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Author(s):

Masoud Bahrami Seif Abad & Mehdi Sharifzadeh

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Some topological results on a globally hyperbolic spacetime with non-compact Cauchy surfaces

Masoud Bahrami Seif Abad[1] and Mehdi Sharifzadeh[2]
[1]Department of Mathematics, Yasouj University,Daneshjo Ave. 75918-74831 Yasouj, Iran
e-mail: *bahrami271@gmail.com*
[2]Department of Mathematics, Yasouj University,Daneshjo Ave. 75918-74831 Yasouj, Iran
e-mail: *msharifzadeh@yu.ac.ir*

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Abstract. This paper studies new topologies on globally hyperbolic spacetimes with non-compact Cauchy surfaces, demonstrating that these topologies are T1 but not Hausdorff, and are contained within manifold topology. The authors have derived certain properties of these topologies and shown that a continuous representation of a globally hyperbolic spacetime with a non-compact Cauchy surface is possible, based on the causally admissible systems of its Cauchy surface.

Key Words: Spacetime, Globally Hyperbolic, Order Isomorphism, Vietoris Topology, Causally Admissible System.
2010 MSC: 06B35, 53C50, 83C99.

1 Introduction and Preliminaries

A time-oriented and causal spacetime M can be seen as a partially ordered set, which means it is a set with a reflexive, antisymmetric, and transitive relation. In this case, the partial ordering is based on causal precedence (you can find the basic definitions and more information about the causal structure of a spacetime in [4], [6], [8]) . So, if p and q are elemenet of $\mathcal{M}$, then $p \leqslant q$ if and only if either $p = q$ or there is a future-pointing causal curve connecting p to q.

Definition 1.1. Let $(X_1, \leqslant_1)$ and $(X_2, \leqslant_2)$ be two partially ordered sets. A mapping $f : X_1 \to X_2$ is said to be
(i) an order-embedding map if $x \leqslant_1 y$ if and only if $f(x) \leqslant_2 f(y)$, for all $x, y \in X_1$.
(ii) an order-isomorphism if it is an order-embedding which maps X_1 onto X_2.

Note that since we assumed that every partial order is reflexive in the above definition an order-embedding is already an injection.

Definition 1.2. Let $(X, \leqslant)$ be a partially ordered set. If there is a family $\mathfrak{C}$ of subsets of a given set E and an order-embedding $f : X \to \mathfrak{C}$ such that for all $x, y \in X$, $x \leqslant y \Leftrightarrow f(x) \subseteq f(y)$, we say that $(X, \leqslant)$ is $\mathfrak{C}$-representable. In addition, if $X, \mathfrak{C}$ are two topological spaces and f is continuous, then we say X is continuously $\mathfrak{C}$-representable.

Example 1.3. Let $\mathcal{M}$ be a globally hyperbolic spacetime with non-compact spacelike Cauchy surface Σ and let " $\leqslant$ " be the causally relation on $\mathcal{M}$. Then, $J^+(\Sigma)$ and $J^-(\Sigma)$ with the causally relation $\leqslant$ are partially ordered sets. Also, the future causally admissible system C^+ and the past causally admissible system C^- corresponding to Σ, with the inclusion relation $\subseteq$, are partially ordered sets. Define $\varphi : J^+(\Sigma) \to C^+$ where $\varphi(p) = S_p^+$ for all $p \in J^+(\Sigma)$ and $\psi : J^-(\Sigma) \to C^-$ where $\psi(p) = S_p^-$ for all $p \in J^-(\Sigma)$. Then, by definitions of C^+ and C^-, Proposition 3.2 in [1] and Theorem 3.1 in [2], φ and ψ are order-isomorphisms between the partially ordered sets. Therefore, $J^+(\Sigma)$ is C^+-representable and $J^-(\Sigma)$ is C^--representable.

Throughout this paper we assume that $\mathcal{M}$ is a globally hyperbolic spacetime with a non-compact, smooth, spacelike Cauchy surface Σ. Let C^+ and C^- be respectively the sets of all future and past causally admissible subsets of $\mathcal{M}$ with respect to Σ. That is

$$C^+ = \{S_p^+ = J^-(p) \cap \Sigma : p \in J^+(\Sigma)\}$$

and

$$C^- = \{S_p^- = J^+(p) \cap \Sigma : p \in J^-(\Sigma)\}$$

and they are called future and past admissible systems respectively. We note that S_p^+ and S_q^- are compact, connected subsets of Σ for each $p \in J^+(\Sigma)$ and each $q \in J^-(\Sigma)$. Let $C = (C^+, C^-)$. It is called causally admissible system on Σ.
Some important properties of the causally admissible subsets are the following (see [2]):

Theorem 1.4. Let Σ be a non-compact Cauchy surface of $\mathcal{M}$;
(i) If $p, q \in J^+(\Sigma)$, then $p \leqslant q$ if and only if $S_p^+ \subseteq S_q^+$.
(ii) If $p, q \in J^-(\Sigma)$, then $p \leqslant q$ if and only if $S_p^- \supseteq S_q^-$.
(iii) if $p \in J^-(\Sigma)$ and $q \in J^+(\Sigma)$, then $p \leqslant q$ if and only if $S_p^- \cap S_q^+ \neq \phi$.

In the following proposition we review some known results about the causally admissible subsets.

Proposition 1.5. *For a spacetime $\mathcal{M}$ with a non-compact Cauchy surface Σ;*
(i) If $p, q \in J^+(\Sigma)$, then $S_p^+ = S_q^+$ if and only if $p = q$.
(ii) If $p, q \in J^-(\Sigma)$, then $S_p^- = S_q^-$ if and only if $p = q$.

Proof. See [1]. □

Some of the most important results in this paper are about the causal or chronological isomorphisms between two spacetimes. Thus we are lead to introduce them as follows.

Definition 1.6. A bijective function $f : \mathcal{M} \to \mathcal{M}'$ between two spacetimes is called a causal isomorphism if $p \leqslant q \Leftrightarrow f(p) \leqslant f(q)$ and a chronological isomorphism if $p \ll q \Leftrightarrow f(p) \ll f(q)$. If there exists a causal isomorphism (chronological isomorphism, resp.) between $\mathcal{M}$ and $\mathcal{M}'$ then we say that $\mathcal{M}$ and $\mathcal{M}'$ are causally isomorphic (chronologically isomorphic, resp.).

In the following we will state some results about the causal isomorphisms which can be found in [5] and [7].

Theorem 1.7. For a bijection $f : \mathcal{M} \to \mathcal{M}'$ between two chronological spacetimes, we have the following properties.
(i) f is a causal isomorphism if and only if f is a chronological isomorphism.
(ii) If f is a causal isomorphism, then f is a smooth conformal diffeomorphism.

Suppose that $\mathcal{M}$ and $\mathcal{M}'$ are globally hyperbolic spacetimes with non-compact Cauchy surfaces Σ and Σ', respectively. Let C^+ and C'^+ be the corresponding future admissible systems for Σ and Σ' respectively, and we denote these by (Σ, C^+) and (Σ', C'^+). Then, since the causal relation is encoded into C through the relation of inclusion, it is not difficult to see the following theorem.

Theorem 1.8. Two spacetimes $\mathcal{M}$ and $\mathcal{M}'$ with non-compact Cauchy surfaces are causally isomorphic if and only if there exists a causally admissible function $f : (\Sigma, C) \to (\Sigma', C')$ between the corresponding causally admissible systems.

Proof. See [1], Theorem 5.4. □

2 Some new topologies on M by using causally admissible systems

Lemma 2.1. *For $p \in I^+(\Sigma)$, let U be any open subset in Σ such that $S^+_p \subseteq U$. Then there exists $q \in I^+(p)$ such that $S^+_p \subset S^+_q \subset U$.*

Proof. See [2], Lemma 4.1. □

Lemma 2.1 has a dual which is obtained by similar arguments as in the case of lemma 2.1

Lemma 2.2. *For $p \in I^-(\Sigma)$, let U be any open subset in Σ such that $S^-_p \subseteq U$. Then there exists $q \in I^-(p)$ such that $S^-_p \subset S^-_q \subset U$.*

Definition 2.3. Let Σ be a non-compact Cuachy surface of a spacetime $\mathcal{M}$ and let U be any open subset in Σ. We set

$$E^+(U) := \{p \in I^+(\Sigma) : S^+_p \subset U\},$$
$$E^-(U) := \{p \in I^-(\Sigma) : S^-_p \subset U\},$$
$$E(U) := E^+(U) \cup U \cup E^-(U).$$

In the following we show that $E^+(U)$, $E^-(U)$ and $E(U)$ are open subsets in $\mathcal{M}$.

Lemma 2.4. *Let U be any open subset in Σ. Then, $E^+(U)$ and $E^-(U)$ are open subsets in $\mathcal{M}$.*

Proof. Let $p \in E^+(U)$. By lemma 2.1 there exists $q \in I^+(p)$ such that $S^+_p \subset S^+_q \subset U$. Now let $r \in I^-(p) \cap I^+(\Sigma)$. Then, $I^+(r) \cap I^-(q)$ is an open neighborhood for p. We want to show that $I^+(r) \cap I^-(q) \subset E^+(U)$.

Let $t \in I^+(r) \cap I^-(q)$. We see that $r \ll t \ll q$. Then, by Theorem 3.1 in [2] (i), $S^+_r \subset S^+_t \subset S^+_q \subset U$ and it implies that $t \in I^+(\Sigma)$ and $S^+_t \subset U$. Therefore, we have $t \in E^+(U)$. Hence, $I^+(r) \cap I^-(q) \subset E^+(U)$ and, $E^+(U)$ is an open set of $\mathcal{M}$.

By lemma 2.2 and similar arguments, we can show that $E^-(U)$ is an open set in $\mathcal{M}$. □

To show that $E(U)$ is an open set in $\mathcal{M}$, we need to express and prove the following lemma. The approach of the proof of the following lemma is similar to the proof of Lemma 4.1 in [2].

Lemma 2.5. *Let U be any open subset in Σ. For each $p \in U$, there exists $q \in E^+(U)$ such that $p \in I^-(q)$ and $I^-(q) \cap \Sigma \subseteq U$.*

Proof. Let $\{q_i\}$ be a sequence in $I^+(p)$ such that q_i converges to p and $q_{i+1} \ll q_i$ for all i. If for some i we have $I^-(q_i) \cap \Sigma \subseteq U$, then the proof is complete. Suppose that for all i, $I^-(q_i) \cap \Sigma \nsubseteq U$. Then, for each i we choose $x_i \in (I^-(q_i) \cap \Sigma) - U$. We know that for each i, $x_i \in (J^-(q_1) \cap \Sigma) - U$ and $J^-(q_1) \cap \Sigma$ is compact therefore there exists subsequence $\{x_{i_k}\}$ of $\{x_i\}$ such that converges to some point $x_0 \in (J^-(q_1) \cap \Sigma) - U$. Without loss of generality, we denote $\{x_{i_k}\}$ by $\{x_i\}$. Since $x_i \ll q_i$ and x_i and q_i converge to x_0 and p, respectively, we have $x_0 \leqslant p$ (since the relation "$\leqslant$" is closed in globally hyperbolic spacetimes). It yields that $x_0 \in S^+_p = \{p\}$ and then, $x_0 = p$ and $x_0 \in U$, which is a contradiction to $x_0 \in (I^-(q_i) \cap \Sigma) - U$. This contradiction stems from the assumption that $I^-(q_i) \cap \Sigma \nsubseteq U$ for each i, and we can conclude that for some i, $I^-(q_i) \cap \Sigma \subseteq U$. We set $q = q_i$ and the proof is complete. □

In the following lemma we state the time dual of lemma 2.5 which can be proved by the similar arguments.

Lemma 2.6. *Let U be any open subset in Σ. For each $p \in U$, there exists $q \in E^-(U)$ such that $p \in I^+(q)$ and $I^+(q) \cap \Sigma \subset U$.*

Note that by lemma 2.5 and lemma 2.6 we can conclude that if U is an nonempty open set in Σ then $E^+(U)$ and $E^-(U)$ are nonempty open sets in $\mathcal{M}$.

Proposition 2.7. *Let U be an open set in Σ. Then, $E(U)$ is an open set in $\mathcal{M}$.*

Proof. Let $p \in E(U)$. If $p \in E^+(U)$ or $p \in E^-(U)$, since $E^+(U)$ and $E^-(U)$ are open in $\mathcal{M}$, p is an interior point of $E(U)$.

Suppose that $p \in U$. By lemma 2.5 and lemma 2.6, there are $a \in E^+(U)$ and $b \in E^-(U)$ such that $p \in I^-(a)$, $I^-(a) \cap \Sigma \subset U$ and $p \in I^+(b)$, $I^+(b) \cap \Sigma \subset U$. Therefore $I^-(a) \cap I^+(b)$ is an open neighborhood of p. We want to show that $I^-(a) \cap I^+(b) \subset E(U)$. For this we have

$$I^-(a) \cap I^+(b) = (I^-(a) \cap I^+(b)) \cap M = I^-(a) \cap I^+(b) \cap (I^+(\Sigma) \cup \Sigma \cup I^-(\Sigma))$$

$$= [I^-(a) \cap I^+(b) \cap I^+(\Sigma)] \cup [I^-(a) \cap I^+(b) \cap \Sigma] \cup [I^-(a) \cap I^+(b) \cap I^-(\Sigma)]$$

On the other hand, we observe that:
(i) $I^-(a) \cap I^+(b) \cap I^+(\Sigma) \subset I^-(a) \cap I^+(\Sigma) \subset E^+(U)$. Because, for each $q \in I^-(a) \cap I^+(\Sigma)$, we have $J^-(q) \subset I^-(a)$ and so $S_q^+ \subset I^-(a) \cap \Sigma \subset U$. Then, $q \in E^+(U)$.
(ii) $I^-(a) \cap I^+(b) \cap \Sigma \subset I^-(a) \cap \Sigma \subset U$.
(iii) $I^-(a) \cap I^+(b) \cap I^-(\Sigma) \subset I^+(b) \cap I^-(\Sigma) \subset U$. Because, for each $q \in I^+(b) \cap I^-(\Sigma)$, we have $J^+(q) \subset I^+(b)$ and so $S_q^- \subset I^+(b) \cap \Sigma \subset U$. Then, $q \in E^-(U)$.
Therefore, by (i), (ii), (iii),

$$I^-(a) \cap I^+(b) \subset E^+(U) \cup U \cup E^-(U) = E(U).$$

Hence, p is an interior point of $E(U)$ and the proof is complete. □

Definition 2.8. Let Σ be a non-compact Cuachy surface of a spacetime $\mathcal{M}$ and let U be any open subset in Σ. We set

$$K^+(U) := \{p \in I^+(\Sigma) : S_p^+ \cap U \neq \phi\}$$
$$K^-(U) := \{p \in I^-(\Sigma) : S_p^- \cap U \neq \phi\}$$
$$K(U) := K^+(U) \cup U \cup K^-(U).$$

In the following we show that $K^+(U)$, $K^-(U)$ and $K(U)$ are open subsets in $\mathcal{M}$. To prove these, we need the following lemmas. Note that by the definitions, $E^+(U) \subset K^+(U)$, $E^-(U) \subset K^-(U)$ and $E(U) \subset K(U)$.

Lemma 2.9. *Let U be an open subset of Σ. If $p \in K^+(U) - E^+(U)$, then $I^-(p) \cap U \neq \phi$.*

Proof. By definitions of $K^+(U)$ and $E^+(U)$ we have $p \in I^+(\Sigma)$, $S_p^+ \cap U \neq \phi$ and $S_p^+ \not\subset U$. Let $q \in S_p^+ \cap U$. Then, $q \in J^-(p)$. If $q \in I^-(p)$, then $q \in I^-(p) \cap U$. Thefore, in this case the proof is complete.
Now let $q \in J^-(p) - I^-(p)$. Then there exists a past directed null curve $\alpha : [0,1] \to \mathcal{M}$ such that $\alpha(0) = p$ and $\alpha(1) = q$. Since $q \in U$ and U is an open subset of Σ, $E(U)$ is an open neighborhood of q in $\mathcal{M}$ (by proposition 2.7, $E(U)$ is an open subset of $\mathcal{M}$). By continuouty there exists $t_0 \in (0,1)$ such that $\alpha(t_0) \in E(U)$. Let $r = \alpha(t_0)$. Since Σ is a spacelike Cuchy surface and α is a past directed null curve that intersect Σ in q, we must have $r \in J^+(\Sigma) - \Sigma = I^+(\Sigma)$ (because, Σ is a spacelike Cauchy surface and α intersect Σ exactly in q). So by the definition of $E(U)$, $r \in E^+(U)$. We choose $s \in I^-(r) \cap \Sigma$. It is possible because $r \in I^+(\Sigma)$, $I^-(r) \cap \Sigma \neq \phi$. Then,

$$s \in I^-(r) \cap \Sigma \subset J^-(r) \cap \Sigma = S_r^+ \subset U.$$

Since $s \ll r$ and $r \leqslant p$, we have $s \ll p$ and $s \in I^-(p)$. Hence, $s \in I^-(p) \cap U$. □

The following lemma is the time dual of lemma 2.9 and one can prove it by the similar arguments as in lemma 2.9.

Lemma 2.10. *Let U be an open subset of Σ. If $p \in K^-(U) - E^-(U)$, then $I^+(p) \cap U \neq \phi$.*

The following lemma is another result about $K^+(U)$.

Lemma 2.11. *Let U be any open set in Σ, $p \in I^+(\Sigma)$ and $I^-(p) \cap U \neq \phi$. Then, $I^+(p) \subset K^+(U)$.*

Proof. Let $q \in I^+(p)$. We see that $I^-(p) \subset J^-(q)$ and since $p \in I^+(\Sigma)$, we have $q \in I^+(\Sigma)$. Then,

$$\phi \neq I^-(p) \cap U \subset J^-(q) \cap U = J^-(q) \cap \Sigma \cap U = S_q^+ \cap U.$$

Therefore, $S_q^+ \cap U \neq \phi$ and $q \in K^+(U)$. This proves that $I^+(p) \subset K^+(U)$. □

The following lemma is the time dual of lemma 2.11, for completeness we state it here.

Lemma 2.12. *Let U be any open subset of Σ, $p \in I^-(\Sigma)$ and $I^+(p) \cap U \neq \phi$. Then, $I^-(p) \subset K^-(U)$.*

Now, we are ready to show that $K^+(U)$, $K^-(U)$ and $K(U)$ are open subsets of $\mathcal{M}$.

Proposition 2.13. *Let U be an open subset in Σ. Then, $K^+(U)$ and $K^-(U)$ are open subsets of $\mathcal{M}$.*

Proof. Let $p \in K^+(U)$. If $p \in E^+(U)$, since $E^+(U) \subset K^+(U)$ and $E^+(U)$ is an open set in $\mathcal{M}$, then p is an interior point of $K^+(U)$. Suppose $p \in K^+(U) - E^+(U)$. In veiw of lemma 2.9, $I^-(p) \cap U \neq \phi$ and we can choose $q \in U$ and $r \in I^+(\Sigma)$ such that $q \ll r \ll p$. Therefore, $q \in I^-(r) \cap U$ and $p \in I^+(r)$. By lemma 2.11, $I^+(r) \subset K^+(U)$. Then, we prove that $p \in I^+(r) \subset K^+(U)$, that is p is an interior point of $K^+(U)$. Hence, $K^+(U)$ is an open set of $\mathcal{M}$.

By lemma 2.10, lemma 2.12 and similar arguments, we can prove that $K^-(U)$ is an open set in $\mathcal{M}$. □

Proposition 2.14. *Let U be an open subset of Σ. Then, $K(U)$ is an open set of $\mathcal{M}$.*

Proof. Let $p \in K(U)$. If $p \in K^+(U)$ or $p \in K^-(U)$, then by proposition 2.13, p is an interior point of $K(U)$. Let $p \in U$. Since $p \in E(U) \subset K(U)$ and $E(U)$ is open in $\mathcal{M}$, p is an interior point of $\mathcal{M}$. Hence, $K(U)$ is an open set in $\mathcal{M}$. □

3 Continuous representation of a spacetime with non-compact Cauchy surfaces

In 2015, Choudhury and Mondal have shown that continuous representation of a globally hyperbolic spacetime with non-compact Cauchy surface is possible in view of the causally admissible systems of its Cauchy surface. The admissible systems have been studied in [1], [2]. They are the building blocks for encoding the causal structure of a globally hyperbolic spacetime with non-compact Cauchy surface into its Cauchy surface. The admissible system C with respect to non-compact spacelike Cauchy surface Σ is the set of all compact and connected subsets of the form $S_p^+ = J^-(p) \cap \Sigma$ for $p \in J^+(\Sigma)$ and $S_q^- = J^+(q) \cap \Sigma$ for $q \in J^-(\Sigma)$. Such subsets are called future and past causally admissible subsets respectively. The sets C^+ and C^- are respectively the sets of all future and past causally admissible subsets of $\mathcal{M}$ with respect to Σ and are called future and past causally admissible systems respectively. It is worthy to note that for compact Cauchy surfaces like in case of Einstein's static universe, which has the compact Cauchy surface, it may happen that $S_p^+ = S_q^+$ with $p \neq q$. However, if Σ is non-compact $S_p^+ = S_q^+$ if and only if $p = q$ (see Proposition 3.2 in [1]). In order to show the continuouty of the representation, Vietoris topology has been employed by Choudhury and Mondal (see Theorem 3.3 in [3]). In the proof of Theorem 3.3 of [3], there are two gaps about the continuity of the map $\varphi : J^+(\Sigma) \to C^+$ where the authors of [3] tried to show that $\varphi^{-1}(O < \mathcal{U} >)$ and $\varphi^{-1}(O < \mathcal{U}, \Sigma >)$ are open subsets of $J^+(\Sigma)$ (which $\mathcal{U}$ is an open subset of spacelike Cauchy surface Σ). In that proof it has been shown that $\varphi^{-1}(O < \mathcal{U} >) = \mathcal{V} \cup \mathcal{U}$ where $\mathcal{V}$ is an open subset of $J^+(\Sigma)$ and the authors implied $\varphi^{-1}(O < \mathcal{U} >)$ is an open subset of $J^+(\Sigma)$. But, the proof still is not clear since $\mathcal{U}$ is an open subset of Σ. Moreover, there is another gap in the proof of Theorem 3.3 in [3] while the authors wanted to establish the assertion that $\varphi^{-1}(O < \mathcal{U}, \Sigma >) = \chi \cup \mathcal{U}$ is an open subset of $J^+(\Sigma)$.

In this note, we fill up these gaps and for this we introduce some open subsets of spacetime $\mathcal{M}$ with respect to U called $E(U)$ and $K(U)$ where U is an open subset of non-compact spacelike Cauchy surface Σ.

Choudhury and Mondal have stated the following theorem (see Theorem 3.3 in [3]) that it shows that $J^+(\Sigma)$ is continuously C^+-representable.

Theorem 3.1. Let $\mathcal{M}$ is a globally hyperbolic spacetime with non-compact spacelike Cauchy surface Σ and let C^+ be considered as a sub-space of $exp\Sigma$ (the set of all closed subsets of Σ) endowed with Vietoris topology and $\varphi : J^+(\Sigma) \to C^+$, the maping defined by $\varphi(p) = S^+_p$ for all $p \in J^+(\Sigma)$, then φ is continuous.

In 2015, Choudhury and Mondal have shown that continuous representation of a globally hyperbolic spacetime with non-compact Cauchy surface is possible in view of the causally admissible systems of its Cauchy surface. The admissible systems have been studied in [1], [2]. They are the building blocks for encoding the causal structure of a globally hyperbolic spacetime with non-compact Cauchy surface into its Cauchy surface. The admissible system C with respect to non-compact spacelike Cauchy surface Σ is the set of all compact and connected subsets of the form $S^+_p = J^-(p) \cap \Sigma$ for $p \in J^+(\Sigma)$ and $S^-_q = J^+(q) \cap \Sigma$ for $q \in J^-(\Sigma)$. Such subsets are called future and past causally admissible subsets respectively. The sets C^+ and C^- are respectively the sets of all future and past causally admissible subsets of $\mathcal{M}$ with respect to Σ and are called future and past causally admissible systems respectively. It is worthy to note that for compact Cauchy surfaces like in case of Einstein's static universe, which has the compact Cauchy surface, it may happen that $S^+_p = S^+_q$ with $p \neq q$. However, if Σ is non-compact $S^+_p = S^+_q$ if and only if $p = q$ (see Proposition 3.2 in [1]). In order to show the continuouty of the representation, Vietoris topology has been employed by Choudhury and Mondal (see Theorem 3.3 in [3]). In the proof of Theorem 3.3 of [3], there are two gaps about the continuity of the map $\varphi : J^+(\Sigma) \to C^+$ where the authors of [3] tried to show that $\varphi^{-1}(O < \mathcal{U} >)$ and $\varphi^{-1}(O < \mathcal{U}, \Sigma >)$ are open subsets of $J^+(\Sigma)$ (which $\mathcal{U}$ is an open subset of spacelike Cauchy surface Σ). In that proof it has been shown that $\varphi^{-1}(O < \mathcal{U} >) = \mathcal{V} \cup \mathcal{U}$ where $\mathcal{V}$ is an open subset of $J^+(\Sigma)$ and the authors implied $\varphi^{-1}(O < \mathcal{U} >)$ is an open subset of $J^+(\Sigma)$. But, the proof still is not clear since $\mathcal{U}$ is an open subset of Σ. Moreover, there is another gap in the proof of Theorem 3.3 in [3] while the authors wanted to establish the assertion that $\varphi^{-1}(O < \mathcal{U}, \Sigma >) = \chi \cup \mathcal{U}$ is an open subset of $J^+(\Sigma)$.
In this note, we fill up these gaps and for this we introduce some open subsets of spacetime $\mathcal{M}$ with respect to U called $E(U)$ and $K(U)$ where U is an open subset of non-compact spacelike Cauchy surface Σ.

Note 3.2. *It is noteworthy that in the proof of the above theorem in [3], there are two gaps about the continuity of the map $\varphi : J^+(\Sigma) \to C^+$, where the authors of [3] tried to show that $\varphi^{-1}(O < \mathcal{U} >)$ and $\varphi^{-1}(O < \mathcal{U}, \Sigma >)$ are open subsets of $J^+(\Sigma)$ where $\mathcal{U}$ is an open subset of the spacelike Cauchy surface Σ. The gap in the proof of $\varphi^{-1}(O < \mathcal{U} >)$ is open subset of $J^+(\Sigma)$, is as the following. The authors of [3] showed that $\varphi^{-1}(O < \mathcal{U} >) = \mathcal{V} \cup \mathcal{U}$ and they showed that $\mathcal{V}$ is an open subset of $J^+(\Sigma)$. Since $\varphi^{-1}(O < \mathcal{U} >) = \mathcal{V} \cup \mathcal{U}$, they concluded $\varphi^{-1}(O < \mathcal{U} >)$ is an open subset of $J^+(\Sigma)$, but $\mathcal{U}$ is an open subset of Σ and we cannot imply that $\mathcal{V} \cup \mathcal{U}$ is open subset of $J^+(\Sigma)$. Similarly, there is another gap in the proof of $\varphi^{-1}(O < \mathcal{U}, \Sigma >) = \chi \cup \mathcal{U}$ is open subset of $J^+(\Sigma)$. In the following we fill up these gaps.*

At first, for filling the mentioned gaps we introduce some open subsets of a globally hyperbolic spacetime.
We start with the following lemmas.
Finally we state the complete proof of Theorem 3.1 as follows.
The proof of theorem 3.1. To prove that φ is continuous we need only to show that inverse image of sub-basic open sets are open.

Let us consider the sub-basic open subset $O_{C^+} < U >= O < U > \cap C^+$ where U is an open subset of Σ. Now, we have

$$O < U > \cap C^+ = \{S^+_p \in C^+ : S^+_p \subset U\}$$

Therefore, we see

$$\varphi^{-1}(O_{C^+} < U >) = \{p \in J^+(\Sigma) : S^+_p \subset U\}$$

Since Σ is a Cauch surface, $J^+(\Sigma) = I^+(\Sigma) \cup \Sigma$. Then, we have

$$\varphi^{-1}(O_{C^+} < U >) = \{p \in I^+(\Sigma) : S^+_p \subset U\} \cup \{p \in \Sigma : S^+_p \subset U\} = E^+(U) \cup U = E(U) \cap J^+(\Sigma)$$

Therefore, $\varphi^{-1}(O_{C^+} < U >)$ is an open subset of $J^+(\Sigma)$ equipped to the sub-space topology.

Next, let us consider the other sub-basic open set $O_{C^+} < U,\Sigma >= O < U,\Sigma > \cap C^+$ where U is an open subset of Σ. Now, set

$$O < U,\Sigma > \cap C^+ = \{S_p^+ \in C^+ : S_p^+ \cap U \neq \phi\}$$

Therefore, we have

$$\varphi^{-1}(O_{C^+} < U,\Sigma >) = \{p \in J^+(\Sigma) : S_p^+ \cap U \neq \phi\}$$

Then, it yields that

$$\varphi^{-1}(O_{C^+} < U,\Sigma >) = \{p \in I^+(\Sigma) : S_p^+ \cap U \neq \phi\} \cup \{p \in \Sigma : S_p^+ \cap U \neq \phi\} = K^+(U) \cup U = K(U) \cap J^+(\Sigma)$$

Therefore, $\varphi^{-1}(O_{C^+} < U,\Sigma >)$ is an open subset of $J^+(\Sigma)$ endowed with the sub-space topology. Hence, we conclude that the mapping φ is continuous.

By a similar approach we can show the time dual of theorem 3.1 as follows.

Theorem 3.3. Let $\mathcal{M}$ is a globally hyperbolic spacetime with non-compact spacelike Cauchy surface Σ and let C^- be considered as a sub-space of $exp\Sigma$ endowed with Vietoris topology. If $\varphi : J^-(\Sigma) \to C^-$, the mapping defined by $\varphi(p) = S_p^-$ for all $p \in J^-(\Sigma)$, then φ is continuous.

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