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Rings in which every nonzero S−weakly prime ideal is weakly prime

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Abstract. In this paper, we introduce and study a new class of rings with multiplicative subset S which we'll call S-ME-rings. A ring R with a multiplicative subset S is said to be S-ME-ring if every non-zero S-weakly prime ideal of R is weakly prime. We next study the possible transfer of the properties of being S-ME-ring in the homomorphic image, in the trivial ring extensions and the amalgamated algebra along an ideal introduced and studied by the authors of [6, 7, 8, 9]. Our results allow us to construct new original class of S-ME-rings subject to various ring theoretical properties.
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1 Introduction

Throughout this paper, all rings considered are assumed to be commutative with non-zero identity and all modules are nonzero unital. As a motivation of this work is to study the rings in which every nonzero S-weakly prime ideal is weakly-prime.

The authors of [2] introduced and studied the concept of weakly-prime ideals. A proper ideal P of R is said to be weakly-prime ideal of R if for every $a, b \in R$ such that $ab \neq 0$, $ab \in P$ implies that either $a \in P$ or $b \in P$. It is shown in [2, Theorem 3] that a proper ideal P is weakly prime if and only if for every $x \in R \backslash P$, $P : x = P \cup (0 : x)$, that is equivalent to say that $P : x = P$ or $P : x = 0 : x$ for every $x \in R \backslash P$. It is shown in [2, Theorem 8] that a ring R has every proper ideal weakly prime if and only if either R is a local ring with unique maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = 0$ or R is isomorphic to direct product of two fields. Next, the authors of [13] defined the S-prime ideals P of a ring R as follows: a proper ideal of R is said to be S-prime if there exists $s \in S$ such that every $a, b \in R$, either $sa \in P$ or $sb \in P$; [13, Definition]. The authors [17] introduced and studied a new class of ideals which called S-weakly prime. A proper ideal P of a ring R with multiplicative subset S is said to be S-weakly prime if there exists $s \in S$ such that the condition holds: for every $a, b \in R$ such that $ab \neq 0$, $ab \in P$, then either $sa \in P$ or $sb \in P$; [17, Definition 2.1].

In [11], A. El Khalfi, N. Mahdou and Y. Zahir introduced the concept of WP-rings. A ring A is called WP-ring if every nonzero weakly prime ideal is prime. Recently, the concept of S-property has an important place in commutative algebra and it draw attention by several authors. The S-weakly prime ideals introduced by the authors of [1, 17] is a generalization of the work of A. Hamed and A. Malek in [13]. Following [17] a proper ideal P is said to be S-weakly prime (where $S \subseteq A$ multiplicative set, and $P \cap S = \emptyset$) if there exists $s \in S$ such that the following condition holds for every $a, b \in A$: $0 \neq ab \in P$ implies that either $sa \in P$ or $sb \in P$. We denote by $\sqrt{0}$ the set for all nilpotent elements of A; $\mathrm{Ann}(I)$ or $(0 : I)$ denote the annihilator of an ideal I; $Reg(A)$ denotes the set of all regular elements of A. If A is an integral domain, we denote its quotients field by $qf(A)$. Let R be a ring and E an R-module. Then $R \propto E$, the trivial ring extension of R by E, is the ring whose additive structure is that of the external direct sum $R \oplus E$ and whose multiplication is defined by

$(a,e)(b,f) := (ab, af+be)$ for all $a,b \in R$ and all $e,f \in E$. (This construction is also known by other terminology and other notation, such as the idealization $R(+)E$) (see [14, 12, 4, 16]).

Let A and B be two rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can consider the following sub-ring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A, j \in J\},$$

called the amalgamation of A with B along J with respect to f (introduced and studied by D'Anna et al. [7, 9]). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana [8] and denoted by $A \bowtie I$).

The present paper contains one main section with introduction. In the main section, we will introduce and study a new class of rings with multiplicative subset S which characterized by the fact that every non-zero S-weakly prime ideal is weakly prime, these rings will be called S-ME-rings. The purpose of this work is to give some methods in order to construct S-ME-rings and to give many examples of this class. We investigate the stability of the S-ME property under homomorphic image, and its transfer to various context of constructions such as trivial rings extensions and amalgamations.

2 Main results

We begin this section by the following main definition stated as follows.

Definition 2.1. A ring R with multiplicative subset S is said to be S-ME-ring if every non-zero S-weakly prime ideal is weakly prime.

Remark 2.2. From [17, Remark 2.2], it is straightforward to see that every ring R with a multiplicative subset $S \subset U(R)$; where $U(R)$ is the units group of R is an S-ME-ring.

Next, we give a sufficient condition for a ring R to be S-ME-ring as follows.

Proposition 2.3. *Let R be a ring with multiplicative subset S. If every proper ideal of R is S-weakly prime, then R is an S-ME-ring.*

Proof. This follows immediately from [1, Proposition 2.26] and Remark 2.2. □

From [2, Theorem 8], it is shown that: every proper ideal of a ring R is weakly prime if and only if either R is a local ring such that its unique maximal ideal $\mathfrak{m}$ satisfies $\mathfrak{m}^2 = 0$ or R is a direct product of two fields. The following Proposition 2.4 allows to give examples of S-ME-rings.

Proposition 2.4. *Let R be a ring. If $(R, \mathfrak{m})$ is a local ring such that $\mathfrak{m}^2 = 0$ or R is a product of two fields, then R is an S-ME-ring for every multiplicative subset S.*

Proof. This is straightforward. □

Now, we study the transfer from S-ME-rings to quotient rings. For this purpose, we recall that if R is a ring with multiplicative subset S and for an ideal I of R, the subset $\overline{S} := \{s + I \mid s \in S\}$ of R/I is a multiplicative subset.

Proposition 2.5. *Let R be a ring with multiplicative subset S and I be an S-weakly prime ideal of R. If R is an S-ME-ring, then R/I is an $\overline{S}$-ME-ring.*

Proof. If Q is an $\overline{S}$-weakly prime ideal of R, then by [17, Proposition 2.6], there exists an S-weakly prime ideal P of R such that $Q = P/I$. But R is supposed an S-ME-ring, so P is a weakly prime in R, then Q is a weakly prime ideal of R/I by [2, Proposition 13]. We proved that R/I is an $\overline{S}$-ME-ring. □

The following Propositions 2.6 and 2.8 establish that the WP-rings and S-WP-rings are not the same.

Proposition 2.6. *Let R be a ring with multiplicative subset S. If both R is a WP-ring and S-ME-ring, then R is an S-WP-ring.*

Proof. Assume that R is both a WP-ring and S-ME-ring. Let P be an S-weakly prime ideal of R. By assumption, R is an S-ME-ring, then P is an S-weakly prime ideal of R, and so P is a prime ideal of R since R is a WP-ring. It follows that P is an S-prime ideal of R, that means R is an S-WP-ring. □

Remark 2.7. The converse of Proposition 2.6 is not true in general: in fact, for $R = \mathbb{Z}[X]$ with multiplicative subset $S = \{2^n \mid n \in \mathbb{N}\}$. It is easy to see that R is both a WP-ring and an S-WP-ring but not S-ME-ring since $P = 4XR$ is an S-weakly prime ideal because P is S-prime by [13, Example 1]. However, P is not weakly prime since $0 \neq 2 \times 2X \in P$ but both $2, 2X \notin P$.

Proposition 2.8. *Let R be a ring with multiplicative subset S. If R is an S-WP-ring and every S-prime ideal of R is prime, then R is a WP-ring.*

Proof. Let P be a weakly prime ideal of R, then P is an S-weakly prime ideal of R. So P is S-prime since R is an S-WP-ring. Then P is prime by assumption. □

Remark 2.9. The converse of the above Proposition 2.8 is not true in general: in fact, for $R = \mathbb{Z}[X]$ with multiplicative subset $S = \{2^n \mid n \in \mathbb{N}\}$. It is easy to see that R is both a WP-ring and an S-WP-ring but R does not satisfy that every S-prime ideal is prime since $P = 4XR$ is an S-weakly prime ideal because it is an S-prime by [13, Example 1]. However, P is not prime since $2.2X \in P$ but $2 \notin P$ and $2X \notin P$.

The following Proposition 2.10 establishes a direct connection between S-weakly prime ideals and prime ideals without invoking the concept of S-prime ideals.

Proposition 2.10. *Let R be a ring with multiplicative subset S. Then the following assertions are equivalent:*

1. *R is an S-ME-domain,*
2. *Every S-weakly prime ideal of R is prime (in particular, R is an S-WP-ring).*
3. *R is a domain and every S-prime ideal of R is prime.*

Proof. (1) ⇒ (2) Assume that R is an S-ME-domain and let P be an S-weakly prime ideal. Then, P is a weakly prime ideal of R since R is an S-ME-ring. So P is prime since R is an integral domain.

(2) ⇒ (1) Assume that every S-weakly prime ideal is prime. As 0 is an S-weakly prime ideal of R by hypothesis, 0 is prime and so R is an integral domain. Let's prove that R is an S-ME-ring. Let P be an S-weakly prime ideal of R, then P is a prime ideal of R by hypothesis, and so P is a weakly prime ideal of R. Therefore, R is an S-ME-ring.

(2) ⇔ (3) Follows from [1, Proposition 17]. □

The next Proposition 2.11 gives a sufficient condition for S-ME-rings.

Proposition 2.11. *Let R be a ring with multiplicative subset S. If every S-prime ideal of R is prime and R is an S-WP-ring, then R is an S-ME-ring.*

Proof. Let I be an S-weakly prime ideal of R. We claim that I is a weakly-prime ideal of R. Since R is an S-WP-ring, then I is S-prime and so I is prime by assumption. It follows that I is a weakly-prime ideal of R, and therefore, R is an S-ME-ring.

□

The converse of Proposition 2.11 is not true in general. To provide such an example, we will need the following theorem, which transfers the S-ME-ring property to the trivial ring extension.

Theorem 2.12. Let A be a ring with multiplicative subset S_0 and let E be an A-module. Set $R = A \propto E$ and $S = S_0 \propto E$. In this setting, the following statements hold.

1. If R is an S-ME-ring, then A is an S_0-ME-ring.
2. If A is an integral domain with quotients field K and E is a K-vector space, then R is an S-ME-ring if and only if A is an S_0-ME-ring.
3. Assume that E is a divisible A-module and A is a domain. If there exists a nonzero S-weakly prime ideal of A, then R is an S-ME-ring if and only if A is an S_0-ME-ring.

To prove theorem, we need the following lemma.

Lemma 2.13. *([5, Proposition 2.20]) Let D be an integral domain and Q a divisible D-module and S be a multiplicative subset of D. Let N be a D-submodule of Q and I be an ideal of D. Then:*

1. *$I \propto Q$ is $(S \propto Q)$-weakly prime if and only if I is S-weakly prime.*
2. *If there exists $s \in S$ such that $sQ \subset N$, then $0 \propto N$ is $(S \propto Q)$-weakly prime.*
3. *If Q/N is S-torsion free D-module, then the following are equivalent:*
   (a) $0 \propto N$ is an $(S \propto Q)$-weakly prime,
   (b) $0 \propto N$ is weakly prime.

Proof of Theorem 2.12. (1) If R is an S-ME-ring, then $0 \propto E$ is a weakly-prime ideal of R by [2, Corollary 19] and so $0 \propto E$ is an S-weakly prime ideal of R. Follows from the isomorphism $\frac{R}{0\propto E} \cong A$ is an $(\overline{S} := \frac{S_0 \propto E}{0\propto E})$-$ME$-ring by Proposition 2.5 above. It follows that A is an S_0-ME-ring.

(2) The necessity is obvious by (1). Let's prove the sufficiency. Assume that A is an S_0-ME-ring. By [3, Corollary 3.4], every ideal J of R is either $I \propto E$ for some ideal I of A or $0 \propto N$ for some A-submodule of E. If $J = 0 \propto N$, then it is easy to see that J is a weakly-prime ideal. If $J = I \propto E$ is an S-weakly prime ideal of R, then I is an S_0-weakly prime ideal of A and for $a, b \in A$ with $ab = 0$ but $sa, sb \notin I$ for each $s \in S_0$, we get $a, b \in Ann(E)$ by [17, Theorem 3.1]. Then I is a weakly-prime ideal of A since A is an S_0-ME-ring. Also, if $ab = 0$ for $a, b \in A$, then $a = 0$ or $b = 0$ since A is an integral domain which implies that either $a \in I$ or $b \in I$. It follows, by [2, Theorem 17], that J is a weakly-prime ideal of R. So, we proved that R is an S-ME-ring.

(3) If R is an S-ME-ring, then A is an S_0-ME-ring by (1).

Conversely, assume that A is an S_0-ME-ring, E is a divisible A-module and there exists an S-weakly prime ideal I of A. By Proposition 2.3, A/I is an $\overline{S_0}$-ME-ring, so $\frac{A\propto E}{I\propto E}$ is an $\overline{S}$-ME-ring since $\frac{A\propto E}{I\propto E} \cong \frac{A}{I}$. But E is divisible, then $E = IE$ and therefore, $\frac{A\propto E}{I\propto E} = \frac{A\propto E}{I\propto IE}$. On other hand, $I \propto IE$ is an S-weakly prime by the above Lemma. Hence, R is an S-ME-ring. □

Now, we establish that the converse of Proposition 2.11 is not true.

Example 2.14. Let $A = \mathbb{R} \propto \mathbb{R}^n$ where $n \geq 2$ and $S \propto R^n$ be a multiplicative subset of A. By Theorem 2.12, A is an S-ME-ring but not a WP-ring by [11, Example 3.4]. By Proposition 2.8, A is not an S-WP-ring and does not satisfy that every S-prime ideal is prime.

In the end of this work, we will try to study the S-ME-rings in the amalgamation of rings a long an ideal. For this purpose, let us fix some notations. Let $f : A \to B$ be a rings homomorphism, let J be an ideal of B and S be a multiplicative subset of A. Define $\mathcal{W}$ the set of all nonzero S-weakly prime ideals I of A which satisfy the following condition: for all $a, b \in A$ such that $ab = 0$ and $sa, sb \notin I$ for every $s \in S$, we have that $f(a)j + f(s)f(b)i + ij = 0$ for each $i, j \in J$. Let K be an ideal of $f(A) + J$. With the [9, Corollary 2.5], we define

$$I \bowtie^f J := \{(i, f(i) + j) \mid i \in I \,\&\, j \in J\}.$$

$$\overline{K}^f := \{(a, f(a) + j) \mid a \in A,\ j \in J \text{ and } f(a) + j \in K\}.$$

$$S' = \{(s, f(s)) \mid s \in S\}.$$

It is easy to check that $I \bowtie^f J$ and $\overline{K}^f$ are ideals of $A \bowtie^f J$. If $0 \notin f(S)$, then S' is a multiplicative subset of $A \bowtie^f J$.

Proposition 2.15. *Let $f : A \to B$ be a rings homomorphism, S a multiplicative subset of A and J be an ideal of B.*

(1) Assume that A is an S-ME-domain. Then, $I \bowtie^f J$ is an S'-weakly prime ideal of $A \bowtie^f J$ if and only if $I \bowtie^f J$ is a weakly prime ideal of $A \bowtie^f J$.

(2) Asssume taht $f(A) + J$ is an $f(S)$-ME-domain. Then, $\overline{K}^f$ is an S'-weakly prime ideal of $A \bowtie^f J$ if and only if $\overline{K}^f$ is a weakly-prime ideal of $A \bowtie^f J$.

Proof. (1) Assume that $I \bowtie^f J$ is an S'-weakly prime ideal of $A \bowtie^f J$. By [17, Theorem 3.6], we have that I is an S-weakly prime ideal of A and for $a, b \in A$ with $ab = 0$ and $sa, sb \notin I$ for every $s \in S$, we have $f(a)j + f(s)f(b)i + ij = 0$ for every $i, j \in J$. Then, I is a weakly prime ideal of A since A is assumed an S-ME-ring. By using [18, Theorem 2.1], we get $I \bowtie^f J$ is a weakly prime ideal of $A \bowtie^f J$.

The converse is straightforward.

(2) Assume that $\overline{K}^f$ is an S'-weakly prime ideal of $A \bowtie^f J$. By [17, Theorem 3.6], we have that K is an $f(S)$-weakly prime ideal of $f(A) + J$ and when $f(s)(f(a) + j), f(s)(f(b) + k) \notin K$ for each $s \in S$, $a, b \in A$, $j, k \in J$ and $(f(a)+j)(f(b)+k) = 0$, then $ab = 0$. So K is a weakly prime ideal of $f(A)+J$ since $f(A)+J$ is an $f(S)$-ME-ring. Thus, $\overline{K}^f$ is a weakly prime ideal of $A \bowtie^f J$ by [18, Theorem 2.1].

The converse is straightforward. □

Proposition 2.16. *Let $f : A \to B$ be a rings homomorphism, S a multiplicative subset of A and J be an ideal of B. Let H be an ideal of $f(A) + J$ such that $f(I)J \subset H \subset J$. If $I \bowtie^f H$ is a weakly prime ideal of $A \bowtie^f J$, then I is a weakly prime ideal of A and for $a, b \in A$ such that $ab = 0$ but $a, b \notin I$, we have $f(a)j + f(b)i + ij = 0$ for every $i, j \in H$. The converse holds if $J \subset H$.*

Proof. Assume that $I \bowtie^f H$ is weakly prime ideal of $A \bowtie^f J$. Let $a, b \in A$ with $0 \neq ab \in I$. Then $(a, f(a))$ and $(b, f(b)) \in I \bowtie^f H$ and $(0,0) \neq (a, f(a))(b, f(b)) \in A \bowtie^f H$. So, $(a, f(a)) \in I \bowtie^f H$ or $(b, f(b)) \in I \bowtie^f H$. Therefore, $a \in I$ or $b \in I$. Next, assume that $a, b \notin I$ with $ab = 0$. Suppose that there exist $i, j \in H$ such that $f(a)j + f(b)i + ij \neq 0$. Then $(0,0) \neq (a, f(a) + i)(b, f(b) + j) \in I \bowtie^f H$, but neither $(a, f(a) + i) \in I \bowtie^f H$ nor $(b, f(b) + j) \in I \bowtie^f H$, a contradiction.

Conversely, let $(a, f(a) + i), (b, f(b) + j) \in A \bowtie^f J$ with $(0,0) \neq (a, f(a) + i)(b, f(b) + j) \in I \bowtie^f H$. If $0 \neq ab$, then $a \in I$ or $b \in I$. So, $(a, f(a) + i) \in I \bowtie^f H$ or $(b, f(b) + j) \in I \bowtie^f H$ (as $J \subseteq H$). If $ab = 0$, then we claim that $a \in I$ or $b \in I$. Deny. By assumption we have, $f(a)j + f(b)i + ij = 0$ for each $i, j \in H$, a contradiction since $(ab, f(a)j + f(b)i + ij) \neq (0,0)$. Hence, $a \in I$ or $b \in I$ and so $(a, f(a) + i) \in I \bowtie^f H$ or $(b, f(b) + j) \in I \bowtie^f H$ (as $J \subseteq H$). □

Theorem 2.17. With the notation above, the following statements hold:

(1) If $A \bowtie^f J$ is an S'-ME-ring, then every ideal in $\mathcal{W}$ is weakly prime.

(2) Assume that $J \neq 0$ and $f^{-1}(J) \neq 0$. If $(A, \mathfrak{m})$ is a local ring with $\mathfrak{m}^2 = 0$ and $(f(A) + J, f(\mathfrak{m} + J))$ is a local ring with $(f(\mathfrak{m}) + J)^2 = 0$, then $A \bowtie^f J$ is an S'-ME-ring.

Proof. (1) Let $I \in \mathcal{W}$. Then $I \bowtie^f J$ is an S'-weakly prime ideal of $A \bowtie^f J$. Indeed, let $(a, f(a)+i)(b, f(b)+j) = (ab, f(ab)+f(a)j+f(b)i+ij) \in I \bowtie^f J \backslash (0,0)$. Hence, $ab \in I$. If $ab \neq 0$, then $sa \in I$ or $sb \in I$ for some $s \in S$. Then, $(s, f(s))(a, f(a)+i) \in I \bowtie^f J$ or $(s, f(s))(b, f(b)+j) \in I \bowtie^f J$. Now, assume that $ab = 0$ with $sa, sb \notin I$ for every $s \in S$. Since $I \in \mathcal{W}$, we get $f(a)j + f(s)f(b)i + ij = 0$, a contradiction with $(a, f(a)+i)(b, f(b)+j) \neq 0$. Since $A \bowtie^f J$ is an S'-ME-ring, we get that $I \bowtie^f J$ is weakly prime. Then I is a weakly prime ideal of A by [18, Theorem 2.1].

(2) Assume that $J \neq 0$, $f^{-1}(J) \neq 0$, $(A, \mathfrak{m})$ be a local ring with $\mathfrak{m}^2 = 0$ and $(f(A)+J, f(\mathfrak{m})+J)$ be a local ring, with $(f(\mathfrak{m})+J)^2 = 0$. We claim that $A \bowtie^f J$ is an S'-ME-ring. Let $I \bowtie^f J$ be an S'-weakly prime ideal of $A \bowtie^f J$, then $I \bowtie^f J$ is a weakly prime ideal of $A \bowtie^f J$ by [18, Theorem 2.15]. □

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