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Author(s):

Ayman Badawi

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Characterizing some commutative rings by divisibility conditions

Ayman Badawi
Department of Mathematics & Statistics, The American University of Sharjah, P.O.
Box 26666, Sharjah, United Arab Emirates,
e-mail:*abadawi@aus.edu*

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Abstract. In this paper, we collect results on divisibility that characterize known commutative rings.
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1 Introduction

Let R be a commutative ring with $1 \neq 0$. Then $Z(R)$ denotes the set of all zerodivisors of R, $Id(R)$ denotes the set of all idempotent elements of R, $U(R)$ denotes the set of all units of R, and $Nil(R)$ denotes the set of all nilpotent elements of R. We recall from [6] and [9] that R is called a *divided* ring if, for every prime ideal P of R, we have $P \subseteq aR$ for every $a \in R \setminus P$. Let $Spec(R)$ denote the set of all prime ideals of R. We say $Spec(R)$ are *linearly ordered* as in [3] if for every prime ideal P_1, P_2 of R we have $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$. We recall from [8] that a commutative ring R is called a *pseudo-valuation* ring abbreviated PVR if, for every prime ideal P of R and for every $a, b \in R$, we have $aP \subseteq bR$ or $bR \subseteq aP$. Recall that R is a *von Neumann regular* if for every $a \in R$, there is $x \in R$ such that $a^2x = a$, that R is π-regular if for every $a \in R$, there are $x \in R$ and an integer $n \geq 1$ such that $a^{2n}x = a^n$, and that R is Boolean if $a^2 = a$ for every $a \in R$. Thus a Boolean ring is von Neumann regular and a von Neumann regular ring is π-regular. Let $\pi - r(R)$ be the set of all π-regular elements of R and $vnr(R)$ be the set of all von Neumann regular elements of R. Recall ([4]) that a proper ideal I of R is called an (m,n)-*closed* ideal of R if $x^m \in I$ implies $x^n \in I$.

In this paper, we collect some divisibility conditions that characterize the mentioned rings above.

2 Results

Recall that $Spec(R)$ are *linearly ordered* as in [3] if for every prime ideal P_1, P_2 of R we have $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$. If I is a proper ideal of R, then $Rad(I) = \{x \in R \mid x^n \in I \text{ for some integer } n \geq 1\}$.

Theorem 2.1. ([5, Theorem 1]) The following statements are equivalent for a commutative ring R.

1. For each $a, b \in R$, there is an $n \geq 1$ such that $a \mid b^n$ or $b \mid a^n$.
2. The prime ideals of R are linearly ordered.
3. The radical ideals of R are linearly ordered.
4. Each proper radical ideal of R is prime.
5. The radical ideals of principal ideals of R are linearly ordered.

Recall from [6] and [9] that R is called a *divided* ring if, for every prime ideal P of R, we have $P \subseteq aR$ for every $a \in R \setminus P$.

Theorem 2.2. ([5, Proposition 2]) For an integral domain R the following are equivalent.

1. For every $a, b \in R$, we have $a \mid b$ or $b \mid a^n$ for some integer $n \geq 1$.
2. For every pair of ideals I, J of R we have $I \subseteq Rad(J)$ or $Rad(J) \subseteq I$.
3. For every $a, b \in R$, we have $aR \subseteq Rad(b)$ or $Rad(b) \subseteq aR$.
4. R is a divided ring.

Theorem 2.3. ([6, Theorem 1]) For a commutative ring R. The following are equivalent.

1. R is a divided ring.
2. For every $a,b \in R$, there is an integer $n \geq 1$, such that $a^nN \subseteq bR$ or $bR \subseteq a^nN$, where N is the set of all nonunit elements of R.

We recall from [8] that a commutative ring R is called a PVR if, for every prime ideal P of R and for every $a,b \in R$, we have $aP \subseteq bR$ or $bR \subseteq aP$.

Theorem 2.4. ([8, Theorem 5]) Let N be the set of all nonunit elements of R. The following statements are equivalent.

1. R is a PVR.
2. For all $a,b \in R$, we have $a \mid b$ (in R) or $b \mid ac$ (in R) for every $c \in N$.
3. For all ideals I and J of R, we have $I \subseteq J$ or $JL \subseteq I$ for every proper ideal L of R.

It is clear that every chained ring is a PVR. Recall that a ring R is a chained ring if for every $a,b \in R$, we have $a \mid b$ (in R) or $b \mid a$ (in R). The following is an example of a PVR that is not a chained ring.

Example 2.5. ([8, Example 10]) Let K be a field and X,Y indeterminates and $I = (X^2, XY, Y^2)$. Then $R = K[X,Y]/I$ has exactly one maximal ideal $M = (X+I, Y+I)$. Since $zM = \{0\}$ for every $z \in M$, R is a PVR. Let $a = x+I$ and $b = y+I$. Since $a \nmid b$ (in R) and $b \nmid a$ (in R), we conclude that R is not a chained ring.

Theorem 2.6. ([7, Corollary 7]) For a commutative ring R. The following are equivalent.

1. R is a PVR.
2. For each $a,b \in R$ and maximal ideal M of R, we have $aM \subseteq bR$ or $bR \subseteq aM$.
3. For each $a,b \in R$, there is a maximal ideal M of R containing $Z(R)$ so that $aM \subseteq bR$ or $bR \subseteq aM$.

Let R be a commutative ring with total quotient ring $T(R)$. Then R is called *additively regular* if for each $x \in T(R)$, there is a $y \in R$ such that $x+y$ is a regular element of $T(R)$. Note that Noetherian rings are additively regular rings. Recall that R is called *root closed* if whenever $x^n \in R$ for some $x \in T(R)$, then $x \in R$. For a commutative ring R, $U(T(R))$ is the set of all units of $T(R)$.

Theorem 2.7. ([2, Proposition 2.2]) Let R be an additively regular ring. The following are equivalent:

1. R is root closed.
2. If $x^n \in R$ for some $x \in U(T(R))$ and $n \geq 1$, then $x \in R$.

Recall that a ring R is called $(2,3)$-closed if whenever $x^2, x^3 \in R$ for some $x \in T(R)$, then $x \in R$.

Theorem 2.8. ([2, Proposition 2.3]) Let R be an additively regular ring. The following are equivalent.

1. R is $(2,3)$-closed.
2. If $x^2, x^3 \in R$ for some $x \in U(T(R))$, then $x \in R$.

Theorem 2.9. ([2, Proposition 2.3]) Let R be an additively regular ring. The following are equivalent.

1. R is integrally closed.
2. If $x \in R'$ for some $x \in U(T(R))$, then $x \in R$, where R' is the integral closure of R in $T(R)$.

Recall ([4]) that a proper ideal I of R is called an (m,n)*-closed* ideal of R if $x^m \in I$ implies $x^n \in I$.

Theorem 2.10. ([4, Theorem 3.1]) Let R be an integral domain, m and n integers with $1 \leq n < m$, and $I = p^kR$, where p is a prime element of R and k is a positive integer. Then the following statements are equivalent.

1. I is an (m,n)-closed ideal of R.
2. $k = ma + r$, where a and r are integers such that $a \geq 0$, $1 \leq r \leq n$, $a(m \; mod \; n) + r \leq n$, and if $a \neq 0$, then $m = n + c$ for an integer c with $1 \leq c \leq n-1$.
3. If $m = bn + c$ for integers b and c with $b \geq 2$ and $0 \leq c \leq n-1$, then $k \in \{1, \ldots, n\}$. If $m = n + c$ for an integer c with $1 \leq c \leq n-1$, then $k \in \bigcup_{h=1}^{n} \{mi + h \mid i \in \mathbb{Z} \text{ and } 0 \leq ic \leq n-h\}$.

Theorem 2.11. ([4, Theorem 3.2]) Let R be an integral domain, n a positive integer, and $I = p^kR$, where p is a prime element of R and k is a positive integer. Then the following statements are equivalent.

1. I is an $(n+1,n)$-closed ideal of R.
2. $k = (n+1)a + r$, where a and r are integers such that $a \geq 0$, $1 \leq r \leq n$, and $a + r \leq n$.
3. $k \in \bigcup_{h=1}^{n} \{(n+1)i + h \mid i \in \mathbb{Z} \text{ and } 0 \leq i \leq n-h\}$.

Moreover, $|\{k \in \mathbb{N} \mid p^kR \text{ is } (n+1,n)\text{-closed}\}| = n(n+1)/2$.

Theorem 2.12. ([4, Corollary 3.3]) Let R be an integral domain and $I = p^k R$, where p is a prime element of R and k is a positive integer. Then I is a $(3,2)$-closed ideal of R if and only if $k \in \{1,2,4\}$.

Theorem 2.13. ([4, Theorem 3.4]) Let R be an integral domain, m and n integers with $1 \leq n < m$, and $I = p_1^{k_1} \cdots p_i^{k_i} R$, where $p_1, \ldots, p_i$ are nonassociate prime elements of R and $k_1, \ldots, k_i$ are positive integers. Then the following statements are equivalent.

1. I is an (m,n)-closed ideal of R.
2. $p_j^{k_j} R$ is an (m,n)-closed ideal of R for every $1 \leq j \leq i$.
3. If $m = bn + c$ for integers b and c with $b \geq 2$ and $0 \leq c \leq n-1$, then $k_j \in \{1, \ldots, n\}$ for every $1 \leq j \leq i$. If $m = n + c$ for an integer c with $1 \leq c \leq n-1$, then $k_j \in \bigcup_{h=1}^{n} \{mv + h \mid v \in \mathbb{Z} \text{ and } 0 \leq vc \leq n-h\}$ for every $1 \leq j \leq i$.

Theorem 2.14. ([4, Corollary 3.5]) Let R be a principal ideal domain, I a proper ideal of R, and m and n integers with $1 \leq n < m$. Then the following statements are equivalent.

1. I is an (m,n)-closed ideal of R.
2. $I = p_1^{k_1} \cdots p_i^{k_i} R$, where $p_1, \ldots, p_i$ are nonassociate prime elements of R and $k_1, \ldots, k_i$ are positive integers, and one of the following two conditions holds.
   (a) If $m = bn + c$ for integers b and c with $b \geq 2$ and $0 \leq c \leq n-1$, then $k_j \in \{1, \ldots, n\}$ for every $1 \leq j \leq i$.
   (b) If $m = n + c$ for an integer c with $1 \leq c \leq n-1$, then $k_j \in \bigcup_{h=1}^{n} \{mv + h \mid v \in \mathbb{Z} \text{ and } 0 \leq vc \leq n-h\}$ for every $1 \leq j \leq i$.

Theorem 2.15. ([4, Corollary 3.6]) Let R be an integral domain, $I = p_1^{k_1} \cdots p_i^{k_i} R$, where $p_1, \ldots, p_i$ are nonassociate prime elements of R and $k_1, \ldots, k_i$ are positive integers, and n a positive integer. Then the following statements are equivalent.

1. I is an $(n+1, n)$-closed ideal of R.
2. $k_j \in \bigcup_{h=1}^{n} \{(n+1)v + h \mid v \in \mathbb{Z} \text{ and } 0 \leq v \leq n-h\}$ for every $1 \leq j \leq i$.

Theorem 2.16. ([4, Corollary 3.7]) Let R be a principal ideal domain, I a proper ideal of R, and n a positive integer. Then the following statements are equivalent.

1. I is a $(n+1, n)$-closed ideal of R.
2. $I = p_1^{k_1} \cdots p_i^{k_i} R$, where $p_1, \ldots, p_i$ are nonassociate prime elements of R and $k_1, \ldots, k_i$ are positive integers, and $k_j \in \bigcup_{h=1}^{n} \{(n+1)v + h \mid v \in \mathbb{Z} \text{ and } 0 \leq v \leq n-h\}$ for every $1 \leq j \leq i$.

The next theorem uses Theorem 2.10 to give an easier criterion to determine when $p^k R$ is (m,n)-closed.

Theorem 2.17. ([4, Theorem 3.8]) Let R be an integral domain, m and n integers with $1 \leq n < m$, and $I = p^k R$, where p is a prime element R and k is a positive integer. Then the following statements are equivalent.

1. I is an (m,n)-closed ideal of R.
2. Exactly one of the following statements holds.
   (a) $1 \leq k \leq n$.
   (b) There is a positive integer a such that $k = ma + r = na + d$ for integers r and d with $1 \leq r, d \leq n-1$.
   (c) There is a positive integer a such that $k = ma + r = n(a+1)$ for an integer r with $1 \leq r \leq n-1$.

For fixed positive integers m and k, we next determine the smallest positive integer n such that $I = p^k R$ is (m,n)-closed. Note that $n \leq m$ since every proper ideal is (m,m)-closed and that I is (m,n')-closed for all positive integers $n' \geq n$. So this determines $\mathcal{R}(p^k R)$. Also, if $m > 1$, then $n = 1$ if and only if $k = 1$, i.e., if and only if I is a prime ideal of R. As usual, $\lfloor x \rfloor$ is the greatest integer, or floor, function.

Theorem 2.18. ([4, Theorem 3.10]) Let R be an integral domain and $I = p^k R$, where p is a prime element of R and k is a positive integer. Let m be a positive integer and n be the smallest postive integer such that I is (m,n)-closed.

1. If $m \geq k$, then $n = k$.
2. Let $m < k$ and write $k = ma + r$, where a is a positive integer and $0 \leq r < m$.
   (a) If $r = 0$, then $n = m$.
   (b) If $r \neq 0$ and $a \geq m$, then $n = m$.
   (c) If $r \neq 0$, $a < m$, and $(a+1) | k$, then $n = k/(a+1)$.
   (d) If $r \neq 0$, $a < m$, and $(a+1) \nmid k$, then $n = \lfloor k/(a+1) \rfloor + 1$.

For fixed positive integers n and k, we next determine the largest positive integer m (or ∞) such that $I = p^kR$ is (m,n)-closed. (If I is (m,n)-closed for every positive integer m, we will say that I is (∞,n)-closed.) Of course, m can also be found using the previous theorem. Clearly, $m \geq n$ since every proper ideal is (n,n)-closed, and I is (m',n)-closed for every positive integer $m' \leq m$.

Theorem 2.19. ([4, Theorem 3.11]) Let R be an integral domain, n a positive integer, and $I = p^kR$, where p is a prime element of R and k is a positive integer.

1. If $n \geq k$, then I is (m,n)-closed for every postive integer m.
2. Let $n < k$ and write $k = na + r$, where a is a positive integer and $0 \leq r < n$. Let m be the largest positive integer such that I is (m,n)-closed.
   - (a) If $a > n$, then $m = n$.
   - (b) If $a = n$ and $r = 0$, then $m = n+1$.
   - (c) If $a = n$ and $r \neq 0$, then $m = n$.
   - (d) If $a < n$, $r = 0$, and $(a-1)|k$, then $m = k/(a-1) - 1$.
   - (e) If $a < n$, $r = 0$, and $(a-1) \nmid k$, then $m = \lfloor k/(a-1) \rfloor$.
   - (f) If $a < n$, $r \neq 0$, and $a|k$, then $m = k/a - 1$.
   - (g) If $a < n$, $r \neq 0$, and $a \nmid k$, then $m = \lfloor k/a \rfloor$.

The previous two theorems easily extend to products of principal prime ideals. In particular, we can calculate $\mathcal{R}(I) = \{(m,n) \in \mathbb{N}\times\mathbb{N} \mid I \text{ is } (m,n)\text{-closed}\}$ for every proper ideal I in a principal ideal domain or every proper principal ideal I in a unique factorization domain.

Theorem 2.20. ([4, Theorem 3.12]) Let R be an integral domain and $I = p_1^{k_1}\cdots p_i^{k_i}R$, where $p_1,\ldots,p_i$ are nonassociate prime elements of R and $k_1,\ldots,k_i$ are positive integers.

1. Let m be a positive integer. If n_j is the smallest positive integer such that $p_j^{k_j}R$ is (m,n_j)-closed for $1 \leq j \leq i$, then $n = \max\{n_1,\ldots,n_i\}$ is the smallest positive integer such that I is (m,n)-closed.
2. Let n be a positive integer. If m_j is the largest positive integer (or ∞) such that $p_j^{k_j}R$ is (m_j,n)-closed for $1 \leq j \leq i$, then $m = \min\{m_1,\ldots,m_i\}$ is the largest positive integer (or ∞) such that I is (m,n)-closed.

Let I be a proper ideal of a commutative ring R, and $\mathcal{R}(I) = \{(m,n) \in \mathbb{N}\times\mathbb{N} \mid I \text{ is } (m,n)\text{-closed}\}$.

Theorem 2.21. ([4, Theorem 4.1]) Let R be a commutative ring, I and J proper ideals of R, and m,n and k positive integers.

1. $(m,n) \in \mathcal{R}(I)$ for all positive integers m and n with $m \leq n$.
2. If $(m,n) \in \mathcal{R}(I)$, then $(m',n') \in \mathcal{R}(I)$ for all positive integers m' and n' with $1 \leq m' \leq m$ and $n' \geq n$.
3. If $(m,n) \in \mathcal{R}(I)$, then $(km,kn) \in \mathcal{R}(I)$.
4. If $(m,n),(n,k) \in \mathcal{R}(I)$, then $(m,k) \in \mathcal{R}(I)$.
5. If $(m,n),(m+1,n+1) \in \mathcal{R}(I)$ for $m \neq n$, then $(m+1,n) \in \mathcal{R}(I)$.
6. If $(n,2),(n+1,2) \in \mathcal{R}(I)$ for an integer $n \geq 3$, then $(n+2,2) \in \mathcal{R}(I)$, and thus $(m,2) \in \mathcal{R}(I)$ for every positive integer m.
7. If $(m,n) \in \mathcal{R}(I)$ for positive integers m and n with $n \leq m/2$, then $(m+1,n) \in \mathcal{R}(I)$, and thus $(k,n) \in \mathcal{R}(I)$ for every positive integer k.
8. $(m,n) \in \mathcal{R}(I)$ for every positive integer m if and only if $(2n,n) \in \mathcal{R}(I)$.
9. $\mathcal{R}(I\times J) = \mathcal{R}(I) \cap \mathcal{R}(J) \subseteq \mathcal{R}(I\cap J)$.

Theorem 2.22. ([4, Theorem 4.3]) Let R be a commutative ring, I a proper ideal of R, and m and n positive integers. Let $f_I(m) = \min\{n \mid I \text{ is } (m,n)\text{-closed}\}$ and $g_I(n) = \sup\{m \mid I \text{ is } (m,n)\text{-closed}\}$.

1. $1 \leq f_I(m) \leq m$.
2. $f_I(m) \leq f_I(m+1)$.
3. If $f_I(m) < m$, then either $f_I(m+1) = f_I(m)$ or $f_I(m+1) \geq f_I(m)+2$.
4. $n \leq g_I(n) \leq \infty$.
5. $g_I(n) \leq g_I(n+1)$.
6. If $g_I(n) > n$, then either $g_I(n+1) = g_I(n)$ or $g_I(n+1) \geq g_I(n)+2$.

Theorem 2.23. ([4, Theorem 5.1]) The following statements are equivalent for a commutative ring R.

1. If $a^2 = b^3$ with $a, b \in R$ regular, then $a = c^3$ and $b = c^2$ for some regular $c \in R$ (i.e., R is seminormal with respect to regular elements of R)
2. If $b^2 \mid a^2$ and $b^3 \mid a^3$ with $a, b \in R$ regular, then $b \mid a$ (i.e., R is $(2,3)$-closed with respect to units of $T(R)$)

Let R be a commutative ring with nonzero identity. Recall that R is a *von Neumann regular* if for every $a \in R$, there is $x \in R$ such that $a^2x = a$, that R is π-regular if for every $a \in R$, there are $x \in R$ and an integer $n \geq 1$ such that $a^{2n}x = a^n$, and that R is Boolean if $a^2 = a$ for every $a \in R$. Thus a Boolean ring is von Neumann regular and a von Neumann regular ring is π-regular. Let $\pi - r(R)$ be the set of all π-regular elements of R and $vnr(R)$ be the set of all von Neumann regular elements of R. If R is a ring, then $Z(R)$ denotes the set of all zerodivisors of R, $Id(R)$ denotes the set of all idempotent elements of R, $U(R)$ denotes the set of all units of R, and $Nil(R)$ denotes the set of all nilpotent elements of R.

Theorem 2.24. ([3, Theorem 2.2]) Let R be a commutative ring. The following statements are equivalent.

1. $a \in vnr(R)$.
2. $a^2u = a$ for some $u \in U(R)$.
3. $a = eu$ for some $e \in Id(R)$ and $u \in U(R)$.
4. $ab = 0$ for some $b \in vnr(R) \setminus \{a\}$ with $a + b \in U(R)$.
5. $ab = 0$ for some $b \in R$ with $a + b \in U(R)$.

Theorem 2.25. ([3, Theorem 2.11]) Let R be a commutative ring with $2 \in U(R)$. The following statements are equivalent.

1. $vnr(R)$ is a subring of R.
2. The sum of any four regular elements of R is a von Neumann regular element of R.
3. Let $u, v, k, m \in U(R)$ with $k^2 = m^2 = 1$. Then $u(1 + k) + v(1 + m) \in vnr(R)$.

Theorem 2.26. ([3, Theorem 2.2]) Let R be a commutative ring. The following statements are equivalent.

1. $a \in \pi - r(R)$.
2. $a^n \in vnr(R)$ for some $n \geq 1$.
3. $a^n = eu$ for some $e \in Id(R)$, $u \in U(R)$, and $n \geq 1$.
4. $a = b + w$ for some $b \in vnr(R)$ and $w \in Nil(R)$.
5. $a = eu + w$ for some $e \in Id(R)$, $u \in U(R)$, and $w \in Nil(R)$.
6. $a^nb = 0$ for some $b \in R$, $n \geq 1$ with $a^n + b \in U(R)$.
7. $ab = Nil(R)$ for some $b \in R$ with $a + b \in U(R)$.

Theorem 2.27. ([3, Proposition 1.1]) Let R be a commutative ring with $1 \neq 0$. The following statements are equivalent.

1. R is a von Neumann regular ring.
2. Let $m \geq 2$ be a fixed integer. If $x \mid y^m$ for $x, y \in R$, then $x \mid y$.
3. Let $x, y \in R$. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid y$.
4. Let $x, y \in R$. If $y^n = xd$ for some integer $n \geq 1$ with $d \in R$ a nonunit, then $x \mid y$.
5. All proper ideals of R are radical ideals.
6. All principal proper ideals of R are radical ideals.

Theorem 2.28. ([3, Proposition 1.2]) Let R be a commutative ring with $1 \neq 0$. The following are equivalent.

1. $T(R)$ is a von Neumann regular ring.
2. Let $x, y \in R$. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid sy$ for some regular element $s \in R$.

Theorem 2.29. ([3, Theorem 2.2]) Let R be a commutative ring. The following statements are equivalent.

1. $T(R)$ is zero-dimensional ring.
2. For each $x \in R$, there is a $y \in R$ and an integer $n \geq 1$ such that $x^ny = 0$ and $x^n + y$ is a regular element of R.
3. For each $x \in R$, there is a $y \in R$ such that $xy \in Nil(R)$ and $x + y$ is a regular element of R.

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