(a, a + i) \mid a \in A, i \in I\Big\}.$$

The following corollaries are consequences of Theorem 2.20 on the transfer of $FMF$-ring property to duplications.

**Corollary 2.21.** *Let $A$ be a ring and $I$ be an ideal of $A$. Then the following assertions hold:*

1. *Assume that $I$ is a pure ideal of $A$. Then $A \bowtie I$ is an $FMF$-ring if and only if so is $A$.*

2. *Assume that $I$ is a regular ideal of $A$. Then $A \bowtie I$ is an $FMF$-ring if and only if so is $A$.*

**Corollary 2.22.** *Let $A$ and $B$ be two integral domains, and let $J$ be an ideal of $B$. Then $A \bowtie^f J$ is an $FMF$-ring.*
*In particular, if $I$ is a nonzero ideal of an integral domain $A$, then $R := A \bowtie I$ is an $FMF$-ring.*

Corollary 2.22 allows us to construct a new original class of $FMF$-rings which are not perfect.

**Example 2.23.** Let $R$ be an integral domain which is not a field and $I$ be an ideal of $R$. Then:

1. $R \bowtie I$ is an $FMF$-ring by Corollary 2.22.
2. $R \bowtie I$ is not perfect by [10, Theorem 2.6].

Now, we provide a new example of an $FMF$-ring which is not arithmetical.

**Example 2.24.** Let $R$ be a domain which is not a Prüfer domain and $I$ be a nonzero ideal of $R$. Then:

1. $R \bowtie I$ is an $FMF$-ring by Corollary 2.22.
2. $R \bowtie I$ is not arithmetical by [11, Corollary 3.8].

Now, we show how one may use Theorem 2.20, to construct new examples of $FMF$-rings which are not Noetherian.

**Example 2.25.** Let $A := \mathbb{Z}_6$ be an $FMF$-ring, $K := \mathbb{Z}_6/3\mathbb{Z}_6$, $E := K^\infty$ be a $K$-vector space and $B := A \propto E$ be the trivial ring extension of $A$ by $E$. Consider the injective ring homomorphism $f : A \hookrightarrow B$ defined by $f(a) = (a, 0)$. Let $J := 3\mathbb{Z}_6 \propto E$ be an ideal of $B$. Clearly $f^{-1}(J) = 3\mathbb{Z}_6$. Then:

1. $A \bowtie^f J$ is an $FMF$-ring.
2. $A \bowtie^f J$ is not Noetherian.

*Proof.* (1) Since $f^{-1}(J) = (3)$ is an idempotent ideal of $A$, then $f^{-1}(J)$ is a pure ideal of $A$. By Theorem 2.20(1), $A \bowtie^f J$ is an $FMF$-ring.
(2) We claim that $A \bowtie^f J$ is not Noetherian. Indeed, $f(A) + J = B$ is not Noetherian since $E$ is not finitely generated. Therefore, by [15, Proposition 5.6], $A \bowtie^f J$ is not Noetherian. □

**Example 2.26.** Let $A$ be an integral domain with quotient field $K$, $E$ be nonzero $K$-vector space such that $dim_K(E) \geq 2$ and $B := A \propto E$ be the trivial ring extension of $A$ by $E$. Consider the ring injective homomorphism $f : A \hookrightarrow B$ defined by $f(a) = (a, 0)$ and let $J := I \propto E$ be a regular ideal of $B$ with $I$ a nonzero regular ideal of $A$. Then:

1. $A \bowtie^f J$ is an $FMF$-ring.
2. $A \bowtie^f J$ is not Noetherian.

*Proof.* (1) Since $A$ is an integral domain, then $A$ is an $FMF$-ring and by Theorem 2.16(2), it follows that $f(A) + J = A \propto 0 + I \propto E = A \propto E = B$ is an $FMF$-ring. Using the fact that $f^{-1}(J)$(resp., $J$) is a regular ideal of $A$ (resp., of $B$), by Theorem 2.20(2), it follows that $A \bowtie^f J$ is an $FMF$-ring.
(2) Using similar arguments as [26, Theorem 3.1(1)], we show that $A \propto E$ is not coherent. Indeed, let $0 \neq f \in E$ and $L = (A \propto E)(0, f)$. Then one can easily check that the principal ideal $L$ is not finitely presented; and therefore $A \propto E$ is not coherent and so $A \propto E$ is not Noetherian. Moreover $f(A) + J = (A \propto 0) + (I \propto E) = ((A + I) \propto E) = A \propto E$ is not Noetherian. Hence, by [15, Proposition 5.6], $A \bowtie^f J$ is not Noetherian since $f(A) + J$ is not Noetherian. □

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