Moroccan Journal of Algebra and Geometry with Applications
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco
Volume 3, Issue 2 (2024), pp 279-287
ISSN: 2820-7114

Title :

Generalized S-prime ideals of commutative rings

Author(s):

Ahmed Hamed

Moroccan Journal of Algebra and Geometry with Applications
Vol. 3(2) (2024), 279–287

Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco **ISSN: 2820-7114**

Generalized S-prime ideals of commutative rings

Ahmed Hamed
Department of Mathematics, Faculty of Sciences, University of Monastir, Monastir, Tunisia
e-mail: *hamed.ahmed@hotmail.fr*

Communicated by Suat Koç
(Received 09 November 2023, Revised 13 February 2024, Accepted 22 February 2024)

Abstract. Let R be a commutative ring with identity and S a multiplicative subset of R. The purpose of this paper is to introduce the concept of generalized S-prime ideals as a new generalization of prime ideals. An ideal P of R disjoint with S is called a generalized S-prime ideal if for all $\alpha, \beta \in R$ there exists an $s \in S$ such that $\alpha\beta \in P$ implies $s\alpha \in P$ or $s\beta \in P$. Several properties, characterizations and examples concerning generalized S-prime ideals are presented. We give a relationship between generalized S-prime ideals of a ring R and those of the idealization ring $R(+)M$. Also, we show that each ideal of R disjoint with S is contained in a minimal generalized S-prime ideal of R. This extends classical well-known result on minimal prime ideals.

Key Words: Generalized S-prime ideal, S-prime ideal.
2010 MSC: 13A15.

1 Introduction

Throughout this article all rings are commutative with non-zero identity. In recent years, the prime ideals and their generalizations have an important place in commutative algebra and they draw attention by several authors. In 2019, the concept of S-prime ideals, which is a generalization of prime ideals, was first initiated by Hamed and Malek in [8]. Let R be a ring and S a multiplicative subset of R. Following [8], an ideal P of R with $P \cap S = \emptyset$ is said to be an *S-prime* ideal of R if there exists $s \in S$ such that for all $a, b \in R$ with $ab \in P$, we have either $sa \in P$ or $sb \in P$. In [8], the authors stated and proved S-version of several classical results on prime ideals. Let P be an ideal of R disjoint with S. It was shown in [8, Corollary 1] that P is S-prime if and only if there exists $s \in S$, such that for all $I_1, \ldots, I_n$ ideals of R, if $I_1 \cdots I_n \subseteq P$, then $sI_j \subseteq P$ for some $j \in \{1, \ldots, n\}$. Let I be an ideal of R and let $P_1, \ldots, P_n$ be S-prime ideals of R. The authors proved that if $I \subseteq \cup_{i=1}^{n} P_i$, then there exist $s \in S$ and $j \in \{1, \ldots, n\}$ such that $sI \subseteq P_j$. This notion was generalized by several authors. According to [10] an ideal I of R with $I \cap S = \emptyset$ is said to be an *S-primary* ideal of R if there exists $s \in S$ such that for all $a, b \in R$ with $ab \in I$, we have either $sa \in I$ or $sb \in \sqrt{I}$. Later, Yetkin Celikel and Hamed generalize the notion of “S-primary ideals" by introducing the concept of “quasi-S-primary" ideals of a commutative ring and study its basic properties. Following [3], a proper ideal I of R disjoint from S is called a *quasi-S-primary* ideal if there exists an (fixed) $s \in S$ such that for all $a, b \in R$ if $ab \in I$, then $sa \in \sqrt{I}$ or $sb \in \sqrt{I}$. Some generalizations of S-prime ideals can be found in [1, 3, 10, 11, 13].

Motivated by the research work on S-prime ideals in [8] and by the above generalizations of prime ideals, in this article, we introduce a new class of a generalization of prime ideals. Let R be a ring and S a multiplicative subset of R. An ideal P of R disjoint with S is called a *generalized S-prime* ideal of R if for all $\alpha, \beta \in R$ there exists an $s \in S$ such that $\alpha\beta \in P$ implies $s\alpha \in P$ or $s\beta \in P$. Note that every S-prime ideal of R is a generalized S-prime ideal of R since the fixed element $s \in S$ in “S-prime" is dependent on the ideal P, but the element $s \in S$ in the concept of “generalized S-prime" is dependent by α and β. Recall from [12], the authors defined the concept of locally S-prime ideals of a ring R as follow: let S a multiplicative subset of a ring R. An ideal P of R disjoint with S is called a *locally*

S-prime ideal of R if whenever $S^{-1}P$ is a prime ideal of $S^{-1}R$. Clearly that every generalized S-prime ideal is locally S-prime (In section 2, we prove that the two concepts are equal).

In the first part of this article, we state and prove some basic properties of generalized S-prime ideals. Clearly, every prime ideal of a ring R disjoint with a multiplicative subset S of R is a generalized S-prime ideal of R. We give an example shows that the converse of the previous implication is not true in general (Example 2.3). Also, we show that if S is a finite multiplicative subset of a ring R, then the notions "S-prime" and "generalized S-prime" coincide (Proposition 2.2). Let R be a commutative ring, S a multiplicative subset of R and I an ideal of R disjoint with S. Let $s \in S$, we denote by $\overline{s}$ the equivalence class of s in R/I. Let $\overline{S} = \{\overline{s} \mid s \in S\}$. Then $\overline{S}$ is a multiplicative subset of R/I. Let P be a proper ideal of R containing I such that $P/I \cap \overline{S} = \emptyset$. We show that P is a generalized S-prime ideal of R if and only if P/I is a generalized $\overline{S}$-prime ideal of R/I (Proposition 2.7). Also, we give a new characterization for an ideal P of R to be a generalized S-prime ideal of R. Let P be an ideal of R disjoint with S. It is proved in Theorem 2.9 that the following conditions are equivalent.

1. P is a generalized S-prime ideal of R.
2. P satisfies:
   (i) for all $\alpha, \beta \in R$, there exists an $s \in S$ such that if $\alpha\beta \in P$, then $s\alpha \in \sqrt{P}$ or $s\beta \in \sqrt{P}$.
   (ii) for each $x \in R$ if $x^2 \in P$, then $sx \in P$ for some $s \in S$.

We end the first part of this paper by giving a relationship between generalized S-prime ideals of a ring R and those of the idealization ring $R(+)M$. First, let us recall the notion of idealization ring $R(+)M$. Let R be a commutative ring with identity and M a unitary R-module. Then the *Nagata's idealization* of M in R (or *trivial extension* of R by M) is a commutative ring

$$R(+)M := \{(r, m) | r \in R \text{ and } m \in M\}$$

under the usual addition and the multiplication defined by $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ for all $(r_1, m_1), (r_2, m_2) \in R(+)M$. It is clear that $(1, 0)$ is the identity of $R(+)M$; and if S is a multiplicative subset of R, then $S(+)M$ and $S(+)0$ are multiplicative subsets of $R(+)M$. For an ideal P of R disjoint with S, we show that $P(+)M$ is a generalized $S(+)M$-prime ideal of $R(+)M$ if and only if P is a generalized S-prime ideal of R equivalent to $P(+)M$ is a generalized $S(+)0$-prime ideal of $R(+)M$ (Proposition2.17).

In the second part of this paper, we study minimal generalized S-prime ideals of a commutative ring. First, we need to collect some necessary notions. Following [8], a multiplicative set S of a commutative ring R is called *weakly anti-Archimedean* if for each family $(s_\alpha)_{\alpha\in\Lambda}$ of elements of S we have

$$(\cap_{\alpha\in\Lambda} s_\alpha R) \cap S \neq \emptyset.$$

Let Q be an ideal of R disjoint with S and P a generalized S-prime ideal of R such that $Q \subseteq P$. We say that P is *minimal over* Q if P is minimal in the set of the generalized S-prime ideals of R containing Q. Let S be a weakly anti-Archimedean multiplicative subset of a ring R. We prove that each ideal of R disjoint with S is contained in a minimal generalized S-prime ideal of R (Theorem 3.3). In the particular case when S consists of units of R, we recover the following well-known result. Every proper ideal of ring R is contained in a minimal prime ideal of R.

2 Characterizations of generalized S-prime ideal

We start this section by introducing the notion of generalized S-prime ideals.

Definition 2.1. Let R be a ring and S a multiplicative subset of R. An ideal P of R disjoint with S is called a generalized S-prime ideal if for all $\alpha, \beta \in R$ there exists an $s \in S$ such that $\alpha\beta \in P$ implies $s\alpha \in P$ or $s\beta \in P$.

Let R be a ring and S a multiplicative closed subset of R. Following [8], an ideal P of R disjoint with S is said to be an S-*prime* ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in P$, then $sa \in P$ or $sb \in P$. Clearly, every S-prime ideal of R is a generalized S-prime ideal of R. Note that the converse is true if S consists of units of R. The next result proved a relationship between the notion "S-prime" and the concept "generalized S-prime."

Proposition 2.2. *Let R be a ring, S a finite multiplicative subset of R and P an ideal of R disjoint with S. Then the following assertions are equivalent.*

1. *P is an S-prime ideal of R.*
2. *P is a generalized S-prime ideal of R.*

Proof. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). Put $S = \{s_1, ..., s_n\}$. Assume that P is a generalized S-prime ideal of R, and let $s = s_1 s_2 \cdots s_n$. Then $s \in S$. Now, let $\alpha, \beta \in R$ such that $\alpha\beta \in P$. By hypothesis, $s_k\alpha \in P$ or $s_k\beta \in P$ for some $s_k \in S$. This implies that $s\alpha = s_1 \cdots s_k \cdots s_n \alpha \in P$ or $s\beta = s_1 \cdots s_k \cdots s_n \beta \in P$; so $s\alpha \in P$ or $s\beta \in P$, and hence P is an S-prime ideal of R. □

We next give an example of a generalized S-prime ideal which is not a prime ideal.

Example 2.3. Let $R = \mathbb{Z}_{24}$ and S be the multiplicative subset of R generated by $\overline{3}$, i.e., $S = \{\overline{1}, \overline{3}, \overline{9}\}$. Let $P = (\overline{6}) = \{\overline{0}, \overline{6}, \overline{12}, \overline{18}\}$. Then P is not a prime ideal of R, since $\overline{3} \cdot \overline{4} \in P$ but neither $\overline{3} \in P$ nor $\overline{4} \in P$. Moreover, $P \cap S = \emptyset$. It is easy to show that for each $\alpha, \beta \in R$ such that $\alpha\beta \in P$, then $\overline{3}\alpha \in P$ or $\overline{3}\beta \in P$ which implies that P is a generalized S-prime ideal of R.

Remark 2.4. Let R be a ring and $T \subseteq S$ be two multiplicatively closed subsets of R. If P is a generalized T-prime ideal of R such that $P \cap S = \emptyset$, then P is also a generalized S-prime ideal of R.

The following example shows that the reverse of the previous Remark is not true in general.

Example 2.5. Let $R = \mathbb{Z}[X]$, $S = \{2^n \mid n \in \mathbb{N} \cup \{0\}\}$ and $T = \{1\} \subseteq U(R)$. Then $T \subseteq S$ are two multiplicative subsets of R. Let $P = 4X\mathbb{Z}[X]$. By [8, Example 1] P is an S-prime ideal of R which implies that P is a generalized S-prime ideal of R. Note that P is not a generalized T-prime ideal of R because $4X \in P$ but neither $1 \cdot 4 \in P$ nor $1 \cdot X \in P$.

Proposition 2.6. *Let R be a ring and $T \subseteq S$ be two multiplicatively closed subsets of R. Assume that for each $s \in S$, there is an element $t \in T$ such that $st \in T$. If P is a generalized S-prime ideal of R, then P a generalized T-prime ideal of R.*

Proof. Let $\alpha, \beta \in R$ such that $\alpha\beta \in P$. There exists an $s \in S$ such that $s\alpha \in P$ or $s\beta \in P$. By hypothesis, there is an element $t \in T$ such that $st \in T$. Let $s' = st \in S$. Then $s'\alpha = st\alpha \in P$ or $s'\beta = st\beta \in P$. Hence P is a generalized T-prime ideal of R. □

Let R be a commutative ring, S a multiplicative subset of R and I an ideal of R disjoint with S. Let $s \in S$, we denote by $\overline{s}$ the equivalence class of s in R/I. Let $\overline{S} = \{\overline{s} \mid s \in S\}$. Then $\overline{S}$ is a multiplicative subset of R/I.

Proposition 2.7. *Let R be a ring, S a multiplicative subset of R and I an ideal of R disjoint with S. Let P be a proper ideal of R containing I such that $P/I \cap \overline{S} = \emptyset$. Then the following assertions are equivalent.*

1. *P is a generalized S-prime ideal of R.*

2. *P/I is a generalized $\overline{S}$-prime ideal of R/I.*

Proof. First, note that $P \cap S = \emptyset$ if and only if $P/I \cap \overline{S} = \emptyset$.

(1) $\Rightarrow$ (2). Assume that P is a generalized S-prime ideal of R, and let $\overline{\alpha}, \overline{\beta} \in R/I$ such that $\overline{\alpha}\overline{\beta} \in P/I$. This implies that $\alpha\beta \in P$; so there exists an $s \in S$ such that $s\alpha \in P$ or $s\beta \in P$. Hence $\overline{s\alpha} \in P/I$ or $\overline{s\beta} \in P/I$.

(2) $\Rightarrow$ (1). Let $\alpha, \beta \in R$ such that $ab \in P$. Then $\overline{ab} \in P/I$. Since P/I is generalized $\overline{S}$-prime ideal of R/I, there exists an $\overline{s} \in \overline{S}$ such that $\overline{s\alpha} \in P/I$ or $\overline{s\beta} \in P/I$; so $s\alpha \in P$ or $s\beta \in P$. This shows that P is a generalized S-prime ideal of R. □

Let R be a ring and S a multiplicative subset of R. The saturation of S is the set $S^* = \{x \in R : xy \in S$ for some $y \in R\}$ is a multiplicative subset of R satisfying $S \subseteq S^*$, see [5]. Our next theorem gives equivalent conditions for an ideal I disjoint with S to be generalized S-prime.

Theorem 2.8. Let P be an ideal of R disjoint with S. Then the following assertions are equivalent.

1. P is a generalized S-prime ideal of R.

2. P is a generalized S^*-prime ideal of R.

3. For any $a \in R$ and any finitely generated ideal I of R, there exists an $s \in S$ such that if $aI \subseteq P$, then $sa \in P$ or $sI \subseteq P$.

Proof. (1) $\Rightarrow$ (2). Since $S \subseteq S^*$, it is sufficient to show that $S^* \cap P = \emptyset$ by Remark 2.4. Assume that $S^* \cap P \neq \emptyset$, and let $t \in S^* \cap P$. Then there exists an $s \in S$ such that $s = tr$ for some $r \in R$. Thus, we conclude $s \in S \cap P$, a contradiction.

(2) $\Rightarrow$ (3). Let $a \in R$ and $I = (\alpha_1, ..., \alpha_n)$ a finitely generated ideal of R such that $aI \subseteq P$. Note that for each $1 \leq i \leq n$, $a\alpha_i \in aI \subseteq P$. Then for each $1 \leq i \leq n$, there exists an $s_i \in S$ such that $s_i a \in P$ or $s_i\alpha_i \in P$. Let $s = s_1 \cdots s_n$. Then $s \in S$. Assume that $sa \notin P$. We show that $sI \subseteq P$. Since $sa \notin P$, $s_i a \notin P$ for each $1 \leq i \leq n$. This implies that for each $1 \leq i \leq n$,

$$s\alpha_i = s_1 \cdots s_i \cdots s_n \alpha_i \in P,$$

because $s_i a \notin P$. Thus $sI \subseteq P$.

(3) $\Rightarrow$(1). Let $\alpha, \beta \in R$ such that $\alpha\beta \in P$. Put $I = \beta R$. Then by hypothesis, there exists an $s \in S$ such that $s\alpha \in P$ or $sI \subseteq P$. Thus $s\alpha \in P$ or $s\beta \in P$, and hence P is a generalized S-prime ideal of R. □

Next we give a new characterization for an ideal P of R disjoint with S to be generalized S-prime.

Theorem 2.9. Let P be an ideal of R disjoint with S. Then the following conditions are equivalent.

1. P is a generalized S-prime ideal of R.

2. P satisfies:

   (i) for all $\alpha, \beta \in R$, there exists an $s \in S$ such that if $\alpha\beta \in P$, then $s\alpha \in \sqrt{P}$ or $s\beta \in \sqrt{P}$.

   (ii) for each $x \in R$ if $x^2 \in P$, then $sx \in P$ for some $s \in S$.

Proof. (1)$\Rightarrow$ (2). Suppose that P is a generalized S-prime ideal of R. We show (i). Let $\alpha, \beta \in R$ such that if $\alpha\beta \in P$. Since P is generalized S-prime, there exists an $s \in S$ such that $s\alpha \in P$ or $s\beta \in P$. This implies that $s\alpha \in \sqrt{P}$ or $s\beta \in \sqrt{P}$, because $P \subseteq \sqrt{P}$. It is clearly that if P is a generalized S-prime ideal of R, then the assertion (ii) hold.

(2) ⇒ (1). We show that P is a generalized S-prime ideal of R. Let $\alpha, \beta \in R$ such that $\alpha\beta \in P$. Then there exists an $s \in S$ such that $s\alpha \in \sqrt{P}$ or $s\beta \in \sqrt{P}$.

Case 1: $s\alpha \in \sqrt{P}$. Then $(s\alpha)^n \in P$ for some $n \in \mathbb{N}$. If n is even, then $n = 2k$ for some $k \in \mathbb{N}$; so by hypothesis, $s_1(s\alpha)^k \in P$ for some $s_1 \in S$. If n is odd, then $n = 2k+1$ for some $k \in \mathbb{N}$, which implies that $(s\alpha)^{2k+2} = (s\alpha)^{n+1} \in P$; so by hypothesis, $s_1(s\alpha)^{k+1} \in P$ for some $s_1 \in S$. Thus there exists an $n_1 < n$, ($n_1 = k$ if n is even and $n_1 = k+1$ if n is odd) and $s_1 \in S$ such that $s_1(s\alpha)^{n_1} \in P$. This implies that $(s_1 s\alpha)^{n_1} \in P$; so there exists an $n_1 < n$, and $t = s_1 s \in S$ such that $(t\alpha)^{n_1} \in P$. If we continue this process, then we obtain, $(t\alpha)^2 \in P$ or $(t\alpha)^3 \in P$ for some $t \in S$. By using the hypothesis (ii), we get $s't\alpha \in P$ for some $s' \in S$. So for $t' = s't \in S$, we have $t'\alpha \in P$.

Case 2: $s\beta \in \sqrt{P}$. In the same way (case 1) we can prove that $t'\beta \in P$ for some $t' \in S$.

This show that P is a generalized S-prime ideal of R, and the proof is completed. □

In the particular case when $S = U(R)$ the set of units of R, we recover the following well-known result.

Corollary 2.10. *Let P be a proper ideal of a ring R. Then the following conditions are equivalent.*

1. *P is a prime ideal of R.*
2. *P satisfies:*
   - **(i)** *for all $\alpha, \beta \in R$, such that $\alpha\beta \in P$ implies $\alpha \in \sqrt{P}$ or $\beta \in \sqrt{P}$.*
   - **(ii)** *for each $x \in R$, $x^2 \in P$ implies $x \in P$.*

Proposition 2.11. *Let R be a commutative ring, S a multiplicative subset of R and P a generalized S-prime ideal of R.*

1. *Let Q be an ideal of R such that $Q \cap S \neq \emptyset$. Then $P \cap Q$ and PQ are generalized S-prime ideals of R.*
2. *Let Q be an ideal of R such that $Q \subseteq P$, then for each $x \in \sqrt{Q}$, there exists an $s \in S$ such that $sx \in P$.*

Proof. (1). Since $P \cap S = \emptyset$, clearly we have $(P \cap Q) \cap S = \emptyset$ and $PQ \cap S = \emptyset$. Let $\alpha\beta \in P \cap Q$. Then $s\alpha \in P$ or $s\beta \in P$ for some $s \in S$. Let $t \in Q \cap S$. Then $st\alpha \in P \cap Q$ or $st\beta \in P \cap Q$. Thus $P \cap Q$ is a generalized S-prime ideal of R. The proof is similar for PQ.

(2). Let $x \in \sqrt{Q}$, then there exists $n \in \mathbb{N}^*$ such that $x^n \in Q \subseteq P$. Thus $x \cdot x \cdots x \in P$ which implies that $s_1 x \in P$ or $s_1(x)^{n-1} \in P$ for some $s_1 \in S$. If $s_1 x \in P$, then the prof is completed. If $s_1(x)^{n-1} \in P$, then $s_1 s_2 x \in P$ or $s_2(x)^{n-2} \in P$ for some $s_2 \in S$. If we continue this process, then we obtain $tx \in P$ for some $t \in S$. □

Proposition 2.12. *Let $f : R \to T$ be a ring homomorphism and S be a multiplicatively closed subset of R such that $f(S)$ does not contain zero. If Q is a generalized $f(S)$-prime ideal of T, then $f^{-1}(Q)$ is a generalized S-prime ideal of R.*

Proof. Note that if $s \in f^{-1}(Q) \cap S$, then $f(s) \in Q \cap S$, which is a contradiction. Hence, $f^{-1}(Q) \cap S = \emptyset$. Let $\alpha, \beta \in R$ such that $\alpha\beta \in f^{-1}(Q)$. Then $f(\alpha\beta) = f(\alpha)f(\beta) \in Q$, and since Q is a generalized S-prime ideal of T, there exists $f(s) \in f(S)$ such that $f(s)f(\alpha) \in Q$ or $f(s)f(\beta) \in Q$. This implies that $s\alpha \in f^{-1}(Q)$ or $s\beta \in f^{-1}(Q)$, and hence $f^{-1}(Q)$ is a generalized S-prime ideal of R. □

Proposition 2.13. *Let S be multiplicatively closed subset of a ring R and P an ideal of R disjoint with S. Then P is a generalized S-prime of R if and only if $S^{-1}P$ is a prime ideal of $S^{-1}R$.*

Proof. Note that $S^{-1}P$ is a proper ideal of $S^{-1}R$ if and only if $S \cap P = \emptyset$. Assume that P is a generalized S-prime of R, and let $\alpha, \beta \in R$ and $s, t \in S$ with $\frac{\alpha}{s}\frac{\beta}{t} \in S^{-1}P$ and $\frac{\alpha}{s} \notin S^{-1}P$. Then $s'\alpha\beta \in P$ for some $s' \in S$. Since P is a generalized S-prime ideal, there exists a $t' \in S$ such that $s't'\beta \in P$ and $t'\alpha \notin P$, because $\frac{\alpha}{s} \notin S^{-1}P$. Thus, $\frac{\beta}{t} = \frac{s't'\beta}{s't't} \in S^{-1}P$, and hence $S^{-1}P$ is a prime ideal of $S^{-1}R$.

Conversely, assume that $S^{-1}P$ is a prime of $S^{-1}R$. Let $\alpha, \beta \in R$ such that $\alpha\beta \in P$. Then $\frac{\alpha}{1}\frac{\beta}{1} \in S^{-1}P$ which implies either $\frac{\alpha}{1} \in S^{-1}P$ or $\frac{\beta}{1} \in S^{-1}P$ since $S^{-1}P$ is prime. Thus there exist $s, t \in S$ such that either $s\alpha \in P$ or $t\beta \in P$. Let $s' = st \in S$. Then we get either $s'\alpha \in P$ or $s'\beta \in P$, and hence P is a generalized S-prime of R. □

Let R be a ring. We denote by $Reg(R)$ the set of regular elements of R. Combining Proposition 2.13 and [8, Remark 1], we get the following result.

Corollary 2.14. *Let $S \subseteq Reg(R)$ be multiplicatively closed subset of a ring R and P an ideal of R disjoint with S such that $(S^{-1}P) \cap R = (P : s)$ for some $s \in S$. Then the following assertions are equivalent.*

1. *P is a generalized S-prime of R.*
2. *P is an S-prime of R.*
3. *$S^{-1}P$ is a prime ideal of $S^{-1}R$.*

Next, we characterize generalized S-prime ideals in a cartesian product of rings.

Theorem 2.15. Let $R = R_1 \times R_2$ and $S = S_1 \times S_2$ where S_1, S_2 are multiplicatively closed subsets and P_1, P_2 be ideals of rings R_1, R_2, respectively. Then the following assertions are equivalent.

1. $P = P_1 \times P_2$ is a generalized S-prime of R.
2. P_1 is a generalized S_1-prime of R_1 and $S_2 \cap P_2 \neq \emptyset$ or P_2 is a generalized S_2-prime of R_2 and $S_1 \cap P_1 \neq \emptyset$.

Proof. (1)⇒ (2). Assume that $P = P_1 \times P_2$ is a generalized S-prime of R. Suppose that $S_1 \cap P_1 = S_2 \cap P_2 = \emptyset$. Let $(\alpha, \beta) \in P$. Then $(\alpha, 1)(1, \beta) \in P$; so there exists an $(s_1, s_2) \in S$ such that $(s_1, s_2)(\alpha, 1) \in P$ or $(s_1, s_2)(1, \beta) \in P$. Thus, we get either $s_2 \in S_2 \cap P_2$ or $s_1 \in S_1 \cap P_1$, a contradiction. Without loss of generality, we may assume that $S_1 \cap P_1 \neq \emptyset$, and we will prove that P_2 is a generalized S_2-prime ideal of R_2. First, $S_2 \cap P_2 = \emptyset$ as $S \cap P = \emptyset$. Let $a, b \in R_2$ such that $ab \in P_2$. Choose $t \in S_1 \cap P_1$. Hence $(t, a)(1, b) \in P$ which implies that $s(t, a) \in P$ or $s(1, b) \in P$ for some $s = (s_1, s_2) \in S$. Therefore, $s_2 a \in P_2$ or $s_2 b \in P_2$, as needed.

(2)⇒ (1). Assume that P_1 is a generalized S_1-prime ideal of R_1 and $S_2 \cap P_2 \neq \emptyset$. Choose $t \in S_2 \cap P_2$. Let $(a_1, a_2)(b_1, b_2) \in P$ for some $a_1, b_1 \in R_1$ and $a_2, b_2 \in R_2$. Hence $a_1 b_1 \in P_1$ which implies that $sa_1 \in P_1$ or $sb_1 \in P_1$. Now set $s' = (s, t) \in S$. Observe that $s'(a_1, a_2)P$ or $s'(b_1, b_2) \in P$, and thus P is a generalized S-prime ideal of R. In the same way one can prove the claim if P_2 is a generalized S_2-prime of R_2 and $S_1 \cap P_1 \neq \emptyset$. □

Using Theorem 2.15, we obtain the following corollary.

Corollary 2.16. *Let $R = R_1 \times \cdots \times R_n$ and $S = S_1 \times \cdots \times S_n$, where S_i's are multiplicatively closed subsets of R_i's for all $i \in \{1, ..., n\}$, respectively. Then $P = P_1 \times \cdots \times P_n$ is a generalized S-prime ideal of R if and only if P_k is a generalized S_k-prime ideal of R_k for some $k \in \{1, ..., n\}$ and $S_j \cap P_j \neq \emptyset$ for all $j \in \{1, ..., n\} \backslash \{k\}$.*

We end this section by giving a relationship between generalized S-prime ideals of a ring R and those of the idealization ring $R(+)M$. First, let us recall the notion of idealization ring $R(+)M$. Let R be a commutative ring with identity and M a unitary R-module. Then the *Nagata's idealization* of M in R (or *trivial extension* of R by M) is a commutative ring

$$R(+)M := \{(r, m) | r \in R \text{ and } m \in M\}$$

under the usual addition and the multiplication defined by $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ for all $(r_1, m_1), (r_2, m_2) \in R(+)M$. It is clear that $(1, 0)$ is the identity of $R(+)M$; and if S is a multiplicative subset of R and N is an R-submodule of M, then $S(+)N$ is a multiplicative subset of $R(+)M$. For an ideal I of R and a submodule N of M, $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. Moreover, the radical of $I(+)N$ is $\sqrt{I(+)N} = \sqrt{I}(+)M$. Note that if S is a multiplicative subset of R, then $S(+)M$ and $S(+)0$ are multiplicative subsets of $R(+)M$.

Proposition 2.17. *Let S be a multiplicative subset of a ring R and M be an R-module. For an ideal P of R disjoint with S, the following statements are equivalent.*

1. $P(+)M$ is a generalized $S(+)M$-prime ideal of $R(+)M$.
2. P is a generalized S-prime ideal of R.
3. $P(+)M$ is a generalized $S(+)0$-prime ideal of $R(+)M$.

Proof. (1)⇒(2). Let $\alpha, \beta \in R$ such that $\alpha\beta \in P$. Then $(\alpha, 0)(\beta, 0) \in P(+)M$, which implies $(s, m)(\alpha, 0) \in P(+)M$ or $(s, m)(b, 0) \in P(+)M$ for some $(s, m) \in S(+)M$. This implies that $s\alpha \in P$ or $s\beta \in P$. Thus P is a generalized S-prime ideal of R.

(2)⇒(3). Assume that $(\alpha, m_1)(\beta, m_2) \in P(+)M$ for some $(a, m_1), (b, m_2) \in R(+)M$. Then $ab \in P$. By hypothesis, there exists an $s \in S$ such that $s\alpha \in P$ or $s\beta \in P$, and thus $(s, 0)(\alpha, m_1) \in P(+)M$ or $(s, 0)(\beta, m_2) \in P(+)M$. Hence $P(+)M$ is a generalized $S(+)0$-prime ideal of $R(+)M$.

(3)⇒(1). Follows from Remark 2.4 as $S(+)0 \subseteq S(+)M$. □

3 Minimal generalized S-prime ideals

Let R be a commutative ring and S a multiplicative set of R. Recall from [2] (or [7]) that R is called an *S-Noetherian ring* if each ideal I of R is S-finite, *i.e.*, there exist an element $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$. We start this section by giving the S-invariant (using the "generalized S-prime" concept) of the Cohen type theorem for S-Noetherian rings.

Theorem 3.1. Let S be a multiplicative subset of R. Then the following conditions are equivalent.

1. R is S-Noetherian.
2. Every generalized S-prime ideal of R is S-finite.
3. Every prime ideal of R is S-finite.

Proof. (1) ⇒ (2) Obvious.
(2) ⇒ (3) Assume that every generalized S-prime ideal of R is S-finite. Let P be a prime ideal of R. Clearly if $P \cap S \neq \emptyset$, then P is S-principal which implies that P is S-finite. Now, if $P \cap S = \emptyset$, then P is an S-prime ideal of R; so by hypothesis, P is S-finite.
(3) ⇒ (1) Follows from [2, Corollary 5]. □

Let R be a commutative ring and S a multiplicative subset of R. Following [2], we say that S is *anti-Archimedean* if $\cap_{n\geq 1} s^n R \cap S \neq \emptyset$ for every $s \in S$. In [8], the authors generalized this notion by introducing the concept of weakly anti-Archimedean multiplicative set. According to [8], a multiplicative set S of a commutative ring R is called *weakly anti-Archimedean* if for each family $(s_\alpha)_{\alpha\in\Lambda}$ of elements of S we have

$$(\cap_{\alpha\in\Lambda} s_\alpha R) \cap S \neq \emptyset.$$

Note that every weakly anti-Archimedean multiplicative set is anti-Archimedean. The converse is not true as was observed in [4, Example 2.7.].

Lemma 3.2. *Let R be a commutative ring, S a weakly anti-Archimedean multiplicative subset of R and $(P_\alpha)_{\alpha\in\Lambda}$ be a chain of generalized S-prime ideals of R. Then $P = \cap_{\alpha\in\Lambda} P_\alpha$ is a generalized S-prime ideal of R.*

Proof. Let $\alpha, \beta \in R$ such that $\alpha\beta \in P$. We prove that there exists an $s \in S$ such that either $s\alpha \in P$ or $s\beta \in P$. Since $\alpha\beta \in P$, then for each $\lambda \in \Lambda$, $\alpha\beta \in P_\lambda$; so there exists an $s_\lambda \in S$ such that $s_\lambda\alpha \in P_\lambda$ or $s_\lambda\beta \in P_\lambda$ since P_λ is a generalized S-prime ideal of R. Now, as S is a weakly multiplicative set, then $(\cap_{\lambda\in\Lambda} s_\lambda R) \cap S \neq \emptyset$. Let $s \in (\cap_{\lambda\in\Lambda} s_\lambda R) \cap S$. Then we have, for each $\lambda \in \Lambda$, $\alpha\beta \in P_\lambda$; so $s_\lambda\alpha \in P_\lambda$ or $s_\lambda\beta \in P_\lambda$. Since $s \in (\cap_{\lambda\in\Lambda} s_\lambda R) \cap S$, then we can write $s = s_\lambda a_\lambda$ for some $a_\lambda \in R$. Thus

$$s\alpha = s_\lambda a_\lambda \alpha \in P_\lambda \text{ or } s\beta = s_\lambda a_\lambda \beta \in P_\lambda.$$

This shows that P is a generalized S-prime ideal of R. □

Let Q be an ideal of R disjoint with S and P a generalized S-prime ideal of R such that $Q \subseteq P$. We say that P is minimal over Q if P is minimal in the set of the generalized S-prime ideals of R containing Q.

Theorem 3.3. Let S be a weakly anti-Archimedean multiplicative subset of a ring R. Then each ideal of R disjoint with S is contained in a minimal generalized S-prime ideal of R.

Proof. Let Q be an ideal of R and let $\mathcal{L}$ be the set of generalized S-prime ideals containing Q. First, we show that $\mathcal{L} \neq \emptyset$. It well-known that for each ideal I of R, there exists a prime ideal of R such that $I \subseteq P$ and $P \cap S = \emptyset$. Now, since Q is an ideal of R, then there exists a prime ideal P of R such that $Q \subseteq P$ and $P \cap S = \emptyset$. It is easy to prove that P is a generalized S-prime ideal of R because every prime ideal disjoint with S is generalized S-prime. This shows that $P \in \mathcal{L}$, and thus $\mathcal{L} \neq \emptyset$. On the other hand, the set $\mathcal{L}$ is ordered by " $\supseteq$ ". Moreover, $\mathcal{L}$ is inductive. Indeed, let $(P_\lambda)_{\lambda\in\Lambda}$ be a chain of elements of $\mathcal{L}$. Put $P = \cap_{\lambda\in\Lambda} P_\lambda$. By Lemma 3.2, P is a generalized S-prime ideal of R. Since for each $\lambda \in \Lambda$, $Q \subseteq P_\lambda$, we get $Q \subseteq P$. Thus P is an upper bound for the chain $(P_\lambda)_{\lambda\in\Lambda}$. Thus by Zorn's Lemma $\mathcal{L}$ has a maximal element for " $\supseteq$ ". Hence Q is contained in a minimal generalized S-prime ideal of R. □

We end this article by recover (using the previous Theorem) the following well-known result on minimal prime ideals.

Corollary 3.4. *Let R be a ring. Then each proper ideal of R is contained in a minimal prime ideal of R.*

Acknowledgments

The author would like to thanks the referee for his/her careful considerations.

References

[1] F. Almahdi, E. M. Bouba and M. Tamekkante, On weakly S-prime ideals of commutative rings, An. St. Uni. Ovidius Constanta 29 (2021), 173-186.

[2] D. D. Anderson and T. Dumitrescu, S-Noetherian rings, Comm. Algebra 30 (2002), 4407-4416.

[3] E.Y. Celikel and A. Hamed, Quasi-S-primary ideals of commutative rings, Comm. Algebra 51(10) (2023), 4285-4298.

[4] D. E. Dobbs, Ahmes Expansions of Formal Laurent Series and a Class of Nonarchimedean Integral Domains, J. Algebra 103 (1986), 193-201.

[5] R. Gilmer, Multiplicative Ideal Theory, *Marcel Dekker*, New York, 1972.

[6] A. Hamed, S-strong Mori domain of $A+XB[X]$, Moroccan J. Algebra Geom. Appl. 1(2) (2022), 183-188.

[7] A. Hamed and S. Hizem, Modules satisfying the S-Noethrian property and S-accr, Comm. Algebra 44 (2016), 1941-1951.

[8] A. Hamed and A. Malek, S-prime ideals of a commutative ring, Beitr. Algebra Geom. 61 (2020), 533-542.

[9] N. Mahdou, A. Mimouni and Y. Zahir, On weakly prime ideals and weak Krull dimension, Moroccan J. Algebra Geom. Appl. 2(2) (2023), 307-314.

[10] S. Visweswaran, Some results on S-primary ideals of a commutative ring, Beitr. Algebra Geom. 63 (2021), 1-20.

[11] E. S. Sevim, T. Arabaci, Ü. Tekir and S. Koç, On S-prime submodules, Turkish J. Math. 43 (2019), 1036-1046.

[12] E.S. Sevim and T. Arabaci, Locally S-prime ideals, Comptes rendus de l'Academie bulgare des Sciences 73(12) (2020), 1650-1657.

[13] G. Ulucak, Ü. Tekir and S. Koç, On-S-2-absorbing submodules and vn-regular modules, An. Univ. Ovidius Const. -Ser. Mat. 28(2) (2020), 239-257.