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Author(s):

Farid Kourki & Rachid Tribak

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On Two Classes of Modules Related to CS Trivial Extensions II

Farid Kourki[1] and Rachid Tribak[2]
Equipe de Recherche: Algèbre & Didactique des Mathématiques,
Centre Régional des Métiers de l'Education et de la Formation
de la Région Tanger-Tétouan-Al Hoceima, (CRMEF-TTH)-Tanger,
Avenue My Abdelaziz, Souani, BP: 3117, Tangier, Morocco
[1] e-mail: *kourkifarid@hotmail.com*
[2] e-mail:*tribak12@yahoo.com*

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Abstract. All rings considered are commutative. Recently, we introduced the notions of weakly IN modules and strongly CS modules in [9]. In this article, we continue the study of these two properties, providing new characterizations and results on the subject. In addition, we introduce and investigate modules and rings satisfying a stronger property than that of being weakly IN.

Key Words: CS-ring, **s**-weakly IN property, strongly CS module (ring), trivial extension, weakly IN module (ring).
2010 MSC: Primary 16D10, 16D60, 16D70; Secondary 16D80.

1 Introduction

Throughout this article, all rings considered are assumed to be commutative rings with an identity and R denotes such a ring. All modules are unital. For any ring R, we denote by $\mathrm{Spec}(R)$, $\mathrm{Max}(R)$ and $\mathrm{Min}(R)$ the set of all prime ideals of R, the set of all maximal ideals of R and the set of all minimal prime ideals of R, respectively. We write $\mathrm{J}(R)$ for the Jacobson radical of R and the nilradical of R (i.e. the set of nilpotent elements of R) is denoted by $\mathrm{Nil}(R)$. Let M be an R-module and let $x \in M$. By $\mathrm{Ann}_R(x)$ and $\mathrm{Ann}_R(M)$ we denote the *annihilator* of x and M, respectively; i.e. $\mathrm{Ann}_R(x) = \{r \in R \mid rx = 0\}$ and $\mathrm{Ann}_R(M) = \{r \in R \mid rM = 0\}$. The notations $N \subseteq M$, $N \leq M$, or $N \subseteq^{ess} M$ mean that N is a subset of M, N is a submodule of M, or N is an essential submodule of M, respectively. By $\mathbb{Z}$ we denote the ring of integer numbers.

In this article, we continue the study of weakly IN modules and strongly CS modules, providing new results on the subject. Recall that a module M is called weakly IN if for any submodules N and L of M with $N \cap L = 0$, $\mathrm{Ann}_R(N) + \mathrm{Ann}_R(L) = R$. On the other hand, a module M is said to be strongly CS if every submodule N of M is essential in a direct summand K having the form $K = eM$ for some $e^2 = e \in R$. It was shown in [9, Theorem 2.8] that every strongly CS module is weakly IN. In addition, we introduce and investigate a stronger form of being weakly IN which will be called the **s**-weakly IN property (SWIN). A module M is said to have the SWIN if for any family $\{N_i \mid i \in I\}$ of submodules of M with $\cap_{i \in I} N_i = 0$, $\sum_{i \in I} \mathrm{Ann}_R(N_i) = R$.

In Section 2, we begin by providing a new characterization of weakly IN modules (Proposition 2.4). Then we characterize weakly IN (and strongly CS) modules having finite uniform dimension (Proposition 2.5 and Corollary 2.6). In Theorem 2.8, we prove that a module M has the SWIN if and only if M is weakly IN and finitely embedded.

In Section 3, we characterize when a direct sum of weakly IN (strongly CS) modules is again weakly IN (strongly CS) (see Theorem 3.2 and Proposition 3.3). Also, we prove an analogous result for modules having the SWIN (Proposition 3.6).

The aim of Section 4 is to investigate when a semisimple module is weakly IN (or strongly CS or has the SWIN). Among other results, we prove in Proposition 4.6 that for the R-module $M = \oplus_{i\in I} R/\mathfrak{m}_i$ where $\mathrm{Max}(R) = \{\mathfrak{m}_i \mid i \in I\}$, M is weakly IN if and only if R is semilocal and M is strongly CS if and only if R is semiperfect.

According to [18], an R-module M is called strongly Kasch if every simple R-module can be embedded in M. In Section 5, we study weakly IN and strongly CS modules which are strongly Kasch. For example, we prove in Corollary 5.9 that a ring R is semilocal (resp. semiperfect) if and only if R has a weakly IN (resp. strongly CS) strongly Kasch R-module.

We conclude the paper by characterizing when a trivial extension has the SWIN (Theorem 2.12).

2 Modules having finite uniform dimension

The notions of weakly IN modules and strongly CS modules were introduced by the authors in [9] and they are defined as follows:

Definition 2.1. Let M be and R-module.

(i) M is called *weakly IN* (for Ikeda-Nakayama) if for any submodules N and L of M with $N \cap L = 0$, there exists $r \in R$ such that $r \in \mathrm{Ann}_R(N)$ and $1 - r \in \mathrm{Ann}_R(L)$, that is, $\mathrm{Ann}_R(N) + \mathrm{Ann}_R(L) = R$.

(ii) M is called *strongly CS* if for any submodule N of M, there exists $e^2 = e \in R$ such that $N \subseteq^{ess} eM$.

Recall that a ring R is called CS if every ideal of R is essential in a direct summand of R. It is clear that R is CS if and only if R, viewed as an R-module, is strongly CS. Moreover, it is shown in [7, Theorem 6] that a ring R is CS if and only if R is a weakly IN R-module. In the following two results we provide some characterizations of weakly IN modules and strongly CS modules. They will be useful later. The first one is taken from [9, Theorem 2.8].

Theorem 2.2. Let R be a ring and let M be a nonzero R-module. Then the following are equivalent:

(i) M is a strongly CS R-module;

(ii) For any submodules N and L of M with $N \cap L = 0$, there exists an idempotent $e \in R$ such that $e \in \mathrm{Ann}_R(N)$ and $1 - e \in \mathrm{Ann}_R(L)$;

(iii) M is a weakly IN R-module and idempotents lift modulo $\mathrm{Ann}_R(M)$.

Theorem 2.3. Let R be a ring and let M be a nonzero R-module. Then the following are equivalent:

(i) M is a weakly IN R-module;

(ii) M is a CS R-module and for any direct summand N of M, there exists $r \in R$ such that $N = rM$ and $r - r^2 \in \mathrm{Ann}_R(M)$;

(iii) For any submodule N of M, there exists $r \in R$ such that $N \subseteq^{ess} rM$ and $r - r^2 \in \mathrm{Ann}_R(M)$;

(iv) M is strongly CS as $R/\mathrm{Ann}_R(M)$-module;

(v) For any submodules N and L of M with $N \cap L = 0$, there exists $r \in R$ such that $r \in \mathrm{Ann}_R(N)$, $1 - r \in \mathrm{Ann}_R(L)$ and $r - r^2 \in \mathrm{Ann}_R(M)$.

Proof. (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) follow from [9, Proposition 2.6].

(iv) ⇒ (v) Use Theorem 2.2((i) ⇒ (ii)).

(v) ⇒ (i) This follows from the definition of a weakly IN module. □

The following result is a generalization of Theorem 2.3((i) ⇔ (v)).

Proposition 2.4. *Let R be a ring and let M be a nonzero R-module. Then the following are equivalent:*

(i) *M is a weakly IN R-module;*

(ii) *For any submodules $N_1, \ldots, N_n$ $(n \geq 2)$ of M with $N_1 \cap \cdots \cap N_n = 0$, we have $\mathrm{Ann}_R(N_1) + \cdots + \mathrm{Ann}_R(N_n) = R$;*

(iii) *For any submodules $N_1, \ldots, N_n$ $(n \geq 2)$ of M with $N_1 \cap \cdots \cap N_n = 0$, there exist $f_1, \ldots, f_n$ in R such that $f_i \in \mathrm{Ann}_R(N_i)$ for all $i \in \{1, 2, \ldots, n\}$ and $\{\overline{f_1}, \ldots, \overline{f_n}\}$ is a complete set of orthogonal idempotents of $R/\mathrm{Ann}_R(M)$, where $\overline{f_i} = f_i + \mathrm{Ann}_R(M)$ for every $1 \leq i \leq n$.*

Proof. (i) ⇒ (iii) Let $N_1, \ldots, N_n$ $(n \geq 2)$ be submodules of M satisfying the condition $N_1 \cap \cdots \cap N_n = 0$. From Theorem 2.3, it follows that for each $k \in \{1, \ldots, n\}$, there exists $e_k \in R$ such that $e_k^2 - e_k \in \mathrm{Ann}_R(M)$ and $N_k \subseteq^{ess} e_k M$. This implies that $N_1 \cap \cdots \cap N_n \subseteq^{ess} e_1 M \cap \cdots \cap e_n M$. But $N_1 \cap \cdots \cap N_n = 0$, so $e_1 M \cap \cdots \cap e_n M = 0$. Therefore $e_n e_{n-1} \cdots e_1 M = 0$ and hence $f = e_n e_{n-1} \cdots e_1 \in \mathrm{Ann}_R(M)$. The following equality is easily checked:

$$(1 - e_1) + (1 - e_2)e_1 + \cdots + (1 - e_n)e_{n-1} \cdots e_1 + e_n e_{n-1} \cdots e_1 = 1.$$

Let $f_1 = 1 - e_1$ and $f_i = (1 - e_i)e_{i-1} \cdots e_1$ for each $i \in \{2, \ldots, n\}$. It is easily seen that $f_i^2 - f_i \in \mathrm{Ann}_R(M)$ and $f_i f_j \in \mathrm{Ann}_R(M)$ for all $i \neq j$ in $\{1, \ldots, n\}$. Moreover, we have $\overline{f}_1 + \overline{f}_2 + \cdots + \overline{f}_n + \overline{f} = \overline{1}$ (in $R/\mathrm{Ann}_R(M)$). Consequently, $\overline{f}_1 + \overline{f}_2 + \cdots + \overline{f}_n = \overline{1}$. Hence, $\{\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_n\}$ is a complete set of orthogonal idempotents of $R/\mathrm{Ann}_R(M)$. In addition, for each $1 \leq i \leq n$, we have $f_i \in \mathrm{Ann}_R(N_i)$ since $f_i \in \mathrm{Ann}_R(e_i M)$ and $N_i \subseteq e_i M$.

(iii) ⇒ (ii) Let $N_1, \ldots, N_n$ $(n \geq 2)$ be submodules of M with $N_1 \cap \cdots \cap N_n = 0$ and let $f_1, \ldots, f_n \in R$ be as in (iii). Since $\{\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_n\}$ is a complete set of orthogonal idempotents of $R/\mathrm{Ann}_R(M)$, $f_1 + f_2 + \cdots + f_n - 1 \in \mathrm{Ann}_R(M)$. Hence, there exists $a \in \mathrm{Ann}_R(M)$ such that $(f_1 + a) + f_2 + \cdots + f_n = 1$. But $f_i \in \mathrm{Ann}_R(N_i)$ for all $i \in \{1, 2, \ldots, n\}$, so $f_1 + a \in \mathrm{Ann}_R(N_1)$ and $\mathrm{Ann}_R(N_1) + \cdots + \mathrm{Ann}_R(N_n) = R$.

(ii) ⇒ (i) This is clear. □

Let n be a positive integer. Recall that an R-module M is said to have *uniform* (or *Goldie*) dimension n (written $u.dimM = n$) if there is an essential submodule $N \subseteq^{ess} M$ that is a direct sum of n uniform submodules, equivalently, n is the supremum of the set of integers k such that M contains a direct sum of k nonzero submodules. If, on the other hand, no such integer n exists, we write $u.dim(M) = \infty$. If $M = 0$ we set $u.dim(M) = 0$. It is clear that $dim(M) = 1$ if and only if M is uniform. In the next two results we characterize when a module having finite uniform dimension is strongly CS or weakly IN.

Proposition 2.5. *Let M be a nonzero R-module with $u.dim(M) = n$ for some positive integer n. Then the following are equivalent:*

(i) *M is a strongly CS;*

(ii) *$M = e_1 M \oplus \cdots \oplus e_n M$ where $\{e_1, \ldots, e_n\}$ a complete set of orthogonal idempotents of R such that $e_i M$ is a uniform submodule of M for all $i = 1, \ldots, n$.*

Proof. (i) ⇒ (ii) We will show this implication by induction on n. If $n = 1$, there is nothing to prove. Assuming the implication to hold for the dimension n, we will prove it for $n+1$. Let M be a strongly CS R-module with $u.dim(M) = n+1$. Then M contains a nonzero uniform submodule N. In view of the fact that M is strongly CS, $N \subseteq^{ess} e_{n+1}M$ for some $e_{n+1}^2 = e_{n+1} \in R$. Note that $M = (1-e_{n+1})M \oplus e_{n+1}M$ and $u.dim(e_{n+1}M) = u.dim(N) = 1$. So $dim((1-e_{n+1})M) = n$ since $u.dim(M) = n+1$ (see [10, Corollary 6.10]). Note that the strongly CS property is inherited by submodules by [9, Proposition 2.4]. Then $(1-e_{n+1})M$ is a strongly CS R-module. Now the induction hypothesis applied to $(1-e_{n+1})M$ yields

$$(1-e_{n+1})M = e_1(1-e_{n+1})M \oplus e_2(1-e_{n+1})M \oplus \cdots \oplus e_n(1-e_{n+1})M,$$

where $\{e_1, \dots, e_n\}$ is a complete set of orthogonal idempotents of R such that each $e_i(1-e_{n+1})M$ $(1 \le i \le n)$ is a uniform submodule of $(1-e_{n+1})M$. Therefore

$$M = e_1(1-e_{n+1})M \oplus e_2(1-e_{n+1})M \oplus \cdots \oplus e_n(1-e_{n+1})M \oplus e_{n+1}M.$$

Moreover, it is easily seen that $\{e_1(1-e_{n+1}), \dots, e_n(1-e_{n+1}), e_{n+1}\}$ is a complete set of orthogonal idempotents of R.

(ii) ⇒ (i) This follows from [9, Theorem 3.3] and the fact that every uniform module is strongly CS. □

Corollary 2.6. *Let M be a nonzero R-module with $u.dim(M) = n$ for some positive integer n. Then the following are equivalent:*

(i) *M is a weakly IN R-module;*

(ii) *$M = r_1M \oplus r_2M \oplus \cdots \oplus r_nM$ for some elements $r_1, \dots, r_n$ of R such that r_iM is a uniform submodule of M for all $(1 \le i \le n)$ and $\{\overline{r}_1, \dots, \overline{r}_n\}$ is a complete set of orthogonal idempotents of $R/\mathrm{Ann}_R(M)$, where $\overline{r}_i = r_i + \mathrm{Ann}_R(M)$ for every $(1 \le i \le n)$.*

Proof. Since $u.dim(M) = n$, it is clear that M considered as an $R/\mathrm{Ann}_R(M)$-module also has uniform dimension n. Now the result follows from Theorem 2.3 and Proposition 2.5. □

We will say that a module M has the **s**-*weakly IN property* (*SWIN*, for short) if for any family $\{N_i \mid i \in I\}$ of submodules of M with $\cap_{i\in I} N_i = 0$, $\sum_{i\in I} \mathrm{Ann}_R(N_i) = R$. Clearly, every module having the SWIN is weakly IN. Also, it is easily seen that every submodule of a module having the SWIN inherits the property.

A module M is said to be *finitely embedded* (or *finitely cogenerated*) if $E(M) \cong E(S_1) \oplus \cdots \oplus E(S_n)$ for suitable simple modules $S_1, \dots, S_n$. A ring R is called *finitely embedded* if the R-module R is finitely embedded.

Remark 2.7. (i) From [10, 19.3A] and [17, 21.4], it follows that being finitely embedded modules is preserved by taking submodules and finite direct sums.

(ii) The following assertions are easy to check.

(a) Every finitely embedded module has finite uniform dimension.

(b) A uniform module U is finitely embedded if and only if U has a simple essential socle.

Theorem 2.8. Let M be a nonzero R-module. Then the following are equivalent:

(i) M has the SWIN;

(ii) M is weakly IN and finitely embedded;

(iii) $M = r_1M \oplus \cdots \oplus r_nM$ where $r_1, \dots, r_n$ are elements of R such that each r_iM has a simple essential socle and $\{\overline{r}_1, \dots, \overline{r}_n\}$ is a complete set of orthogonal idempotents of $R/\mathrm{Ann}_R(M)$, where $\overline{r}_i = r_i + \mathrm{Ann}_R(M)$ for every $(1 \le i \le n)$.

Proof. (i) $\Rightarrow$ (ii) It is clear that M is weakly IN. To prove that M is finitely embedded, let $\{N_i \mid i \in I\}$ be a family of submodules of M with $\cap_{i\in I} N_i = 0$. So $\sum_{i\in I} \mathrm{Ann}_R(N_i) = R$ and hence $\sum_{j\in J} \mathrm{Ann}_R(N_j) = R$ for some finite subset J of I. Therefore $\cap_{j\in J} N_j = 0$. Now using [10, Proposition 19.1], we infer that M is finitely embedded.

(ii) $\Rightarrow$ (iii) Since M is finitely embedded, $u.dim(M) < \infty$. Then, by Corollary 2.6, $M = r_1M \oplus r_2M \oplus \cdots \oplus r_nM$ for some elements $r_1, \ldots, r_n$ of R such that each r_iM is a uniform submodule of M and $\{\overline{r}_1, \ldots, \overline{r}_n\}$ is a complete set of orthogonal idempotents of $R/\mathrm{Ann}_R(M)$, where $\overline{r}_i = r_i + \mathrm{Ann}_R(M)$ for every $(1 \le i \le n)$. For each i, since r_iM is uniform and finitely embedded, r_iM has a simple essential socle (see Remark 2.7(ii)).

(iii) $\Rightarrow$ (i) Since each r_iM $(1 \le i \le n)$ is finitely embedded, it follows from Remark 2.7(i) that M is finitely embedded. To show that M has the SWIN, take a family $\{N_i \mid i \in I\}$ of submodules of M with $\cap_{i\in I} N_i = 0$. By [10, Proposition 19.1], there exists a finite subset F of I such that $\cap_{i\in F} N_i = 0$. But M is weakly IN by Corollary 2.6, so $\sum_{i\in F} \mathrm{Ann}_R(N_i) = R$ by Proposition 2.4. Hence $\sum_{i\in I} \mathrm{Ann}_R(N_i) = R$. This completes the proof. □

Corollary 2.9. *Let M be a faithful R-module. Then the following are equivalent:*

(i) *M has the SWIN;*

(ii) *M is weakly IN and finitely embedded;*

(iii) *M is strongly CS and finitely embedded;*

(iv) *$M = e_1M \oplus \cdots \oplus e_nM$ where $\{e_1, \ldots, e_n\}$ a complete set of orthogonal idempotents of R such that each e_iM $(1 \le i \le n)$ has a simple essential socle.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) follow from Theorem 2.8.

(ii) $\Leftrightarrow$ (iii) comes from the fact that a faithful module is weakly IN if and only if it is strongly CS (see Theorem 2.3). □

Recall that a ring R is called *subdirectly irreducible* if the intersection of all its nonzero ideals is a nonzero ideal, equivalently, R has a simple essential socle (see [11]). If the R-module R has the SWIN (is finitely embedded), we will say that the ring R has the SWIN (is finitely embedded). Using Corollary 2.9, we obtain the next corollary which characterizes rings having the SWIN.

Corollary 2.10. *Let R be a ring. Then the following are equivalent:*

(i) *The ring R has the SWIN;*

(ii) *R is CS and finitely embedded;*

(iii) *$R = R_1 \times R_2 \times \cdots \times R_n$ is a finite direct product of subdirectly irreducible rings R_i $(1 \le i \le n)$.*

Recall that a ring R is called a *valuation* ring if any two ideals of R are comparable. It is clear that every valuation ring is local. The following example shows that none of the two conditions in the statement (ii) of Corollary 2.10 implies the other.

Example 2.11. (i) Let R be a local ring with maximal ideal $\mathfrak{m}$ and let $N = E(R/\mathfrak{m})$. Consider the R-module $M = N \oplus N$. Then M is finitely embedded but not weakly IN since M is not square-free (see Lemma 4.3). Moreover, note that M is a faithful R-module since $\mathrm{Ann}_R(N) = 0$ by [14, Proposition 2.26 Corollary 2]. Let $A = R \propto M$, the trivial extension of R by M. Note that A is a finitely embedded ring by [8, Theorem 2.10]. However, A is not a CS ring by [9, Theorem 5.4].

(ii) Let R be a valuation domain which is not a field (e.g., we can take $R = \mathbb{Z}_{(2)}$, the localization of $\mathbb{Z}$ at the maximal ideal $2\mathbb{Z}$). Then R is not artinian. By [10, Proposition 19.4], there exists an ideal

I of R such that R/I is not a finitely embedded R-module. Consider the ring $A = R/I$. Clearly A is a valuation ring. Then A is a uniform A-module and hence A is a CS ring. On the other hand, it is clear that A is not a finitely embedded ring.

Let R be a ring and let M be an R-module. The abelian group $R \oplus M$ can be endowed with the following product: $(a,x)(b,y) = (ab, ay + bx)$. The result is a ring called the trivial extension of R by M denoted by $R \propto M$. Moreover, R becomes a subring of $R \propto M$ and M an ideal such that $M^2 = 0$.

After characterizing CS trivial extensions in [9], we next characterize when a trivial extension has the SWIN.

Theorem 2.12. Let M be an R-module. Then the following assertions are equivalent:

(i) The ring $A = R \propto M$ has the SWIN;

(ii) The R-module $M \oplus \mathrm{Ann}_R(M)$ has the SWIN;

(iii) R satisfies the following two conditions:

  (a) $\mathrm{Ann}_R(M)$ is a direct summand of R, and

  (b) both the R-modules M and $\mathrm{Ann}_R(M)$ have the SWIN.

Proof. The proof follows from Theorem 2.8, Corollary 2.10, [8, Theorem 2.10] and [9, Theorem 5.4]. In fact, the ring A has the SWIN if and only if A is CS and finitely embedded (see Corollary 2.10). Moreover, A is CS if and only if $M \oplus \mathrm{Ann}_R(M)$ is weakly IN if and only if M and $\mathrm{Ann}_R(M)$ are weakly IN R-modules and $\mathrm{Ann}_R(M)$ is a direct summand of R by [9, Theorem 5.4]. In addition, note that A is finitely embedded if and only if $M \oplus \mathrm{Ann}_R(M)$ is finitely embedded if and only if M and $\mathrm{Ann}_R(M)$ are finitely embedded by [8, Theorem 2.10]. Now apply Theorem 2.8. □

The proof of the next corollary follows immediately from Theorem 2.12.

Corollary 2.13. *Let M be a faithful R-module. Then the following conditions are equivalent:*

(i) *The ring $A = R \propto M$ has the SWIN;*

(ii) *M has the SWIN.*

3 Direct sums of modules

In [9, Section 3], we characterized when a finite direct sum of weakly IN (strongly CS) modules is again weakly IN (strongly CS). In this section, we continue our investigations by focusing on the question of when these two concepts are preserved under an arbitrary direct sum (finite or infinite).

In Theorem 3.2 we shall characterize when a direct sum of modules is weakly IN. First we prove the following lemma.

Lemma 3.1. *Let a module $M = \oplus_{i \in I} M_i$ be a direct sum of submodules M_i $(i \in I)$. Assume that the following condition holds:*

(∗) $\cap_{i \in I} \mathrm{Ann}_R(N_i) + \cap_{i \in I} \mathrm{Ann}_R(L_i) = R$ *for all submodules N_i and L_i of M_i with $N_i \cap L_i = 0$ for all $i \in I$.*
Then M satisfies the following two conditions:

(i) $\mathrm{Ann}_R(M_j) + \mathrm{Ann}_R(M_k) = R$ *for all distinct j, k in I.*

(ii) $N = \oplus_{i \in I}(N \cap M_i)$ *for every submodule N of M.*

Proof. (i) Let $j \neq k$ in I and let $i \in I$. We put $N_j = M_j$ and $N_i = 0$ if $i \neq j$. Also, we put $L_k = M_k$ and $L_i = 0$ if $i \neq k$. It is clear that $N_i \cap L_i = 0$ for all $i \in I$. By (∗), we have $\cap_{i\in I} \mathrm{Ann}_R(N_i) + \cap_{i\in I} \mathrm{Ann}_R(L_i) = R$. Hence $\mathrm{Ann}_R(M_j) + \mathrm{Ann}_R(M_k) = R$. This proves (i).

(ii) follows from (i) and [4, Lemma 2.6]. □

Theorem 3.2. Let a module $M = \oplus_{i\in I} M_i$ be a direct sum of submodules M_i $(i \in I)$. Then the following statements are equivalent:

(i) M is a weakly IN module;

(ii) $\cap_{i\in I} \mathrm{Ann}_R(N_i) + \cap_{i\in I} \mathrm{Ann}_R(L_i) = R$ for all submodules N_i and L_i of M_i with $N_i \cap L_i = 0$ for all $i \in I$.

Proof. (i) ⇒ (ii) For each $i \in I$, let N_i and L_i be submodules of M_i such that $N_i \cap L_i = 0$. Set $N = \oplus_{i\in I} N_i$ and $L = \oplus_{i\in I} L_i$. Then N and L are submodules of M such that $N \cap L = \oplus_{i\in I}(N_i \cap L_i) = 0$. From the fact that M is weakly IN, we deduce that $\mathrm{Ann}_R(N) + \mathrm{Ann}_R(L) = R$. Consequently, $\cap_{i\in I} \mathrm{Ann}_R(N_i) + \cap_{i\in I} \mathrm{Ann}_R(L_i) = R$.

(ii) ⇒ (i) Let N and L be two submodules of M such that $N \cap L = 0$. By Lemma 3.1, $N = \oplus_{i\in I}(N \cap M_i)$ and $L = \oplus_{i\in I}(L \cap M_i)$. Since $N \cap L = 0$, it follows that $(N \cap M_i) \cap (L \cap M_i) = 0$ for all $i \in I$. By (ii), $\cap_{i\in I} \mathrm{Ann}_R(N \cap M_i) + \cap_{i\in I} \mathrm{Ann}_R(L \cap M_i) = R$. Therefore $\mathrm{Ann}_R(N) + \mathrm{Ann}_R(L) = R$ and hence M is weakly IN. □

Proposition 3.3. *Let a module $M = \oplus_{i\in I} M_i$ be a direct sum of submodules M_i $(i \in I)$. Then the following statements are equivalent:*

(i) *M is strongly CS;*

(ii) *M satisfies the following two conditions:*

(a) $\mathrm{Ann}_R(M_i) + \mathrm{Ann}_R(M_j) = R$ *for all distinct $i, j \in I$, and*

(b) *for any family $\{N_i\}_{i\in I}$ where $N_i \leq M_i$ $(i \in I)$, there exists $e^2 = e \in R$ such that $N_i \subseteq^{ess} eM_i$ for all $i \in I$.*

Proof. (i) ⇒ (ii) (a) By Theorem 2.2, M is weakly IN. Now (a) follows from the definition of a weakly IN module.

(b) Let $N_i \leq M_i$ $(i \in I)$ and set $N = \oplus_{i\in I} N_i$. Since M is strongly CS, there exists $e^2 = e \in R$ such that $\oplus_{i\in I} N_i \subseteq^{ess} eM = \oplus_{i\in I} eM_i$. This implies that $N_i \subseteq^{ess} eM_i$ for all $i \in I$.

(ii) ⇒ (i) Let N be a submodule of M. Using (a) and [4, Lemma 2.6], we get $N = \oplus_{i\in I}(N \cap M_i)$. By (b), there exists $e = e^2 \in R$ such that $N \cap M_i \subseteq^{ess} eM_i$ for all $i \in I$. Therefore $\oplus_{i\in I}(N \cap M_i) \subseteq^{ess} \oplus_{i\in I} eM_i = eM$ (see [17, 17.4]). Thus $N \subseteq^{ess} eM$ and hence M is strongly CS. □

Let I_1 and I_2 be two nonempty subsets of a set I. Then $\{I_1, I_2\}$ is said to be a partition of I if $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = I$.

Corollary 3.4. *Let a module $M = \oplus_{i\in I} M_i$ be a direct sum of submodules M_i $(i \in I)$. Consider the following two conditions:*

(i) *M is a weakly IN R-module.*

(ii) *M_i is a weakly IN R-module for all $i \in I$ and for any partition $\{I_1, I_2\}$ of I we have $\cap_{i\in I_1} \mathrm{Ann}_R(M_i) + \cap_{i\in I_2} \mathrm{Ann}_R(M_i) = R$.*

Then (i) ⇒ (ii). If, moreover, each M_i $(i \in I)$ is a uniform R-module, then (ii) ⇒ (i).

Proof. (i) $\Rightarrow$ (ii) Suppose that M is a weakly IN R-module. Then clearly every submodule of M is weakly IN. In particular, each M_i ($i \in I$) is a weakly IN R-module. Now take a partition $\{I_1, I_2\}$ of I and let $(N_i)_{i\in I}$ and $(L_i)_{i\in I}$ be such that N_i and L_i are submodules of M_i for all $i \in I$ with $N_i = M_i$ if $i \in I_1$ and $N_i = 0$ if $i \in I_2$; $L_i = 0$ if $i \in I_1$ and $L_i = M_i$ if $i \in I_2$. Then $N_i \cap L_i = 0$ for all $i \in I$ and so $\cap_{i\in I} \mathrm{Ann}_R(N_i) + \cap_{i\in I} \mathrm{Ann}_R(L_i) = R$ since M is weakly IN (Theorem 3.2). Consequently, $\cap_{i\in I_1} \mathrm{Ann}_R(M_i) + \cap_{i\in I_2} \mathrm{Ann}_R(M_i) = R$.

(ii) $\Rightarrow$ (i) Suppose that (ii) is satisfied and that each M_i ($i \in I$) is uniform. Let us prove that M is weakly IN by using Theorem 3.2. For each $i \in I$, let N_i and L_i be submodules of M_i such that $N_i \cap L_i = 0$. Since each M_i is uniform, it follows that either $N_i = 0$ or $L_i = 0$ for all $i \in I$. Let $I_1 = \{i \in I \mid L_i \neq 0\}$ and $I_2 = \{i \in I \mid L_i = 0\}$. Clearly $I_1 \cup I_2 = I$. If $I_1 = \emptyset$ or $I_2 = \emptyset$, then it is clear that $\cap_{i\in I} \mathrm{Ann}_R(N_i) + \cap_{i\in I} \mathrm{Ann}_R(L_i) = R$. So without loss of generality we can assume that both I_1 and I_2 are nonempty. Then $\{I_1, I_2\}$ is a partition of I. So, by hypothesis, $\cap_{i\in I_1} \mathrm{Ann}_R(M_i) + \cap_{i\in I_2} \mathrm{Ann}_R(M_i) = R$. Note that $\cap_{i\in I} \mathrm{Ann}_R(L_i) = \cap_{i\in I_1} \mathrm{Ann}_R(L_i)$ and $\cap_{i\in I} \mathrm{Ann}_R(N_i) = \cap_{i\in I_2} \mathrm{Ann}_R(N_i)$ since $N_i = 0$ for all $i \in I_1$. Since $\cap_{i\in I_1} \mathrm{Ann}_R(M_i) \subseteq \cap_{i\in I_1} \mathrm{Ann}_R(L_i)$ and $\cap_{i\in I_2} \mathrm{Ann}_R(M_i) \subseteq \cap_{i\in I_2} \mathrm{Ann}_R(N_i)$, we have $\cap_{i\in I} \mathrm{Ann}_R(N_i) + \cap_{i\in I} \mathrm{Ann}_R(L_i) = R$. This completes the proof. □

Corollary 3.5. *Let a module $M = \oplus_{i\in I} M_i$ be a direct sum of submodules M_i ($i \in I$). Consider the following two conditions:*

(i) *M is a strongly CS R-module.*

(ii) *M_i is a strongly CS R-module for all $i \in I$ and for every partition $\{I_1, I_2\}$ of I, there exists an idempotent e of R such that $e \in \cap_{i\in I_1} \mathrm{Ann}_R(M_i)$ and $1 - e \in \cap_{i\in I_2} \mathrm{Ann}_R(M_i)$.*

Then (i) $\Rightarrow$ (ii). If, moreover, each M_i ($i \in I$) is a uniform R-module, then (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii) Suppose that M is strongly CS. Then each M_i is a strongly CS R-module by [9, Proposition 2.4]. Now let $\{I_1, I_2\}$ be a partition of I and put $N = \oplus_{i\in I_1} M_i$ and $L = \oplus_{i\in I_2} M_i$. Then $N \cap L = 0$ and so the exists $e^2 = e \in R$ such that $e \in \mathrm{Ann}_R(N) = \cap_{i\in I_1} \mathrm{Ann}_R(M_i)$ and $1 - e \in \mathrm{Ann}_R(L) = \cap_{i\in I_2} \mathrm{Ann}_R(M_i)$ (see Theorem 2.2).

(ii) $\Rightarrow$ (i) Suppose that the indexed set $(M_i)_{i\in I}$ satisfies condition (ii) and that each M_i is uniform. For each $i \in I$, let N_i be a submodule of M_i. If $N_i \neq 0$ for every $i \in I$, then $N_i \subseteq^{ess} 1M_i$ for all $i \in I$. Also, if $N_i = 0$ for all i, then $N_i \subseteq^{ess} 0M_i$ for all $i \in I$. Now assume that $I_1 = \{i \in I \mid N_i = 0\} \neq \emptyset$ and $I_2 = \{i \in I | N_i \neq 0\} \neq \emptyset$. Then it is clear that $\{I_1, I_2\}$ is a partition of I. So, by hypothesis, there exists an idempotent e of R such that $e \in \cap_{i\in I_1} \mathrm{Ann}_R(M_i)$ and $1 - e \in \cap_{i\in I_2} \mathrm{Ann}_R(M_i)$. Therefore $0 = N_i \subseteq^{ess} eM_i = 0$ for all $i \in I_1$ and $N_i \subseteq^{ess} M_i = eM_i$ for all $i \in I_2$ since each M_i is uniform. In addition, note that it follows easily from (ii) that $\mathrm{Ann}_R(M_j) + \mathrm{Ann}_R(M_k) = R$ for all distinct $j \neq k$ in I. Now using Proposition 3.3, we conclude that M is a strongly CS R-module. □

Proposition 3.6. *Let a module $M = \oplus_{i\in I} M_i$ be a direct sum of submodules M_i ($i \in I$). Then the following are equivalent:*

(i) *M has the SWIN;*

(ii) *M satisfies the following three conditions:*

  (a) *I is a finite set,*

  (b) *M_i has the SWIN for all $i \in I$, and*

  (c) *$\mathrm{Ann}_R(M_j) + \mathrm{Ann}_R(M_k) = R$ for all distinct $j, k \in I$.*

Proof. (i) $\Rightarrow$ (ii) By Theorem 2.8, M is finitely embedded and so M has finite uniform dimension. Therefore I is a finite set. It is clear that each M_i has the SWIN since this property is closed under submodules. Moreover, using the definition of a module having the SWIN, it follows easily that condition (c) is also satisfied.

(ii) $\Rightarrow$ (i) For any $i \in I$, it is clear that M_i is weakly IN since M_i has the SWIN. Using (c) and the fact that I is finite, we obtain that M is weakly IN by [9, Theorem 3.1]. Moreover, taking into account Theorem 2.8, it follows from condition (b) that each M_i is finitely embedded. Hence M is finitely embedded by Remark 2.7(i). Applying again Theorem 2.8, we infer that M has the SWIN. □

4 Semisimple modules

In this section, we continue the study of direct sums of weakly IN modules and strongly CS modules. But we will restrict our attention to direct sums of simple modules.

Let $\mathfrak{p}$ be a prime ideal of R. It is not difficult to see that $R/\mathfrak{p}$ is a uniform R-module and $\mathrm{Ann}_R(R/\mathfrak{p}) = \mathfrak{p}$. Moreover, it is clear that the R-module $R/\mathfrak{p} \oplus R/\mathfrak{p}$ is not weakly IN.

Before dealing with semisimple modules, we prove the following three lemmas. The first one follows easily from Corollary 3.4.

Lemma 4.1. *Let $\{\mathfrak{p}_i\}_{i\in I}$ be a family of distinct prime ideals of R and let $M = \oplus_{i\in I} R/\mathfrak{p}_i$. Then the following are equivalent:*

(i) *M is a weakly IN R-module;*

(ii) *For every partition $\{I_1, I_2\}$ of I, $\cap_{i\in I_1}\mathfrak{p}_i + \cap_{i\in I_2}\mathfrak{p}_i = R$.*

Next, we provide an analogue of the previous lemma for strongly CS modules.

Lemma 4.2. *Let $\{\mathfrak{p}_i\}_I$ be a family of distinct prime ideals of R and let $M = \oplus_{i\in I} R/\mathfrak{p}_i$. Then the following are equivalent:*

(i) *M is a strongly CS R-module;*

(ii) *For every partition $\{I_1, I_2\}$ of I, there exists an idempotent e of R such that $e \in \cap_{i\in I_1}\mathfrak{p}_i$ and $1 - e \in \cap_{i\in I_2}\mathfrak{p}_i$.*

(iii) *Idempotents lift modulo $\cap_{i\in I}\mathfrak{p}_i$ and for every partition $\{I_1, I_2\}$ of I, $\cap_{i\in I_1}\mathfrak{p}_i + \cap_{i\in I_2}\mathfrak{p}_i = R$.*

Proof. (i) $\Leftrightarrow$ (ii) Use Corollary 3.5.

(i) $\Leftrightarrow$ (iii) This follows from Theorem 2.2 and Lemma 4.1. □

An R-module M is called *square-free* if it contains no nonzero submodule isomorphic to $N \oplus N$ for some submodule N of M, equivalently, if K and L are two submodules of M such that $K \cap L = 0$ and $K \cong L$, then $K = L = 0$.

Lemma 4.3. *If M is weakly IN R-module, then M is square-free.*

Proof. Suppose that M is weakly IN and let N and L be two submodules of M such that $N \cap L = 0$ and $N \cong L$. Then $\mathrm{Ann}_R(N) + \mathrm{Ann}_R(L) = R$. But $N \cong L$, so $\mathrm{Ann}_R(N) = \mathrm{Ann}_R(L)$ and consequently $\mathrm{Ann}_R(N) = \mathrm{Ann}_R(L) = R$. This implies that $N = L = 0$. This proves the lemma. □

Proposition 4.4. *Let $M = \oplus_{i\in I} S_i^{(A_i)}$ be a semisimple R-module, where S_i ($i \in I$) are pairwise nonisomorphic simple R-modules and A_i ($i \in I$) are nonempty sets. Then the following are equivalent:*

(i) *M is a weakly IN R-module;*

(ii) *M is square-free (that is, each A_i ($i \in I$) is a singleton set) and for every partition $\{I_1, I_2\}$ of I, $\cap_{i\in I_1} \mathrm{Ann}_R(S_i) + \cap_{i\in I_2} \mathrm{Ann}_R(S_i) = R$.*

Proof. For each $i \in I$, there exists $\mathfrak{m}_i \in \mathrm{Max}(R)$ such that $S_i \cong R/\mathfrak{m}_i$.

(i) $\Rightarrow$ (ii) By Lemma 4.3, M is square-free. Now the second part of (i) follows from Lemma 4.1.

(ii) $\Rightarrow$ (i) Use Lemma 4.1. □

Proposition 4.5. *Let $M = \oplus_{i\in I} S_i^{(A_i)}$ be a semisimple R-module, where S_i ($i \in I$) are pairwise nonisomorphic simple R-modules and A_i ($i \in I$) are nonempty sets. Then the following are equivalent:*

(i) *M is a strongly CS R-module;*

(ii) *M is square-free and for every partition $\{I_1, I_2\}$ of I, there exists an idempotent e of R such that $e \in \cap_{i\in I_1} \mathrm{Ann}_R(S_i)$ and $1 - e \in \cap_{i\in I_2} \mathrm{Ann}_R(S_i)$.*

Proof. For each $i \in I$, let $\mathfrak{m}_i \in \mathrm{Max}(R)$ such that $S_i \cong R/\mathfrak{m}_i$.

(i) $\Rightarrow$ (ii) Note that M is weakly IN (Theorem 2.2) and hence M is square-free by Lemma 4.3. The second part of (ii) follows from Lemma 4.2.

(ii) $\Rightarrow$ (i) This follows from Lemma 4.2. □

Note that the implication (d) $\Rightarrow$ (e) in the next result was also proved by Smith in a general setting (see [15, Theorem 6.6]).

Proposition 4.6. *Let R be a ring and let $\mathrm{Max}(R) = \{\mathfrak{m}_i \mid i \in I\}$. Consider the R-module $M = \oplus_{i\in I} R/\mathfrak{m}_i$. Then the following hold true:*

(i) *The following are equivalent:*

  (a) *Every square-free semisimple R-module is weakly IN;*

  (b) *M is a weakly IN R-module;*

  (c) *M has the SWIN;*

  (d) *$\mathfrak{m}_j + \cap_{i\in I\setminus\{j\}} \mathfrak{m}_i = R$ for all $j \in I$.*

  (e) *R is semilocal (i.e., I is finite).*

(ii) *The following are equivalent:*

  (a) *Every square-free semisimple R-module is strongly CS;*

  (b) *M is a strongly CS R-module;*

  (c) *R is semiperfect.*

Proof. (i) (a) $\Rightarrow$ (b) This is clear.

(b) $\Rightarrow$ (d) Let $j \in I$. It is clear that $\{\{j\}, I \setminus \{j\}\}$ is a partition of I. Now apply Lemma 4.1 to get (d).

(d) $\Rightarrow$ (e) Let $\overline{R} = R/\mathrm{J}(R)$ and for any ideal $\mathfrak{a}$ of R which contains $\mathrm{J}(R)$, let $\overline{\mathfrak{a}} = \mathfrak{a}/\mathrm{J}(R)$. Let $j \in I$. We have $\overline{\mathfrak{m}}_j + \cap_{i\neq j} \overline{\mathfrak{m}}_i = \overline{R}$. But $\overline{\mathfrak{m}}_j \cap (\cap_{i\neq j} \overline{\mathfrak{m}}_i) = \cap_{i\in I} \overline{\mathfrak{m}}_i = \overline{\mathrm{J}(R)} = 0$, so $\overline{\mathfrak{m}}_j \oplus \cap_{i\neq j} \overline{\mathfrak{m}}_i = \overline{R}$. Thus, every maximal ideal of $\overline{R}$ is a direct summand. By [13, Theorem 3.2], $\overline{R}$ is a semisimple ring and consequently R is semilocal.

(e) $\Rightarrow$ (c) This follows from Proposition 3.6.

(c) $\Rightarrow$ (a) It is clear that M is weakly IN. Hence (a) follows from the fact that every square-free semisimple R-module is isomorphic to a direct summand of M.

(ii) (a) $\Rightarrow$ (b) This is clear since M is a square-free semisimple R-module.

(b) $\Rightarrow$ (c) Since M is strongly CS, M is weakly IN and idempotents lift modulo $\mathrm{Ann}_R(M) = \mathrm{J}(R)$ by Theorem 2.2. Moreover, R is a semilocal ring by (i). Hence R is semiperfect.

(c) ⇒ (a) Suppose that R is a semiperfect ring. Then R is semilocal and idempotents lift modulo $J(R) = \mathrm{Ann}_R(M)$. By (i), M is weakly IN. Therefore M is strongly CS by Theorem 2.2. Now (a) follows from [9, Proposition 2.4] and the fact that every square-free semisimple R-module is isomorphic to a submodule of M. □

By [13, Theorem 3.2], every maximal ideal of a ring R is a direct summand if and only if R is a finite direct product of fields (that is, R is semisimple). In the following lemma we give a dual result. Recall that a ring R is called *PIF* (*PIP*) if every principal ideal is flat (projective). Note that PIP rings are often called *PP*-rings or *Rickart* rings (see for example [10, p. 260-261]). It is clear that every PIP ring is PIF ([10, Proposition 4.3]).

Lemma 4.7. *The following are equivalent for a ring R:*

(i) *Every minimal prime ideal of R is a direct summand;*

(ii) *R is PIF and* $\mathrm{Min}(R)$ *is finite;*

(iii) *R is PIP and* $\mathrm{Min}(R)$ *is finite;*

(iv) *R is a finite direct product of integral domains.*

Proof. The equivalences (ii) ⇔ (iii) ⇔ (iv) follow from [12, Proposition 2.2].

(i) ⇒ (ii) Let $\mathfrak{p} \in \mathrm{Spec}(R)$. It is well known that

$$Min(R_{\mathfrak{p}}) = \{\mathfrak{q}R_{\mathfrak{p}} \mid \mathfrak{q} \in \mathrm{Min}(R) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Let $\mathfrak{q} \in \mathrm{Min}(R)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Then there exists $e^2 = e \in R$ such that $\mathfrak{q} = eR$. Therefore $\mathfrak{q}R_{\mathfrak{p}} = \frac{e}{1}R_{\mathfrak{p}}$ and $(\frac{e}{1})^2 = \frac{e}{1}$. Consequently, $\mathfrak{q}R_{\mathfrak{p}}$ is a direct summand of $R_{\mathfrak{p}}$. But $R_{\mathfrak{p}}$ is an indecomposable ring (since it is local) and $\mathfrak{q}R_{\mathfrak{p}} \neq R_{\mathfrak{p}}$, so $\mathfrak{q}R_{\mathfrak{p}} = 0$. Therefore $R_{\mathfrak{p}}$ is an integral domain and hence R is PIF by [12, Proposition 2.1]. Now to prove that $\mathrm{Min}(R)$ is finite, take a minimal prime ideal $\mathfrak{p}$ of R. Thus $\mathfrak{p}$ is direct summand of R and hence $\mathfrak{p}$ is finitely generated. Therefore any minimal prime ideal over 0 is finitely generated. By a Theorem of D.D. Anderson (see [3]), R has only finitely many minimal prime ideals.

(iii) ⇒ (i) Since any projective module is flat, it follows that R is PIF. Now apply [12, Proposition 2.14]. □

Recall that a ring R is called an *mp-ring* if every prime ideal of R contains a unique minimal prime ideal, or, equivalently, if every maximal ideal of R contains a unique minimal prime ideal (see [1]).

Proposition 4.8. *Let R be a ring and let* $\mathrm{Min}(R) = \{\mathfrak{p}_i \mid i \in I\}$. *Consider the R-module $M = \oplus_{i\in I} R/\mathfrak{p}_i$. Then the following are equivalent:*

(i) *M is a weakly IN R-module;*

(ii) *M is a strongly CS R-module;*

(iii) *R is an mp-ring and* $\mathrm{Min}(R)$ *is finite;*

(iv) *$R/\mathrm{Nil}(R)$ is a finite direct product of integral domains.*

Proof. (i) ⇔ (ii) Use [9, Corollary 2.10] and the fact that $\mathrm{Ann}_R(M) = \mathrm{Nil}(R)$.

(i) ⇒ (iv) Let $j \in I$. Since M is weakly IN, $\mathfrak{p}_j + \cap_{i\neq j}\mathfrak{p}_i = R$ (Lemma 4.1). Let $\overline{R} = R/\mathrm{Nil}(R)$. For any ideal $\mathfrak{a}$ of R which contains $\mathrm{Nil}(R)$, let $\overline{\mathfrak{a}} = \mathfrak{a}/\mathrm{Nil}(R)$. Note that $\overline{\mathfrak{p}}_j + \cap_{i\neq j}\overline{\mathfrak{p}}_i = \overline{R}$ and $\overline{\mathfrak{p}}_j \cap (\cap_{i\neq j}\overline{\mathfrak{p}}_i) = \cap_{i\in I}\overline{\mathfrak{p}}_i = \overline{\mathrm{Nil}(R)} = 0$. Thus $\overline{\mathfrak{p}}_j \oplus \cap_{i\neq j}\overline{\mathfrak{p}}_i = \overline{R}$. It follows that every minimal prime ideal of $\overline{R}$ is a direct summand. Using Lemma 4.7, we infer that $\overline{R}$ is a finite direct product of integral domains.

(iv) $\Rightarrow$ (iii) By Lemma 4.7, $R/\text{Nil}(R)$ is PIF and $Min(R/\text{Nil}(R))$ is finite. This implies that R is an mp-ring (see [12, Proposition 2.1]) and that $\text{Min}(R)$ is finite.

(iii) $\Rightarrow$ (i) Let $\text{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$. So $M = R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_n$. Since R is an mp-ring, it follows easily that $\mathfrak{p}_i + \mathfrak{p}_j = R$ for all $i \neq j \in \{1, \dots, n\}$. Therefore M is weakly IN by [9, Theorem 3.1]. □

Proposition 4.9. *Let R be a ring and let $\text{Spec}(R) = \{\mathfrak{p}_i \mid i \in I\}$. Consider the R-module $M = \oplus_{i\in I} R/\mathfrak{p}_i$. Then the following are equivalent:*

(i) *M is a weakly IN R-module;*

(ii) *M is a strongly CS R-module;*

(iii) *M has the SWIN;*

(iv) *R is semiperfect and $dim(R) = 0$;*

(v) *R is semilocal and $dim(R) = 0$;*

(vi) *$R/\text{Nil}(R)$ is semisimple.*

Proof. (i) $\Leftrightarrow$ (ii) Use [9, Corollary 2.10] and the fact that $\text{Ann}_R(M) = \text{Nil}(R)$.

(ii) $\Rightarrow$ (iv) Suppose that M is strongly CS and let $H = \oplus_{\mathfrak{m}\in \text{Max}(R)} R/\mathfrak{m}$. Then H is isomorphic to a submodule of M and consequently H is also strongly CS. By Proposition 4.6(ii), R is semiperfect. To prove that $dim(R) = 0$, assume to the contrary that R has a prime ideal $\mathfrak{p}$ which is not maximal. Then $\mathfrak{p} \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in \text{Max}(R)$. Consider the R-module $T = R/\mathfrak{p} \oplus R/\mathfrak{m}$. Since T is isomorphic to a submodule of M, it follows that T is strongly CS and hence T is a weakly IN R-module. Now using Lemma 4.1, we conclude that $\mathfrak{p} + \mathfrak{m} = R$, a contradiction. Hence $dim(R) = 0$.

(iv) $\Rightarrow$ (vi) This follows from the fact that $\text{J}(R) = \text{Nil}(R)$.

(vi) $\Rightarrow$ (v) Since $R/\text{Nil}(R)$ is semisimple, it follows easily that $dim(R) = dim(R/\text{Nil}(R)) = 0$ and $R/\text{J}(R)$ is semisimple.

(v) $\Rightarrow$ (iii) This follows from Proposition 4.6(i) and the fact that every prime ideal of R is maximal.

(iii) $\Rightarrow$ (i) This follows from the definitions of a weakly IN module and a module having the SWIN. □

Corollary 4.10. *Let R be a ring with $\text{Spec}(R) = \{\mathfrak{p}_i \mid i \in I\}$ and $\text{Min}(R) = \{\mathfrak{p}_j \mid j \in J\}$ ($J \subseteq I$). Consider the R-modules $M_1 = \oplus_{i\in I} R/\mathfrak{p}_i$ and $M_2 = \oplus_{j\in J} R/\mathfrak{p}_j$. Then the following are equivalent:*

(i) *M_1 has the SWIN;*

(ii) *M_2 has the SWIN;*

(iii) *R is semilocal and $dim(R) = 0$.*

Proof. (i) $\Rightarrow$ (iii) This follows from Proposition 4.9.

(iii) $\Rightarrow$ (ii) Apply Proposition 4.9 and use the fact that M_2 is isomorphic to a submodule of M_1.

(ii) $\Rightarrow$ (i) Let $j \in J$. As $R/\mathfrak{p}_j$ is a submodule of M_2, $R/\mathfrak{p}_j$ has the SWIN. Therefore $R/\mathfrak{p}_j$ is finitely embedded by Theorem 2.8. But $R/\mathfrak{p}_j$ is a uniform module, so $E(R/\mathfrak{p}_j) \cong E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of R. This yields $\mathfrak{p}_j = \mathfrak{m}$ (see for example [14, Lemma 2.31 Corollary]). It follows that $dim(R) = 0$. This implies that $\text{Spec}(R) = \text{Min}(R) = \text{Max}(R)$ and so $M_1 = M_2$ has the SWIN. □

5 Strongly Kasch modules

In this section, we investigate strongly Kasch modules which are weakly IN (or strongly CS). Recall that a ring R is called *Kasch* if every simple R-module embeds in R, equivalently, $\mathrm{Ann}_R(\mathfrak{m}) \neq 0$ for every maximal ideal $\mathfrak{m}$ of R (see [10, Definition 8.26 and Proposition 8.27]. This concept was generalized to a module theoretic setting by Albu and Wisbauer [2]. A module M is said to be *Kasch* provided M contains a copy of every simple module in $\sigma[M]$, where $\sigma[M]$ is the full subcategory of Mod-R consisting of all R-modules that are subgenerated by M. Another way to generalize the notion of Kasch rings to modules appeared in [18]. Following Zhu, an R-module M is is said to be *strongly Kasch* if every simple R-module embeds in M. It is clear that a ring R is Kasch if and only if the R-module R is Kasch if and only if the R-module R is strongly Kasch.

Proposition 5.1. *Let M be an R-module and let $\mathrm{Max}(R) = \{\mathfrak{m}_i \mid i \in I\}$. Then the following assertions are equivalent for a nonzero R-module M:*

(i) *M is a strongly Kasch R-module;*

(ii) *$Hom_R(N,M) \neq 0$ for every finitely generated nonzero R-module N;*

(iii) *$\mathrm{Ann}_M(\mathfrak{m}_i) \neq 0$ for all $i \in I$;*

(iv) *$\mathrm{Ann}_M(\mathrm{Ann}_R(N)) \neq 0$ for any nonzero finitely generated R-module N;*

(v) *$E(M)$ is a strongly Kasch R-module;*

(vi) *$E(M)$ is an injective cogenerator;*

(vii) *$\oplus_{i\in I} R/\mathfrak{m}_i$ is isomorphic to a submodule of M;*

(viii) *$E(\oplus_{i\in I} R/\mathfrak{m}_i)$ is isomorphic to a summand of $E(M)$.*

Proof. The equivalences (i) ⇔ (ii) ⇔ (iii) ⇔ (v) ⇔ (vi) come from [19, Theorem 3.9].

(ii) ⇔ (iv) This follows from [16, Theorem 2.1].

(v) ⇔ (viii) By [5, Corollary 3.3].

(i) ⇒ (vii) Let $i \in I$ and let S_i be a simple submodule of M such that $S_i \cong R/\mathfrak{m}_i$. It is easily seen that the sum $\Sigma_{i\in I} S_i$ is direct. Hence $\oplus_{i\in I} R/\mathfrak{m}_i$ is isomorphic to a submodule of M since $\oplus_{i\in I} R/\mathfrak{m}_i \cong \oplus_{i\in I} S_i$.

(vii) ⇒ (viii) This is immediate. □

For an R-module M, let $\mathcal{Z}(M)$ denote the set of zero divisors of M, that is, the set of elements a of R such that $\mathrm{Ann}_M(a) \neq 0$.

Remark 5.2. (i) Let R be a ring and let $\mathrm{Max}(R) = \{\mathfrak{m}_i \mid i \in I\}$. From Proposition 5.1, it follows that the R-modules $M_1 = \oplus_{i\in I} R/\mathfrak{m}_i$, $M_2 = E(\oplus_{i\in I} R/\mathfrak{m}_i)$ and $M_3 = \oplus_{i\in I} E(R/\mathfrak{m}_i)$ are strongly Kasch.

(ii) Let M be a strongly Kasch R-module and let $a \in R \setminus \mathcal{Z}(M)$. Then $\mathrm{Ann}_M(aR) = 0$ and hence $\mathrm{Ann}_M(\mathrm{Ann}_R(R/aR)) = 0$. Using Proposition 5.1, we infer that $aR = R$. Therefore a is invertible.

Lemma 5.3. *Let M be a strongly Kasch R-module. Then the following assertions are equivalent:*

(i) *M is a weakly IN R-module;*

(ii) *$N \subseteq^{ess} \mathrm{Ann}_M(\mathrm{Ann}_R(N))$ for every submodule N of M.*

In this case, $Soc(M) \subseteq^{ess} M$.

Proof. (i) ⇒ (ii) Let N be a submodule of M. Clearly $N \subseteq \mathrm{Ann}_M(\mathrm{Ann}_R(N))$. Now let L be a submodule of M such that $L \subseteq \mathrm{Ann}_M(\mathrm{Ann}_R(N))$ and $L \cap N = 0$. Since M is weakly IN, $\mathrm{Ann}_R(N) + \mathrm{Ann}_R(L) = R$. But $\mathrm{Ann}_R(N) \subseteq \mathrm{Ann}_R(L)$, so $\mathrm{Ann}_R(L) = R$. Consequently, $L = 0$.

(ii) ⇒ (i) Let N and L be two submodules of M such that $N \cap L = 0$. By hypothesis, $N \subseteq^{ess} \mathrm{Ann}_M(\mathrm{Ann}_R(N))$ and $L \subseteq^{ess} \mathrm{Ann}_M(\mathrm{Ann}_R(L))$. This implies that $N \cap L \subseteq^{ess} \mathrm{Ann}_M(\mathrm{Ann}_R(N)) \cap \mathrm{Ann}_M(\mathrm{Ann}_R(L))$ and so $\mathrm{Ann}_M(\mathrm{Ann}_R(N)) \cap \mathrm{Ann}_M(\mathrm{Ann}_R(L)) = 0$. Therefore,

$$\mathrm{Ann}_M(\mathrm{Ann}_R(N) + \mathrm{Ann}_R(L)) = \mathrm{Ann}_M(\mathrm{Ann}_R(N)) \cap \mathrm{Ann}_M(\mathrm{Ann}_R(L)) = 0.$$

Since M is strongly Kasch, it follows from Proposition 5.1 that $\mathrm{Ann}_R(N) + \mathrm{Ann}_R(L) = R$.

To prove the last part, suppose that M is a weakly IN strongly Kasch R-module and let $0 \neq x \in M$. Let $\mathfrak{m}$ be a maximal ideal of R containing $\mathrm{Ann}_R(x)$. Then $\mathrm{Ann}_M(\mathfrak{m}) \subseteq \mathrm{Ann}_M(\mathrm{Ann}_R(x))$. Since M is strongly Kasch, there exists $0 \neq y \in M$ such that yR is a simple submodule of M and $\mathfrak{m} = \mathrm{Ann}_R(y)$. Hence, $\mathrm{Ann}_M(\mathrm{Ann}_R(y)) \subseteq \mathrm{Ann}_M(\mathrm{Ann}_R(x))$. This yields $yR \subseteq \mathrm{Ann}_M(\mathrm{Ann}_R(x))$ as $yR \subseteq \mathrm{Ann}_M(\mathrm{Ann}_R(y))$. Moreover, from the implication (i) ⇒ (ii), we deduce that $xR \subseteq^{ess} \mathrm{Ann}_M(\mathrm{Ann}_R(x))$. Consequently, $xR \cap yR \neq 0$. But yR is simple, so $yR \subseteq xR$. Thus, $\mathrm{Soc}(M) \subseteq^{ess} M$. □

Lemma 5.4. *Let R be a semilocal ring having exactly n maximal ideals for some positive integer n. Let M be a nonzero weakly IN R-module. Then M has finite uniform dimension with $u.dim(M) \leq n$.*

Proof. Suppose that M has an independent family $\{N_1, \ldots, N_{n+1}\}$ of $n+1$ nonzero submodules of M. Since $N_1 \neq 0$, there exists a maximal ideal $\mathfrak{m}_1$ of R such that $\mathrm{Ann}_R(N_1) \subseteq \mathfrak{m}_1$. Similarly, $\mathrm{Ann}_R(N_2) \subseteq \mathfrak{m}_2$ for some maximal ideal $\mathfrak{m}_2$ of R. Since $N_1 \cap N_2 = 0$ and M is weakly IN, we have $\mathrm{Ann}_R(N_1) + \mathrm{Ann}_R(N_2) = R$. Therefore $\mathfrak{m}_1 \neq \mathfrak{m}_2$. We continue in this fashion to obtain maximal ideals $\mathfrak{m}_i$ such that $\mathrm{Ann}_R(N_i) \subseteq \mathfrak{m}_i$ $(1 \leq i \leq n)$ and $\mathfrak{m}_j \neq \mathfrak{m}_k$ for $j \neq k$. Note that $\mathrm{Max}(R) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. Also, we must have $\mathrm{Ann}_R(N_{n+1}) \subseteq \mathfrak{m}_k$ for some $k \in \{1, \ldots, n\}$. But this contradicts the fact that $\mathrm{Ann}_R(N_k) \subseteq \mathfrak{m}_k$ and $\mathrm{Ann}_R(N_k) + \mathrm{Ann}_R(N_{n+1}) = R$. It follows that $u.dim(M) \leq n$. □

Theorem 5.5. Let M be a nonzero strongly Kasch R-module which is weakly IN (strongly CS). Then R is a semilocal (semiperfect) ring. Moreover, in both cases M is a finitely embedded R-module having the SWIN.

Proof. Let $\mathrm{Max}(R) = \{\mathfrak{m}_i \mid i \in I\}$. Since M is strongly Kasch, $\oplus_{i \in I} R/\mathfrak{m}_i$ is isomorphic to a submodule of M (Proposition 5.1). But M is weakly IN (strongly CS). Thus the R-module $\oplus_{i \in I} R/\mathfrak{m}_i$ is also weakly IN (strongly CS). Hence, R is semilocal (semiperfect) by Proposition 4.6. Using Lemma 5.4, we get that M has finite uniform dimension. Moreover, $\mathrm{Soc}(M) \subseteq^{ess} M$ by Lemma 5.3. This implies that $\mathrm{Soc}(M)$ is finitely generated. Therefore M is finitely embedded (see [14, Proposition 3.18]) and hence M has the SWIN by Theorem 2.8. □

Proposition 5.6. *Let R be a ring and let M be a nonzero R-module. Then the following are equivalent:*

(i) *M is weakly IN and strongly Kasch;*

(ii) *M has the SWIN and every $a \in R \setminus \mathcal{Z}(M)$ is invertible.*

Proof. (i) ⇒ (ii) This is clear by Remark 5.2(ii) and Theorem 5.5.

(ii) ⇒ (i) By Theorem 2.8, M is weakly IN and finitely embedded. Therefore there exist finitely many maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ of R such that $E(M) \cong E(R/\mathfrak{m}_1) \oplus E(R/\mathfrak{m}_2) \oplus \cdots \oplus E(R/\mathfrak{m}_n)$. It is clear that $\mathcal{Z}(M) = \mathfrak{m}_1 \cup \mathfrak{m}_2 \cup \cdots \cup \mathfrak{m}_n$. We claim that $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$. To prove this, let $\mathfrak{m}$ be a maximal ideal of R and let $a \in \mathfrak{m}$. Since a is not invertible, $a \in \mathcal{Z}(M)$. It follows that $\mathfrak{m} \subseteq \mathcal{Z}(M) = \mathfrak{m}_1 \cup \mathfrak{m}_2 \cup \cdots \cup \mathfrak{m}_n$ and so $\mathfrak{m} = \mathfrak{m}_i$ for some $i \in \{1, 2, \ldots, n\}$, proving the claim. Now apply Proposition 5.1 to conclude that M is strongly Kasch. □

The next corollary is an immediate consequence of Proposition 5.6.

Corollary 5.7. *The following are equivalent for a ring R:*

(i) *R is CS and Kasch;*

(ii) *R has the SWIN and every regular element of R is invertible.*

The following example shows that each of the implication “Kasch ⇒ CS”and its converse need not be true.

Example 5.8. (i) Let R, $\mathfrak{m}$, A and M be as in Example 2.11(i). It is clear that A is a local ring with maximal ideal $(\mathfrak{m}, M)$. Let $x = 1 + \mathfrak{m} \in R/\mathfrak{m} \leq M$. Since $(0, x)(\mathfrak{m}, M) = 0$, it follows that A is a Kasch ring by Proposition 5.1. However, A is not CS.

(ii) Let R be an integral domain which is not a field. Then it is clear that R is a CS ring but not a Kasch ring.

Corollary 5.9. *Let R be a ring. Then the following equivalences hold:*

(i) *R is semilocal if and only if R has a weakly IN strongly Kasch R-module.*

(ii) *The following are equivalent:*

  (a) *R is semiperfect;*

  (b) *R has a strongly CS strongly Kasch R-module;*

  (c) *R has a faithful strongly CS strongly Kasch R-module;*

  (d) *R has a faithful weakly IN strongly Kasch R-module.*

Proof. Let $\mathrm{Max}(R) = \{\mathfrak{m}_i \mid i \in I\}$ and let $N = \oplus_{i \in I} R/\mathfrak{m}_i$. Then N is a strongly Kasch R-module by Remark 5.2.

(i) (⇒) Since R is semilocal, N is weakly IN by Proposition 4.6(i).

(⇐) Let M be a weakly IN strongly Kasch R-module. Then N is isomorphic to a submodule of M by Proposition 5.1. This implies that N is weakly IN and so R is semilocal by Proposition 4.6(i).

(ii) (c) ⇔ (d) This follows from the fact that a faithful R-module is weakly IN if and only if it is strongly CS (see [9, Corollary 2.10]).

(c) ⇒ (b) This is evident.

(b) ⇒ (a) Let M be a strongly CS strongly Kasch R-module. By Proposition 5.1, N is isomorphic to a submodule of M and so N is also strongly CS. From Proposition 4.6(ii), it follows that R is semiperfect.

(a) ⇒ (c) Since R is semiperfect, $R = R_1 \times R_2 \times \cdots \times R_n$ where each R_i is a local ring. For each $1 \leq i \leq n$, let $\mathfrak{n}_i$ be the maximal ideal of R_i and let E_i denote the injective hull of the simple R_i-module $R_i/\mathfrak{n}_i$. By Proposition 5.1, E_i is a strongly Kasch R_i-module. Since each M_i is a uniform R_i-module, it is clear that each M_i is a strongly CS R_i-module. Note that each E_i is a faithful R_i-module by [14, Proposition 2.26 Corollary 2]. Now consider the R-module $M = E_1 \times E_2 \times \cdots \times E_n$. Then M is a strongly CS R-module by induction and using [9, Lemma 3.2]. Moreover, M is a faithful R-module since $\mathrm{Ann}_R(M) = \mathrm{Ann}_{R_1}(E_1) \times \mathrm{Ann}_{R_2}(E_2) \times \cdots \times \mathrm{Ann}_{R_n}(E_n) = 0$. Note that an ideal $\mathfrak{a}$ of R is maximal if and only if there exists an integer $k \in \{1, 2, \ldots, n\}$ such that $\mathfrak{a}$ has the form $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \cdots \times \mathfrak{a}_n$ with $\mathfrak{a}_k = \mathfrak{n}_k$ and $\mathfrak{a}_i = R_i$ for all $i \neq k$. It follows easily from Proposition 5.1 that M is a strongly Kasch R-module. This finishes the proof. □

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