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A treed domain need not be valtreed

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Abstract. If $3 \leq n \leq \infty$, there exists a quasi-local treed domain which has Krull dimension n and is not a valtreed domain. A consequence is that the class of valtreed domains fits properly between the class of treed domains and the class of going-down domains. Although the class of treed domains that are not going-down domains is stable under the classical $D+M$ construction, the class of valtreed domains that are not going-down domains is markedly unstable under that construction.
Key Words: Commutative ring, integral domain, prime ideal, treed domain, valuation domain, maximal ideal, valtreed domain, going-down domain, field extension, integral closure, left topology, $D+M$ construction.
2010 MSC: Primary 13B21, 13G05; Secondary 13A15, 13F05, 54A10.

In memory of my masters thesis advisor, Wilbur J. Jónsson

1 Introduction

All rings and algebras considered in this note are commutative and unital. Although our applications are to (commutative integral) domains, the proof of our main result, Theorem 2.1, depends in part on results on spectral sets due to Hochster [20], with which familiarity is assumed. The proof of Theorem 2.1 also depends in part on the left topology of a poset (in the sense of [3, Exercice 1, page 89]). We assume some familiarity with that exercise, although the reader may also find it useful to have access to the final paragraph of the Introduction of [13], where some background on the left topology and on some associated concepts from [7] is reprised and augmented. For the sake of completeness and convenience, we will begin Section 2 with a restatement of much of that just-mentioned paragraph. One of those just-mentioned concepts, which also plays a crucial role in the proof of Theorem 2.1, is that of an L-spectral set (where, of course, “L" stands for “L(eft topology)"). The definition of that concept and the statement of some of the results about it that will be needed in the proof of Theorem 2.1 will be recalled either at the start of Section 2 or during the proof of Theorem 2.1. Because the statement of Theorem 2.1 is somewhat technical, we will give a full statement of that result three paragraphs hence, after providing some additional technical background. Suffice it to say at this point that the main consequence of Theorem 2.1 is the fact the class of valtreed domains (which was recently introduced in [9]) fits properly between the class of treed domains and the class of going-down domains.

Recall from [4], [14] that a domain R is said to be a *going-down domain* if $R \subseteq T$ satisfies GD for each domain T containing R as a subring. (We follow [22, page 28] in letting GD denote the going-down property of ring extensions.) The most familiar examples of going-down domains are domains of (Krull) dimension at most 1 and Prüfer domains (cf. [4]). It is known [14, Theorem 1] that in determining whether a domain R is a going-down domain, it suffices to check that GD is satisfied by each ring extension of the form $R \subseteq T$ where T is a valuation overring of R. (As usual, if R is a domain with quotient field K, then an *overring* of R is an R-subalgebra of K; and a *valuation overring* of R is a valuation domain which is an overring of R.) One easy (and well known) consequence of the definition of a going-down domain is that the property of being a going-down domain is a local property of domains (in the sense that a domain R is a going-down domain if and only if R_M is a going-down domain for each maximal ideal M of R). It follows that any valuation domain is a going-down domain (since each of its localizations is a Prüfer domain). A relevant associated property is that of a treed domain. Recall from [4] that a ring R is said to be *treed* if the set Spec(R) (consisting of all the prime ideals of R) is, when considered as a poset under inclusion, a tree; that is, if no maximal ideal of R can contain a pair of incomparable prime ideals of R. It follows easily from familiar facts about the behavior of prime ideals in rings of fractions that the property of being a treed domain is a local property of domains. It was shown in [4, Theorem 2.2] that any going-down domain is treed. (A quicker proof of that fact became available by using a subsequent characterization of going-down domains in [14, Theorem 1].) Recently, it was shown in [9, Corollary 2.8] that the integral closure of any going-down domain R can be expressed in several interesting (and often equivalent) ways as an intersection of valuation overrings of R. One such expression is given in condition (g) of [9, Corollary 2.6]. For the moment, let us simply say that if a domain R is such that each of its maximal ideals has height at least 2, then R satisfies (*) if its integral closure R' (in a quotient field K of R) can be expressed as in condition (g) of [9, Corollary 2.8]. Thus, the problem of finding a new characterization of going-down domains reduces naturally to the following question: can one find a property, let us call it $\mathfrak{P}$ here, such that a quasi-local domain (R, M) of dimension at least 2 is a going-down domain if and only if R satisfies both (*) and $\mathfrak{P}$? That problem was recently solved in [9, Corollary 2.16], where $\mathfrak{P}$ was called the property of R being a "valtreed domain". One should note that [9, Corollary 2.16] is a sharp result, in a sense that is explained in the next two sentences. In [9, Example 2.11] (see also [9, Proposition 2.10], the first three sentences of the proof of [9, Theorem 2.15] and [18, Corollary 19.7 (2)]), an example was given of a quasi-local two-dimensional domain R which is not a going-down domain (indeed, it is not even a treed domain) although its integral closure satisfies condition (g) of [9, Corollary 2.8], along with many of the other above-mentioned expressions as an intersection of valuation overrings of R; it follows from [9, Corollary 2.16] that R is not a valtreed domain although R does satisfy (*). Moreover, an example due to W. J. Lewis, which was first presented in [15, Example 4.4] with Dr. Lewis' kind permission, is of a quasi-local two-dimensional (hence, necessarily valtreed) domain R which is not a going-down domain; it follows from [9, Corollary 2.16] that R does not satisfy (*), although R is a valtreed domain. Two paragraphs hence, we will say more, including the definition of a valtreed domain (from which, it will be clear why any two-dimensional treed domain is valtreed) and a precise statement of Theorem 2.1. First, it will be convenient to devote the next paragraph to some notational conventions that have not yet been explicitly introduced here.

All subrings, inclusions of rings, ring extensions and subalgebras considered here are unital ring extensions. As usual, if R is a ring, Spec(R) (resp., Max(R)) denotes the set of all prime (resp., maximal) ideals of R; any comments of a "dimensional" nature refer to the (classical) Krull dimension, denoted by $\dim(R)$, of an ambient ring R; $\subset$ and $\supset$ denote proper inclusion and proper containment, respectively; and $|S|$ denotes the cardinal number of a set S. If R is a domain with quotient field K and if R is a subring of a field F (that is, if F is a field extension of K), then $X_F(R)$ denotes the set of valuation domains of F containing R (as a subring), and R'_F denotes the integral closure of R in F; the set of valuation overrings of R is denoted by $X(R)$ $(= X_K(R))$; and, of course, $R'_K = R'$. If R, K, F are as above, with $P \in \mathrm{Spec}(R)$ and $M \in \mathrm{Max}(R)$ such that $P \subseteq M$, it is useful to consider the set

$$\mathcal{T}_{F,R,M,P} := \{W \in X_F(R) \mid W \text{ is centered on } M \text{ and there exists } \mathcal{P} \in \mathrm{Spec}(W) \text{ such that } \mathcal{P} \cap R = P\};$$

(if $F = K$, the just-displayed set is denoted simply by $\mathcal{T}_{R,M,P}$). Indeed, a standard result in multiplicative ideal theory [18, Theorem 19.8] states that if R is a domain and $Q \in \mathrm{Spec}(R)$, then $(R_Q)' = \cap\{V \in X_F(R) \mid$ the center of V on R is $Q\}$. (Recall that if R, K, F and Q are as above, with $(W, \mathcal{N}) \in X_F(R)$, one says that W is *centered on* Q (*in* R), or equivalently that *the center of* W *on* R *is* Q, if $\mathcal{N} \cap R = Q$.) In particular, if (R,M) is a quasi-local domain, then $R^{'}$ is the intersection of all the valuation overrings of R that are centered on M. Moreover, we can next explain in detail two items that were mentioned above. First, a domain R satisfies the property that was called (*) in the preceding paragraph if and only if $R^{'}$ can be expressed as

$$\cap_M (\cap_{P \subset M} (\cap \{W \in X_F(R) \mid W \in \mathcal{T}_{F,R,M,P}\})),$$

where M ranges over the elements of Max(R) and, for each such M, P ranges over the elements of Spec(R) satisfying $0 \subset P \subset M$. Second, the "$\mathcal{T}_{F,R,M,P}$" notation plays a critical role in the following definition. Let R, K and F be as above. Then the domain R is said to be an *F-valtreed domain* if, for each $M \in$ Max(R) and each pair of nonzero prime ideals P_1 and P_2 of R that are each properly contained in M, one has that $\mathcal{T}_{F,R,M,P_1} = \mathcal{T}_{F,R,M,P_2}$; if $F = K$, one says "valtreed domain" instead of "F-valtreed domain". It is obvious that any treed domain of dimension at most 2 is (vacuously) an F-valtreed domain, regardless of whether the given domain is quasi-local. It was shown in [9, Lemma 2.14 (b)] that if F is any field containing R as a subring, then R is an F-valtreed domain if and only if R is a valtreed domain. This interesting fact will be needed in the proof of Theorem 2.1, as that proof will involve a domain T containing a domain R of interest and we will need to consider the quotient field of that ring T without knowing whether T is an overring of R.

Only two more pieces of information are needed before we can finish motivating the statement of Theorem 2.1. Part (a) of [9, Lemma 2.14] established that any going-down domain is a valtreed domain; and part (c) of [9, Lemma 2.14] established that any valtreed domain is a treed domain. As the above-mentioned example of Lewis is a quasi-local two-dimensional domain (hence, valtreed domain) which is not a going-down domain, the only thing still missing from a proof of the assertion in the second sentence of the Abstract of this note is an example of a treed domain which is not a valtreed domain. That example, in turn, is provided by Theorem 2.1, which states that for each integer $n \geq 3$, there exists a quasi-local treed domain R such that dim(R) = n and R is not a valtreed domain.

As indicated by the first sentence of the Abstract, the assertion that is established in Theorem 2.1 also holds in case $n = \infty$. That fact is established in Corollary 2.4. While topological studies of spectral sets (aided substantially by an appeal to a realization theorem of Hochster [20]) suffice to give a proof of Theorem 2.1, the method of proof of Corollary 2.4 is quite different, as it involves a study of the behavior of the "valtreed domain" property with respect to the classical $D + M$ construction (as well as an appeal to the finitistic result established in Theorem 2.1). The following two additional consequences of that study seem especially noteworthy. It is shown in Corollary 2.3 (b) that the class of valtreed domains that are not going-down domains is remarkably unstable with respect to the classical $D + M$ construction. (I would go so far as to view that particular instability as being pathological, as it stands in sharp contrast to the fact that the classes of treed domains and of going-down domains are each preserved and reflected by the classical $D + M$ construction.) That instability is actually proved as a consequence of Corollary 2.3 (a), a result which gives, *i.a.*, a new characterization of going-down domains in terms of valtreed $D + M$ constructions.

Theorem 2.1 gives a negative answer to one of the questions that were raised in [9, Remark 2.17 (c)], namely, whether a treed domain must be valtreed. However, we do not yet know the answers to some other questions that were raised in [9, Remark 2.17]. Remark 2.5 reflects on some possible extensions of (or alternate approaches to) the results in this note, while also revisiting the questions and suggestions for future related research that were offered in parts (c)-(f) of [9, Remark 2.17].

Any unexplained material is in standard references, such as [18], [22].

2 Results

We move at once to our main result. The analogue of Theorem 2.1 for the case where $n = \infty$ will be proven in Corollary 2.4.

Theorem 2.1. For each integer $n \geq 3$, there exists a quasi-local treed domain R such that $\dim(R) = n$ and R is not a valtreed domain.

Proof. We will first prove the assertion in case $n = 3$, and then we will show how to modify that proof for any integer $n \geq 4$.

We begin by defining a poset $(Y, \leq)$, with $Y := \{y_0, y_1, y_2, y_3\}$, by imposing the requirements that $y_3 < y_2 < y_0$ and $y_3 < y_1$ (and $|Y| = 4$), with no other occurrences of "<". (As usual, a statement of the form "$a < b$" means "$a \leq b$ and $a \neq b$".) We claim that Y is a spectral set. This claim means that there is a ring D such that $(Y, \leq)$ is order-isomorphic to the poset structure $(\mathrm{Spec}(D), \leq_D)$ that is induced by the Zariski topology on $\mathrm{Spec}(D)$ (where, as in [20, page 53, lines 13-14], if P and Q are prime ideals of D, $P \leq_D Q$ means that Q is in the Zariski-topology closure of $\{P\}$). Moreover, we claim that $(Y, \leq)$ is an L-spectral set, in the sense of [12, page 229]. This second (and stronger) claim means that Y^L, the topological space obtained by imposing the left topology on the poset Y, is a spectral space. (Recall from [3, Exercice 1, page 89] that an open basis for Y^L consists of the sets of the form $v^{\downarrow}$ as v runs through the elements of Y, where $v^{\downarrow} := \{u \in Y \mid u \leq v\}$; and recall from [20, page 43, second paragraph] that a *spectral space* is a topological space that is homeomorphic to $\mathrm{Spec}(E)$ with the Zariski topology for some ring E.) It is straightforward to check that the poset structure induced on Y by the left topology on Y is precisely $(Y, \leq)$. (Because Y is finite, the preceding assertion is also an immediate consequence of either [7, Main Theorem] or [7, Corollary 2.6].) Consequently, every L-spectral set is a spectral set. Thus, we need only prove the second claim (as the first claim will then follow.) To that end, one need only verify the four order-theoretic conditions (α) - (δ) in the characterization of L-spectral sets in [12, Theorem 2.4]. Since Y is finite, it is evident that the following three conditions
(α) each nonempty linearly ordered subset of Y has a least upper bound,
(γ) Y has only finitely many maximal elements, and
(δ) for each pair of distinct elements $u, v \in Y$, there exist only finitely many elements of Y which are maximal in the set of common lower bounds of u and v
all hold in Y. Moreover, checking (β) amounts to the (rather easy) verification that each nonempty lower-directed subset Z of Y has a greatest lower bound z such that $\{y \in Y \mid z \leq y\} = \{y \in Y \mid w \leq y$ for some $w \in Z\}$. In fact, as noted in [12, Remark 2.5 (a)], it is a matter of universal algebra that, after one has checked the other three conditions, the task of checking (β) can be replaced with checking that every strictly decreasing sequence in Y stabilizes (and, since Y is finite, it is trivial to check *that*). This completes the proof of the above two claims. It will also be useful to record that the following is a list of all (seven of) the open sets of Y^L:

$$\emptyset, Y, \{y_0, y_2, y_3\}, \{y_1, y_3\}, \{y_2, y_3\}, \{y_3\}, \text{ and } \{y_1, y_2, y_3\}.$$

We leave it to the reader to check the just-displayed list by using the open basis $\{y^{\downarrow} \mid y \in Y\}$ for Y^L.

Next, define a four-element linearly ordered poset $X := \{x_0, x_1, x_2, x_3\}$ by imposing the requirements that $x_3 < x_2 < x_1 < x_0$. (Since X has a unique maximal element, the eventual treed domain A will be automatically quasi-local.) By considering the above-mentioned conditions (α)-(δ) in [12, Theorem 2.4], one shows easily that any finite linearly ordered set is an L-spectral set. In particular, X is an L-spectral set (and hence, by the above comments, also a spectral set). It will be useful to record that the following is a list of all (five of) the open sets of X^L:

$$\emptyset, X, \{x_1, x_2, x_3\}, \{x_2, x_3\}, \text{ and } \{x_3\}.$$

It is straightforward to check the just-displayed list by using the open basis $\{x^{\downarrow} \mid x \in X\}$ for X^L.

Next, define the function $\varphi : Y \to X$ by $\varphi(y_i) = x_i$, for $i = 0, 1, 2, 3$. Observe that φ is surjective and order-preserving. (We have also arranged that φ is *not* an order-isomorphism. That fact will play an important role in our proof that the eventual domain A is not valtreed. The reader may have already perceived how we have arranged the weaker conclusion that the eventual extension of domains $A \subseteq B$ will not satisfy GD, the point being that although $x_2 \leq x_1$ and $\varphi(y_1) = x_1$, there does not exist $y \in Y$ such that $y \leq y_1$ and $\varphi(y) = x_2$. This single fact captures most of the novelty that distinguishes the construction here of Y, X and φ from the construction of the similarly denoted quantities in the easier proof in [13, pages 3-5].) One could use the above lists of open sets to check that φ is continuous when viewed as a map $Y^L \to X^L$ (as it is very clear that the inverse image under φ of any open set in X^L is an open set in Y^L). However, this detail can be avoided by appealing to [12, Lemma 2.6 (a)], which states that any order-preserving map of posets is continuous when these posets are each equipped with the left topology. Next, recall from [20, page 43] that a map h of spectral spaces is said to be a *spectral map* if h is continuous and the inverse image under h of any quasi-compact open subset of the codomain of h is quasi-compact (and open). Since φ is a continuous function between finite spectral spaces, we can also conclude that φ is a spectral map (the point being that the finitude of X and Y ensures that every subset of X (resp., Y) is quasi-compact). In short, $\varphi : Y^L \to X^L$ is both spectral and surjective.

The above data are made to order for the realization assertion in [20, Theorem 6 (b)]. This result states that when Spec is viewed as a contravariant functor from the category of commutative rings (and ring homomorphisms) to the category of spectral spaces (and spectral maps), then Spec is invertible on the (nonfull) subcategory of all spectral spaces and surjective spectral maps. In particular, one infers the existence of a ring homomorphism $f : A \to B$ and homeomorphisms $\alpha : \mathrm{Spec}(A) \to X$, $\beta : \mathrm{Spec}(B) \to Y$, (where Spec($A$) and Spec($B$) are each endowed with the Zariski topology) such that $\alpha \circ \mathrm{Spec}(f) = \varphi \circ \beta$. It follows that Spec($f$) inherits the "surjective" property of φ. Moreover, since the homeomorphisms α, β are necessarily order-isomorphisms, it also follows that Spec(f) has all the order-theoretic properties of φ. Hence, A is a quasi-local three-dimensional treed ring. (Furthermore, f does not satisfy the "homomorphism version" of the GD property.)

We next reduce to the case of injective f. Indeed, the First Isomorphism Theorem gives the factorization $f = j \circ \pi$, where $\pi : A \to A/\ker(f)$ is the canonical projection and $j : A/\ker(f) \hookrightarrow B$ is the canonical injection. Note that Spec(π) is a homeomorphism (hence, an order-isomorphism), the key point being that $P \supseteq \ker(f)$ for each prime ideal P of A. (To see this, take a prime ideal Q of B such that $P = \mathrm{Spec}(f)(Q) = f^{-1}(Q)$ and observe that $\ker(f) = f^{-1}(\{0\}) \subseteq f^{-1}(Q)$. We have now shown that Spec(π) is a continuous bijection. To conclude that Spec(π) is a homeomorphism, it is enough to note that a standard homomorphism theorem ensures that Spec(π) is a closed map.) As $\mathrm{Spec}(j) = (\mathrm{Spec}(\pi))^{-1} \circ \mathrm{Spec}(f)$, we see that Spec($j$) has all the order-theoretic properties of Spec(f) and, hence, all the order-theoretic properties of φ. (In particular, j does not satisfy GD.) By *abus de langage*, we henceforth replace f with j, viewed as an inclusion (and thus replace A with $A/\ker(f)$). Notice also that (either the "old" or the "new") A is a quasi-local three-dimensional treed ring, thanks to the order-isomorphism α and the construction of X.

Since f does not satisfy GD, we see via [10, Lemma 3.2 (a)] that the injection $f_{\mathrm{red}} : A_{\mathrm{red}} \to B_{\mathrm{red}}$ of associated reduced rings also does not satisfy GD. (Recall that if E is any ring, then $E_{\mathrm{red}} := E/\sqrt{E}$, where $\sqrt{E}$ denotes the set of all nilpotent elements of E. It is well known that applying the Spec functor to the canonical projection $E \to E_{\mathrm{red}}$ produces a homeomorphism. Of course, f_{red} is defined by $a + \sqrt{A} \mapsto f(a) + \sqrt{B}$.) By more *abus de langage*, we replace f with f_{red}, which is now viewed as an inclusion. Observe that (the "new") A is a quasi-local, three-dimensional treed ring. Moreover, we have now reduced to the case in which both A and B are reduced rings (that is, rings with no nonzero nilpotents) each having a unique minimal prime ideal, that is, (integral) domains.

The last three paragraphs were taken, after some light editing, from a proof in [13]. That paper's presentation had aimed only to produce a "spectral construction" of a treed domain that is not a going-down domain. The domain constructed "spectrally" in [13] was minimal for its purpose, in the sense that (it was quasi-local and) its dimension was 2. As all two-dimensional treed domains are valtreed, we have needed to devise a more complicated construction here. That is why our poset X needed to have cardinality (at least) 4 (and our poset Y needed to have at least *that* cardinality if we were to appeal to [20, Theorem 6 (b)], as that result requires φ to be surjective). At this point, in order to next show that/why the present (quasi-local, three-dimensional, treed) domain A is not a valtreed domain, we must (once again) deviate from the proofs in [13].

Thanks to the above instances of *abus de langage* and the above homeomorphisms α and β, we have domains $A \subset B$, such that $\mathrm{Spec}(A) = \{0_A, P, Q, M\}$ and $\mathrm{Spec}(B) = \{0_B, \mathcal{P}, N_1, N_2\}$, with the only proper inclusions in these prime spectra being given by $0_A \subset P \subset Q \subset M$, $0_B \subset \mathcal{P} \subset N_1$, and $0_B \subset N_2$. (In detail, α and β effect the following correspondences: $0_A \leftrightarrow x_3$, $P \leftrightarrow x_2$, $Q \leftrightarrow x_1$, $M \leftrightarrow x_0$, $0_B \leftrightarrow y_3$, $\mathcal{P} \leftrightarrow y_2$, $N_1 \leftrightarrow y_0$, and $N_2 \leftrightarrow y_1$.) In addition, the element-wise definition of φ ensures that (of course, $0_B \cap A = 0_A$, as well as)

$$\mathcal{P} \cap A = P,\ N_2 \cap A = Q \text{ and } N_1 \cap A = M.$$

In particular, in $\mathrm{Spec}(B)$, we have that $\mathcal{P}$ is "adjacent to" N_1 (in the sense that there is no prime ideal of B contained strictly between $\mathcal{P}$ and N_1), and also that 0_B is adjacent to both $\mathcal{P}$ and N_2.

Next, choose any valuation overring $(W, \mathcal{N})$ of B whose center on B is $\mathcal{P}$ (cf. [18, Theorem 19.6]). Since $\mathcal{P} \subset N_1$, an application of [18, Corollary 19.7 (2)] gives that there exists a valuation overring $(\mathcal{V}, \mathfrak{N})$ of B such that $\mathcal{V} \subset \mathcal{W}$ and $\mathfrak{N} \cap B = N_1$. Let $\mathfrak{N}^*$ denote the intersection of all the prime ideals of $\mathcal{V}$ that intersect B in N_1. Since all valuation domains are quasi-local treed rings, $\mathfrak{N}^*$ is an intersection of a (nonempty) chain of prime ideals of $\mathcal{V}$. It follows that $\mathfrak{N}^*$ is a prime ideal of $\mathcal{V}$ [22, Theorem 9] and that $\mathfrak{N}^* \cap B = N_1$. Note that $\mathcal{N}$ is also a prime ideal of $\mathcal{V}$. Moreover, $\mathcal{N} \subset \mathfrak{N}^*$. Indeed, this is the only possibility, in view of the facts that $\mathcal{N}$ and $\mathfrak{N}^*$ are comparable under inclusion, with $\mathcal{N} \cap B = \mathcal{P} \subset N_1 = \mathfrak{N}^* \cap B$.

Put $\mathcal{V}^* := \mathcal{V}_{\mathfrak{N}^*}$. We have that $\mathcal{N} \subset \mathfrak{N}^*$ as prime ideals of $\mathcal{V}^*$. We claim that any prime ideal $\mathfrak{I}$ of $\mathcal{V}^*$ that is contained strictly between $\mathcal{N}$ and $\mathfrak{N}^*$ must intersect B in $\mathcal{P}$. To see this, it suffices to observe (via standard propeties of valuation domains) that $\mathfrak{I}$ is a prime ideal of $\mathcal{V}$, use the definition of $\mathfrak{N}^*$, and also use the fact that $\mathcal{N} \cap B = \mathcal{P}$ and $N_1 = \mathfrak{N}^* \cap B$ are adjacent in $\mathrm{Spec}(B)$. Consequently, since the prime ideals of (the valuation domain) $\mathcal{V}^*$ are linearly ordered by inclusion, it follows that, besides N_1 and $\mathcal{P}$, any *other* element in the image of the canonical contraction map $\mathrm{Spec}(\mathcal{V}^*) \to \mathrm{Spec}(B)$ must be a subset of $\mathcal{P}$. By the functoriality of the Spec functor, we can now conclude that *the only possible* elements in the image of the canonical contraction map $\mathrm{Spec}(\mathcal{V}^*) \to \mathrm{Spec}(A)$ are $\mathfrak{N}^* \cap A = (\mathfrak{N}^* \cap B) \cap A = N_1 \cap A = M$, $\mathcal{N} \cap A = (\mathcal{N} \cap B) \cap A = \mathcal{P} \cap A = P$, and some other elements each of which is a subset of $\mathcal{P} \cap A = P$. Notice that Q is not in this list (since $P \subset Q \subset M$). Letting F denote the quotient field of B, we have just proven that $\mathcal{V}^* \notin \mathcal{T}_{F,A,M,Q}$. On the other hand, we *do* have that $\mathcal{V}^* \in \mathcal{T}_{F,A,M,P}$ (since $\mathfrak{N}^* \cap A = M$ and $\mathcal{N} \cap A = P$).

Although this paragraph is not required by the demands of logic, it will show that an interesting set is nonempty. Observe that $(W, \mathcal{N})$ is a valuation ring of F whose center on B is $\mathcal{P}$. Hence, the center of W on A is $\mathcal{P} \cap A = P$. Since $P \subset Q \subset M$ in $\mathrm{Spec}(A)$, an application of [18, Corollary 19.7 (2)] gives that there exists a valuation ring $\mathfrak{W}$ of F such that $\mathfrak{W} \subset W$, and some chain of three prime ideals of $\mathfrak{W}$ (with smallest element $\mathcal{N}$) contracts to the chain $P \subset Q \subset M$ in $\mathrm{Spec}(A)$. It follows that the center of $\mathfrak{W}$ on A is M. We can now conclude that $\mathfrak{W} \in \mathcal{T}_{F,A,M,Q}$. In particular, $\mathcal{T}_{F,A,M,Q} \neq \emptyset$.

We saw two paragraphs ago that $\mathcal{V}^* \in \mathcal{T}_{F,A,M,P} \setminus \mathcal{T}_{F,A,M,Q}$. Thus, P and Q are distinct nonzero prime ideals of the domain A, each of which is a subset of a (the!) maximal ideal M of A, such that $\mathcal{T}_{F,A,M,P} \neq \mathcal{T}_{F,A,M,Q}$. Therefore, A is not an F-valtreed domain (by the very definition of an F-valtreed domain, which was recalled in the Introduction). Hence, by [9, Lemma 2.14 (b)], the (quasi-local three-dimensional treed) domain A is not a valtreed domain. Taking $R := A$ thus completes the proof for the case $n = 3$.

Next, we sketch how to modify the above proof in order to handle the case of an integer $n \geq 4$. Begin by adjoining a set of $n-3$ (new) elements y_{-j} to Y (for $j = 1, 2, \ldots, n-3$), thereby creating a set Y_n having Y as a sub-poset, with these new elements satisfying only the following instances of "<":

$$y_3 < y_{-(n-3)} < \ldots < y_{-2} < y_{-1} < y_2.$$

Next, similarly adjoin a set of $n-3$ (new) elements x_{-j} to X (for $j = 1, 2, \ldots, n-3$), thereby creating a set X_n having X as a sub-poset, with these new elements satisfying only the following instances of "<":

$$x_3 < x_{-(n-3)} < \ldots < x_{-2} < x_{-1} < x_2.$$

Next, define a function $\varphi_n : Y_n \to X_n$ which extends φ by also requiring that $\varphi_n(y_{-j}) = x_{-j}$ for all $j = 1, 2, \ldots, n-3$.

The above proofs from the case $n = 3$ carry over with only minor modifications for the present context where $n \geq 4$. One thus gets a ring extension of domains $A_n \subset B_n$ where A_n (resp., B_n) differs in its prime spectrum from that of A (resp., B) only by having adjoined a chain of $n-3$ prime ideals

$$P_{-(n-3)} \subset \ldots \subset P_{-2} \subset P_{-1} \text{ (resp., } \mathcal{P}_{-(n-3)} \subset \ldots \subset \mathcal{P}_{-2} \subset \mathcal{P}_{-1})$$

contained strictly between 0_A and P (resp., strictly between 0_B and $\mathcal{P}$) such that $\mathcal{P}_{-j} \cap A_n = P_{-j}$ for all j such that $1 \leq j \leq n-3$. Apart from minor modifications to the argumentation that applied to the earlier case (where $n = 3$), the main reason that the earlier proof also carries over to the present context is that we have arranged that whenever $1 \leq j \leq n-3$, the newly adjoined prime ideal $\mathcal{P}_{-j}$ satisfies $\mathcal{P}_{-j} \cap A_n \subset P \subset Q$, so that Q is not lain over by any of the "new" prime ideals $\mathcal{P}_{-j}$.

The upshot of the above adjunctions and reasoning in the last two paragraphs (with Y_n, X_n and φ_n having respectively replaced/extended Y, X and φ from the argument that had treated the case $n = 3$) is that the domain A_n is quasi-local, treed and n-dimensional but is not a valtreed domain. The proof is complete. □

As explained in the Introduction, when Theorem 2.1 is combined with [4, Theorem 2.2], an example of Lewis [15, Example 4.4] and [9, Lemma 2.14 (a), (c)], one gets that the class of valtreed domains fits strictly between the class of treed domains and the class of going-down domains. We will use the classical $D+M$ construction (which is recalled in the next paragraph) to show that this "fit" is somewhat uncomfortable, in the sense that it does not exhibit some stability properties that may have been expected on the basis of some earlier results about treed domains and going-down domains.

It is well known that if K is a field and $1 \leq m \leq \infty$, then there exists a valuation domain (V, M) of the form $V = K + M$ such that $\dim(V) = m$ (cf. the proofs of [18, Theorem 18.3 and Corollary 18.5]). Valuation domains having this form $K + M$, with $M \neq 0$, have appeared often in the literature since the mid-1930s, especially in the construction of (counter)examples having the form $D + M$ for suitable proper subrings D of K. For instance, I. J. Papick and the author proved in [14, Corollary] that a domain D is a going-down domain if every (equivalently, some) corresponding $D + M$ (arising as above from a valuation domain $K + M$) is a going-down domain. It is clear from well known facts about the prime ideals of this "classical $D + M$ construction" (cf. [18, Exercise 12 (1)-(2), page 202]) that a domain D is treed if and only if every (equivalently, some) corresponding $D + M$ is a treed domain. It follows that a domain D is a treed domain but not a going-down domain if and only if every (equivalently, some) corresponding $D + M$ is a treed domain but not a going-down domain. However, Corollary 2.3 (b) will establish that the class of valtreed domains that are not going-down domains is spectacularly unstable under the classical $D + M$ construction.

The main goal of the next several results is to study when D and/or $D + M$ are/is valtreed. Given their announced purpose, these results will not need to address the situation where D is a field, say k.

Indeed, when (V,M) is a valuation domain of the form $V = K+M$ where K is a field and $D = k \subseteq K$ is a field extension, then $D+M = k+M$ is a pseudo-valuation domain by [19, Example 2.1] and, hence, a going-down domain: see [6, page 560 and Proposition 2.1]. As any going-down domain is valtreed [9, Lemma 2.14 (a)] (hence, treed, by [9, Lemma 2.14 (b)]) and any field is (vacuously) valtreed, the subcase of the classical $D+M$ construction where D is a field cannot be expected to shed new light on the relations among the classes of treed, valtreed or going-down domains. Our examination of these relations via the classical $D+M$ construction begins with some fundamental technical information developed in parts (a) and (b) of Proposition 2.2, while the process of inferring results from that information begins in Proposition 2.2 (c)-(d).

Proposition 2.2. *Let D be a domain, with quotient field $k \supset D$, and let $k \subseteq K$ be a field extension. Let (V,M) be a valuation domain of the form $V = K+M$ with $M \neq 0$. Let F be the quotient field of V. Consider the domain $R := D+M$. Let $\mathcal{M} \in \mathrm{Max}(R)$. Necessarily, $\mathcal{M} = \mathfrak{M}+M$ for some (uniquely determined nonzero) $\mathfrak{M} \in \mathrm{Max}(D)$. Let $\mathcal{P}$ be a nonzero prime ideal of R such that $\mathcal{P} \subset \mathcal{M}$. Necessarily, either* (i) *$0 \subset \mathcal{P} \subseteq M$ with $\mathcal{P} \in \mathrm{Spec}(V)$ or* (ii) *$\mathcal{P} = \mathfrak{P}+M$ for some (uniquely determined) $\mathfrak{P} \in \mathrm{Spec}(D)$ such that $\mathfrak{P} \subset \mathfrak{M}$. Then:*

(*a*) *In case* (i),

$$\mathcal{T}_{R,\mathcal{M},\mathcal{P}} = \mathcal{T}_{R,\mathcal{M},\mathcal{M}} = \mathcal{T}_{K,D,\mathfrak{M},\mathfrak{M}} + M \; (= \{E+M \mid E \in \mathcal{T}_{K,D,\mathfrak{M},\mathfrak{M}}\}).$$

(*b*) *In case* (ii),

$$\mathcal{T}_{R,\mathcal{M},\mathcal{P}} = \mathcal{T}_{K,D,\mathfrak{M},\mathfrak{P}} + M \; (= \{\mathcal{W}+M \mid \mathcal{W} \in \mathcal{T}_{K,D,\mathfrak{M},\mathfrak{P}}\}).$$

(*c*) *If R is a valtreed domain, then D is a valtreed domain.*

(*d*) *If D is a treed domain but not a valtreed domain, then R is a treed domain but not a valtreed domain.*

Proof. This paragraph will serve to justify the two "Necessarily" comments in the statement of this result. The basic facts about the prime spectrum of the classical $D+M$ construction, as in [18, Exercise 12, (1)-(3), page 202], ensure that each prime ideal of R is comparable with M (under inclusion); that the prime ideals of R which are contained in M are precisely the prime ideals of V; and that each prime (resp., maximal) ideal of R that contains M is uniquely expressible in the form $\mathfrak{P}+M$ for some prime (resp., maximal) ideal $\mathfrak{P}$ of D. Of course, if such $\mathfrak{P}$ is in $\mathrm{Max}(D)$, then $\mathfrak{P} \neq 0$ since we have assumed that D is not a field.

We claim that if $(W,N) \in X(R)$ $(= X_F(R))$ is such that $N \cap R = \mathcal{M}$, then W does not contain V as a subset. Deny. Then $W = V_Q$ for some $Q \in \mathrm{Spec}(V)$ (by [22, Theorem 65], since V is a valuation domain and R has the same quotient field as V), whence $Q \subseteq N$. Also since V is a valuation domain, it follows that $N = QV_Q = Q$. Hence,

$$\mathfrak{M}+M = \mathcal{M} = N \cap R = Q \cap R = (Q \cap V) \cap R \subseteq M \cap R = M \subset \mathfrak{M}+M$$

(since $\mathfrak{M} \neq 0$). This (desired) contradiction proves the above claim.

Next, we combine the above claim, the catalog of overrings of any classical $D+M$ construction in [2, Theorem 3.1], and the characterization in [18, Exercise 13 (2), page 203] of the classical $D+M$ constructions that are valuation domains. The upshot is that if $(W,N) \in X(R)$ satisfies $N \cap R = \mathcal{M}$, then $W = E+M$ for some (uniquely determined) ring $E \in X_K(D) \setminus \{K\}$, that is, for some (uniquely determined) valuation domain E of K such that $D \subseteq E \subset K$.

(a) Assume that $0 \subset \mathcal{P} \subseteq M$ with $\mathcal{P} \in \mathrm{Spec}(V)$. It is clear from the definitions that $\mathcal{T}_{R,\mathcal{M},\mathcal{P}} \subseteq \mathcal{T}_{R,\mathcal{M},\mathcal{M}}$. We will next prove the reverse inclusion; that is, assuming that $(W,N) \in \mathcal{T}_{R,\mathcal{M},\mathcal{M}}$, we will show that $(W,N) \in \mathcal{T}_{R,\mathcal{M},\mathcal{P}}$. By assumption, $W \in X(R)$ and $N \cap R = \mathcal{M}$; our task is to find a prime ideal of W that meets R in $\mathcal{P}$. By the above comments, W can be (uniquely) expressed as $E+M$ for some valuation domain E of K such that $D \subseteq E \subset K$; and $\mathcal{P} \in \mathrm{Spec}(E+M) = \mathrm{Spec}(W)$. Thus, $\mathcal{P} = \mathcal{P} \cap R$ is in the image of the canonical contraction map $\mathrm{Spec}(W) \to \mathrm{Spec}(R)$. In particular, we have shown that $\mathcal{P}$ is a prime ideal of W that meets R in $\mathcal{P}$. This completes the proof of the first equality asserted in (a).

Next, consider any (W,N) where $W \in \mathcal{T}_{R,\mathcal{M},\mathcal{M}}$ is expressed as $E+M$ as above. We will show that $E \in \mathcal{T}_{K,D,\mathfrak{M},\mathfrak{M}}$. As we already know that $E \in X_K(D)$, it remains only to prove that the (unique) maximal ideal of E meets D in $\mathfrak{M}$. Let I denote the maximal ideal of E. Then $I+M$ is the maximal ideal of $E+M$ and we know that $(I+M)\cap R = \mathcal{M}$. Since

$$(I\cap D)+M = (I+M)\cap(D+M) = (I+M)\cap R = \mathcal{M} = \mathfrak{M}+M,$$

we get $I\cap D = \mathfrak{M}$, as desired.

The preceding paragraph established that $\mathcal{T}_{R,\mathcal{M},\mathcal{M}} \subseteq \mathcal{T}_{K,D,\mathfrak{M},\mathfrak{M}}+M$. To complete the proof of (a), one need only prove the reverse inclusion. To that end, consider any $(E,I)\in\mathcal{T}_{K,D,\mathfrak{M},\mathfrak{M}}$, and put $W := E+M$. As $E\in X_K(D)$, we have $W\in X(R)$; also, $I\cap D=\mathfrak{M}$. Moreover, the unique maximal ideal of W is $I+M$, and

$$(I+M)\cap R = (I+M)\cap(D+M) = (I\cap D)+M = \mathfrak{M}+M = \mathcal{M}.$$

Thus, $W\in\mathcal{T}_{R,\mathcal{M},\mathcal{M}}$. This completes the proof of (a).

(b) Assume that $\mathcal{P} = \mathfrak{P}+M$ for some (uniquely determined) $\mathfrak{P}\in\mathrm{Spec}(D)$ such that $\mathfrak{P}\subset\mathfrak{M}$. Suppose first that $(W,N)\in\mathcal{T}_{R,\mathcal{M},\mathcal{P}}$. Then $W\in X(R)$, $N\cap R=\mathcal{M}$, and some prime ideal J of W satisfies $J\subseteq\mathcal{M}$ and $J\cap R=\mathcal{P}$. By some of the discussion in the proof of (a) (including the claim that was proved there), $W=E+M$ for some (uniquely determined) valuation domain (E,I) of K such that $D\subseteq E\subset K$. We claim that $E\in\mathcal{T}_{K,D,\mathfrak{M},\mathfrak{P}}$. Since $N=I+M$, we have

$$(I\cap D)+M = (I+M)\cap(D+M) = N\cap R = \mathcal{M} = \mathfrak{M}+M,$$

and so $I\cap D=\mathfrak{M}$. To prove the above claim, it remains only to find a prime ideal of E that is contained in $\mathfrak{M}$ and meets D in $\mathfrak{P}$. As $J\supseteq J\cap R=\mathcal{P}=\mathfrak{P}+M$ for some (uniquely determined) $\mathfrak{P}\in\mathrm{Spec}(D)$ such that $\mathfrak{P}\subset\mathfrak{M}$, we have $J\supseteq M$. Thus, $J=\mathfrak{Q}+M$ for some prime ideal $\mathfrak{Q}$ of E. Then

$$(\mathfrak{Q}\cap D)+M = (\mathfrak{Q}+M)\cap R = J\cap R = \mathcal{P} = \mathfrak{P}+M,$$

whence $\mathfrak{Q}\cap D=\mathfrak{P}$. Since $\mathfrak{Q}+M=J\subseteq\mathcal{M}=\mathfrak{M}+M$, we get that $\mathfrak{Q}\subseteq\mathfrak{M}$. Hence, $\mathfrak{Q}$ is the prime ideal of E that we had sought. This completes the proof of the above claim. We have now proven half of the assertion in (b).

It remains to prove that if $(E,I)\in\mathcal{T}_{K,D,\mathfrak{M},\mathfrak{P}}$, then $E+M\in\mathcal{T}_{R,\mathcal{M},\mathcal{P}}$. As $E\in X_K(D)$, we get that $W := E+M\in X(R)$. By hypothesis, $I\cap D=\mathfrak{M}$, and so the maximal ideal of W, namely $N := I+M$, satisfies

$$N\cap R = (I+M)\cap(D+M) = (I\cap D)+M = \mathfrak{M}+M = \mathcal{M}.$$

Also by hypothesis, some prime ideal $\mathfrak{Q}$ of E satisfies $\mathfrak{Q}\cap D=\mathfrak{P}$. Consider $J := \mathfrak{Q}+M\in\mathrm{Spec}(W)$. We have

$$J\cap R = (\mathfrak{Q}+M)\cap(D+M) = (\mathfrak{Q}\cap D)+M = \mathfrak{P}+M = \mathcal{P}.$$

We have shown that $W\in\mathcal{T}_{R,\mathcal{M},\mathcal{P}}$. This completes the proof of (b).

(c) We will prove the contrapositive of the assertion; that is, we will assume that D is not valtreed and we will then prove that R is not valtreed. By [9, Lemma 2.14 (b)], the assumption ensures (in fact, is equivalent to) D not being a K-valtreed domain. Therefore, there exist a (necessarily nonzero) maximal ideal $\mathfrak{M}$ of D and distinct nonzero prime ideals, $\mathfrak{P}_1$ and $\mathfrak{P}_2$, (of D) both of which are properly contained in $\mathfrak{M}$, such that $\mathcal{T}_{K,D,\mathfrak{M},\mathfrak{P}_1}\neq\mathcal{T}_{K,D,\mathfrak{M},\mathfrak{P}_2}$. For $i\in\{1,2\}$, consider

$$\mathcal{P}_i := \mathfrak{P}_i+M\in\mathrm{Spec}(D+M)=\mathrm{Spec}(R).$$

Note that each $\mathcal{P}_i$ is nonzero and is properly contained in the maximal ideal $\mathfrak{M}+M=\mathcal{M}$ of R; also note that $\mathcal{P}_1\neq\mathcal{P}_2$. So, to complete the proof, it will be enough to show that $\mathcal{T}_{R,\mathcal{M},\mathcal{P}_1}\neq\mathcal{T}_{R,\mathcal{M},\mathcal{P}_2}$. By (b), this is equivalent to showing that

$$\mathcal{T}_{K,D,\mathfrak{M},\mathfrak{P}_1}+M \neq \mathcal{T}_{K,D,\mathfrak{M},\mathfrak{P}_2}+M.$$

As *that* is, in turn, equivalent to (showing that) $\mathcal{T}_{K,D,\mathfrak{M},\mathfrak{p}_1} \neq \mathcal{T}_{K,D,\mathfrak{M},\mathfrak{p}_2}$, the proof of (c) is complete.

(d) It is well known that D is treed if and only if $D+M$ $(=R)$ is treed. The conclusion that R is not valtreed follows from the assumption that D is not valtreed (by the contrapositive of (c)). The proof is complete. □

The "descent" result in Proposition 2.2 (c) leads naturally to the question of whether there is a corresponding "ascent" result, that is, whether the converse of Proposition 2.2 (c) holds in general. A casual glance may suggest that the proof of Proposition 2.2 (c) could be reversed. However, that is, in fact, decidedly **not** the case: see Corollary 2.3 (b) below.

Corollary 2.3 (a) gives our most important application of Proposition 2.2. On the one hand, the equivalence (3) $\Leftrightarrow$ (1) in Corollary 2.3 (a) shows how the "valtreed domain" concept can be used to give a new characterization of going-down domains. That observation stands in some contrast to the fact that a valtreed domain need not be a going-down domain. (As explained in the second paragraph of the Introduction, the above-mentioned example of Lewis is a valtreed domain but not a going-down domain.) On the other hand, Corollary 2.3 (a) immediately implies Corollary 2.3 (b), a result that reveals a surprisingly fragility within the class of valtreed domains. More specifically, while we saw in Proposition 2.2 (d) that the class of treed domains that are not valtreed domains is stable under the classical $D+M$ construction, Corollary 2.3 (b) shows that the class of valtreed domains that are not going-down domains is as unstable as possible under the classical $D+M$ operation, in the sense that if D is *any* member of that class, then $D+M$ is *not* a member of that class!

Corollary 2.3. *Let D be a domain, with quotient field k, and let $k \subseteq K$ be a field extension. Suppose also that $D \subset K$. Let (V,M) be a valuation domain of the form $V = K+M$ with $M \neq 0$. Then:*

(a) The following three conditions are equivalent:

(1) $D+M$ is a valtreed domain;

(2) $D+M$ is a going-down domain;

(3) D is a going-down domain.

(b) If D is a valtreed domain but not a going-down domain, then $D+M$ is not a valtreed domain.

Proof. It will be convenient to let R denote the domain $D+M$.

(a): (3) $\Leftrightarrow$ (2) by [14, Corollary]; and (2) $\Rightarrow$ (1) since any going-down domain is a valtreed domain [9, Lemma 2.14 (a)]. Therefore, it will suffice to show that (1) $\Rightarrow$ (3).

Let us suppose, then, that R $(= D+M)$ is a valtreed domain; our task is to prove that D is a going-down domain. Without loss of generality, $\dim(D) \geq 2$. By the characterization of going-down domains via condition (4) in [9, Theorem 2.6], this task can be reformulated as follows: to show that for each $\mathcal{M} \in \mathrm{Max}(R)$ and each $\mathcal{P} \in \mathrm{Spec}(R)$ such that $0 \subset \mathcal{P} \subset \mathcal{M}$, $\mathcal{T}_{R,\mathcal{M},\mathcal{P}} = \mathcal{T}_{R,\mathcal{M},\mathcal{M}}$. (Note that some of the notation from that cited result has been harmlessly changed here in order to be more suggestive of the notation that was used above in Proposition 2.2; note also that the field F that plays a role in the just-cited condition (4) is being taken here to be the quotient field of R, that is, the quotient field of V.) As it is trivial that $\mathcal{T}_{R,\mathcal{M},\mathcal{P}} \subseteq \mathcal{T}_{R,\mathcal{M},\mathcal{M}}$, it will suffice to prove the reverse inclusion.

As recalled in the first paragraph of the proof of Proposition 2.2, $\mathcal{M}$ can be (uniquely) expressed as $\mathfrak{M}+M$ with $\mathfrak{M} \in \mathrm{Max}(D)$; and either (j) $\mathcal{P}$ is a prime ideal of V (and hence $\mathcal{P} \subseteq M$) or (jj) $\mathcal{P}$ can be (uniquely) expressed as $\mathfrak{P}+M$ with $\mathfrak{P} \in \mathrm{Spec}(D)$ such that $0 \subset \mathfrak{P} \subset \mathfrak{M}$. In case (j), our task of showing that $\mathcal{T}_{R,\mathcal{M},\mathcal{M}} \subseteq \mathcal{T}_{R,\mathcal{M},\mathcal{P}}$ is dispatched immediately by appealing to the first assertion in Proposition 2.2 (a). So, we are left to address case (jj).

In case (jj), $\mathcal{P} = \mathfrak{P}+M$, with $0 \subset \mathfrak{P} \subset \mathfrak{M}$ in $\mathrm{Spec}(D)$. Then, by appealing to parts (a) and (b) of Proposition 2.2, we can reformulate the task of showing that $\mathcal{T}_{R,\mathcal{M},\mathcal{M}} \subseteq \mathcal{T}_{R,\mathcal{M},\mathcal{P}}$ as being the task of showing that

$$\mathcal{T}_{K,D,\mathfrak{M},\mathfrak{M}} + M \subseteq \mathcal{T}_{K,D,\mathfrak{M},\mathfrak{P}} + M.$$

Observe that M and $\mathcal{P}$ are nonzero prime ideals of R which are each properly contained in the maximal ideal $\mathcal{M}$ of R. Hence, it follows from the assumption that R is a valtreed domain that

$$\mathcal{T}_{R,\mathcal{M},M} = \mathcal{T}_{R,\mathcal{M},\mathcal{P}}.$$

The just-displayed equality can be reformulated, by applying Proposition 2.2 (b) twice, as

$$\mathcal{T}_{K,D,\mathfrak{M},0} + M = \mathcal{T}_{K,D,\mathfrak{M},\mathfrak{p}} + M.$$

Note that $\mathcal{T}_{K,D,\mathfrak{M},0} = \{(W,N) \in X_K(D) \mid N \cap D = \mathfrak{M}\} = \mathcal{T}_{K,D,\mathfrak{M},\mathfrak{M}}$. Therefore, the last display can be rewritten as

$$\mathcal{T}_{K,D,\mathfrak{M},\mathfrak{M}} + M = \mathcal{T}_{K,D,\mathfrak{M},\mathfrak{p}} + M.$$

Replacing the equals sign "=" in the last display with "$\subseteq$" establishes an inclusion which we saw above would be (and, hence, now is) enough to complete the proof of (a).

(b) Of course, (b) is immediate from (a). The proof is complete. □

Note that the conclusion in Proposition 2.3 (b) would hold even without the assumption that D is a valtreed domain. (We expect that many readers would have found the addition of that assumption to be somewhat natural at that point, in view of Proposition 2.2 (d).) The above formulation of Proposition 2.3 (b) was chosen in order to stand in contrast to the formulation of Proposition 2.2 (d).

As promised at the opening of this section, we can now give the infinitistic counterpart of Theorem 2.1.

Corollary 2.4. *There exists a quasi-local treed domain R such that* $\dim(R) = \infty$ *and R is not a valtreed domain.*

Proof. Using the first part of the proof of Theorem 2.1, choose a quasi-local treed domain D such that $\dim(D) = 3$ and D is not a valtreed domain. Let K be any field containing D as a subring. (For instance, take K to be the quotient field of D.) Take (V,M) to be any infinite-dimensional valuation domain of the form $V = K + M$. Then $R := D + M$ has the asserted properties. Indeed, [18, Exercise 12 (4), page 203] ensures that $\dim(R) = \dim(D) + \dim(V) = 3 + \infty$ $(= \infty)$; R inherits the "quasi-local treed domain" property from D; and R is not valtreed, by Proposition 2.2 (c). The proof is complete. □

The penultimate paragraph of the Introduction has already explained the several purposes of our final remark.

Remark 2.5. (a) The use of the classical $D + M$ construction in the proof of Corollary 2.4 suggests an alternate way to finish the proof of Theorem 2.1 for the case where $4 \leq n < \infty$. To wit: let D denote the three-dimensional treed domain which is not a valtreed domain that was found in the first part of the proof of Theorem 2.1. Next, take (V,M) to be any $(n-3)$-dimensional valuation domain of the form $V = K + M$, where K is any field containing D as a subring. Then, by reasoning as in the proof of Corollary 2.4, we get that $R := D + M$ is an n-dimensional treed domain but not a valtreed domain.

While the proof in the preceding paragraph lessened the amount of topological reasoning that would be needed in an alternate proof of Theorem 2.1, note that we would have needed to delay the presentation of this alternate proof until we had proven Proposition 2.2 (since the alternate proof's assertion that $D + M$ is not valtreed depended on Proposition 2.2 (c)). For the purposes of presenting the results in this note to a class or a seminar, that sort of delay may not be inappropriate, as the proof of the case $n = 3$ of Theorem 2.1 suffices to establish the titular assertion of this note.

(b) Let us next consider a couple of other possible approaches to potential proofs of Corollary 2.4. For the first of these, we begin by recalling that the case where $4 \leq n < \infty$ had been proved in Theorem 2.1 with the use of suitable sequences

$$y_3 < y_{-(n-3)} < \ldots < y_{-2} < y_{-1} < y_2 \text{ and}$$

$$x_3 < x_{-(n-3)} < \ldots < x_{-2} < x_{-1} < x_2.$$

In the spirit of that proof, it would be natural to try to (further) enlarge the sets Y and X by inserting a strictly decreasing (infinite) sequence $y_{-1} > y_{-2} > \ldots$ (resp., $x_{-1} > x_{-2} > \ldots$) strictly between y_3 and y_2 (resp., strictly between x_3 and x_2), with y_{-1} adjacent to y_2 (resp., x_{-1} adjacent to x_2) but no element of the enlarged poset Y_∞ (resp., X_∞) adjacent to y_3 (resp., x_3). There may be topological or order-theoretic impediments to such an approach to the case $n = \infty$. To wit: neither Y_∞ nor X_∞ is an L-spectral set, as each of these posets fails to satisfy condition (β) from [12, Theorem 2.4], the point being that Y_∞ and X_∞ each contain a strictly decreasing *infinite* sequence (cf. [12, Remark 2.5 (a)]). Each of the sets Y_∞ and X_∞ *can* be shown to be a spectral set, with their above orderings being induced by what Lewis and J. Ohm dubbed the "$C(m)$ topology" in [21, page 824]: see [21, Lemma 3.1 and Theorem 3.2]. Interested readers wishing to handle the case $n = \infty$ by modifying the method of proof of Theorem 2.1 are advised to begin by discerning all the differences (which must exist) between the $C(m)$ topology and the left topology on Y_∞ and X_∞.

In another vein, I wish to point out that it is perhaps not surprising that certain familiar methods that deal with finite sets of data may not easily adapt to contexts involving infinite sequences. For instance, as evidenced by the calculations in papers such as [24] and [11], the spectral nuances of iterating various inverse limits can be surprisingly intricate.

Lastly, let us consider whether a construction of A. M. S. Doering and Y. Lequain in [16, Example D] could be used to prove Corollary 2.4. That example constructed an infinite-dimensional quasi-local domain (say, let us call it (E, N) here) that was a subring of a one-dimensional valuation domain (let us call it $(H, \mathcal{N})$ here) such that the center of H on E is N (that is, such that $\mathcal{N} \cap E = N$). Some readers might still then be tempted to try to finish an analysis of the case $n = \infty$ by using this data set and modifying the reasoning in the proof of Theorem 2.1. However, I doubt that one could use the data from [16, Example D] in this way, as it seems likely from the construction in [16, Example D] of the infinite-dimensional quasi-local domain (which we have called E) that E is not treed (cf. [22, Theorem 144]).

In yet another vein, I would like to raise the question whether gluing methods in the spirit of [16, Example D] could be used to give an (another) alternate proof of at least some of Theorem 2.1.

(c) For more than 40 years, there have been many reasons for researchers to study certain kinds of pullbacks that are more general than the classical $D + M$ construction. I would advise anyone with an interest in the possibility of generalizing Proposition 2.2 and Corollary 2.3 to the context of such more general pullbacks to peruse [17]. As that work gave a complete topological and order-theoretic description of the prime spectrum of a pullback (see [17, Theorem 1.4] and its consequences), I suggest that the first order of such a program should be to find a pullback-theoretic generalization of the result that we cited from [2] that would perhaps be strong enough to ensure that each relevant ring is comparable under inclusion with the analogue of $K + M$. With such a result in hand, the desired generalization of Proposition 2.2 would (probably) easily ensue. I am not aware of any interesting examples that could be constructed via such information but could not be constructed via the classical $D + M$ construction.

(d) We have often had occasion to mention the example of Lewis that was presented in [15, Example 4.4]. For a somewhat more elaborate example, consider the ring constructed in [8, Example 2.3]. That ring is a quasi-local two-dimensional treed domain (hence, a valtreed domain) which is not a going-down domain and which has the property (notably lacking in Lewis' example) that each of its overrings is a treed domain. It would be interesting to find other examples, or a characterization, of domains having these properties exhibited by the domain that was constructed in [8, Example 2.3].

(e) In part (c) of [9, Remark 2.17], we raised the question of whether a treed domain must be a valtreed domain. Theorem 2.1 and Corollary 2.4 have completely answered that question in the negative. Accordingly, it seems appropriate to reiterate the suggestion from [9, Remark 2.17 (c)] to develop some partial converses of [9, Lemma 2.14 (c)]. In particular, it may be useful, for some applications/contexts, to devise a property (♯) such that valtreed domains are precisely the treed domains that satisfy (♯).

As very little time has passed since I wrote [9], very few people have even had an opportunity to read that paper yet. So, apart from what has been presented in this paper, I am unaware of any (other) updates that can be reported in regard to the questions that were raised in [9]. On re-reading the suggestions for future research that were made in parts (d)-(f) of [9, Remark 2.17], I continue to find all of them to be pertinent and timely, and so I would (continue to) direct interested readers to those comments for specific details, references, etc.

I will close with the following two paragraphs. The first of these is brief and asks only one question of the reader. The final paragraph suggests what would likely be a deeper research program.

To persons who have read [9, Remark 2.17 (c)-(f)] **and** all of Remark 2.5 (e) to this point: I would ask you to ponder whether the results in this note and the above comments here in (e) have affected the level of urgency that you attribute to the suggestions from [9, Remark 4.17 (c)-(f)].

As explained in the Introduction, the essential new piece in a characterization of going-down domains was achieved in [9, Corollary 2.16], where it was shown that a quasi-local domain (R,M) of dimension at least 2 is a going-down domain if and only if R is a valtreed domain that satisfies condition (g) of [9, Corollary 2.8]. (That condition stipulated that R' is expressible as the intersection of a certain set of valuation overrings of R.) When one considers the combined effect of [9, Corollary 2.16] and the present work, perhaps the deepest question that arises is the following. Can one replace “valtreed" with “treed" in the above statement of [9, Corollary 2.16]? If one were able to show that the answer to this question is negative by modifying the proof of Theorem 2.1 (which included an appeal to a realization theorem of Hochster [20, Theorem 6 (b)]), it seems to me that one would also have succeeded in extending the explicit results in [20] in a poset-theoretic way that realizes ring extensions that are integral closures of other ring extensions. I would expect that any such result on integrality which would have been obtained via the methods of category theory would be of fundamental interest to commutative ring theorists. On the other hand, if one were able to show that the answer to the above question is positive, it seems to me that one would likely also have developed new interesting results connecting valuation domains and integral closures. Perhaps it is not unrealistic to hope that any resolution of the above question could then be used to settle the questions about the possible ascent of the “going-down domain" property under integral closures that were raised in [5].

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