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On ϕ-1-absorbing primary submodules

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Abstract. In this article, we introduce ϕ-1-absorbing primary submodules of modules over commutative rings. Let R be a commutative ring with a nonzero identity and M be a nonzero unital module. $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function where $\mathcal{S}(M)$ is the set of all submodules of M. A proper submodule N of M is said to be a ϕ-1-absorbing primary submodule if whenever $abm \in N - \phi(N)$ for some nonunit elements $a, b \in R$ and $m \in M$, then $ab \in (N :_R M)$ or $m \in M\text{-}rad(N)$, where $M\text{-}rad(N)$ is the prime radical of N. Many properties and characterizations of ϕ-1-absorbing primary submodules are given. We also give the relations between ϕ-1-absorbing primary submodules and other classical submodules such as ϕ-prime, ϕ-primary, ϕ-2-absorbing primary submodules.
Key Words: ϕ-prime submodule, 1-absorbing primary submodule, (weakly) pseudo primary submodule, ϕ-1-absorbing primary submodule.
2010 MSC: Primary 13A15; Secondary 16D60.

1 Introduction

Throughout this article, we focus only on commutative rings with identity and nonzero unital modules. Let R will always denote such a ring and M will denote such an R-module. A submodule N of M is called a proper submodule if $N \neq M$. Let N be a proper submodule of M. The ideal $(N :_R M)$ (or briefly $(N : M)$) is defined as $\{a \in R : aM \subseteq N\}$. A submodule N of M is said to be idempotent if $(N :_R M)N = N$. The radical of N, denoted by $M\text{-}rad(N)$, is defined as the intersection of all prime submodules of M containing N.

Recall that a proper submodule N of M is said to be prime if $am \in N$ for $a \in R$ and $m \in M$ implies that either $a \in (N : M)$ or $m \in N$. We say that M is a prime module if the zero submodule of M is a prime submodule of M. Over the years, prime submodules and its generalizations have an important place in commutative algebra and they draw attention by many authors. See for example, [8], [13], [23] and [29]. In [9], Atani and Farzalipour studied on weakly primary submodule which is a generalization of primary submodule. A proper submodule N of an R-module M is said to be a weakly primary if $0 \neq am \in N$ implies $a \in \sqrt{(N : M)}$ or $m \in N$ where $a \in R$, $m \in M$. They also defined the notion of weakly prime submodules in [15]. A proper submodule N of an R-module M is called a weakly prime if $0 \neq am \in N$ implies $a \in (N : M)$ or $m \in N$ where $a \in R, m \in M$. They investigated many properties and gave some characterizations of weakly prime and weakly primary submodules. In [31], Zamani introduced ϕ-prime submodules. Let $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function where $\mathcal{S}(M)$ is the set of all submodules of M. A proper submodule N of M is called ϕ-prime if whenever $a \in R$ and $m \in M$ such that $am \in N - \phi(N)$, then either $a \in (N :_R M)$ or $m \in N$. Mostafanasab et al. introduced 2-absorbing primary submodules in [22]. A proper submodule N of an R-module M is called a 2-absorbing primary submodule of M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in M\text{-}rad(N)$ or $bm \in M\text{-}rad(N)$ or $ab \in (N : M)$. They gave the relations among 2-absorbing ideals, 2-absorbing submodules, 2-absorbing primary ideals and 2-absorbing primary submodules. Recently, in 2017, Darani et al. defined weakly 2-absorbing primary submodule that is a generalization of 2-absorbing primary submodules [14]. A proper submodule N of an R-module M is called a weakly 2-absorbing primary submodule of M if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in M\text{-}rad(N)$ or $bm \in M\text{-}rad(N)$ or $ab \in (N : M)$. In [21], Moradi and Ebrahimpour defined ϕ-2-absorbing primary submodules. A proper submodule N of an R-module M is called a ϕ-2-absorbing primary submodule of M if whenever $a,b \in R$ and $m \in M$ and $abm \in N - \phi(N)$, then $am \in N$ or $bm \in N$ or $ab \in \sqrt{(N :_R M)}$. They characterized ϕ-2-absorbing primary submodules and investigated ϕ-2-absorbing primary submodules of some well-known modules. Currently, in [26], Yetkin Celikel introduced the concept of 1-absorbing primary submodules. A proper submodule N of an R-module M is called a 1-absorbing primary submodule of M if whenever nonunit elements $a,b \in R$ and $m \in M$ and $abm \in N$, then $ab \in (N : M)$ or $m \in M\text{-}rad(N)$. The author gave some properties and characterizations of this type of submodules. More recently, Yetkin Celikel et al., in [28] defined the notion of weakly 1-absorbing primary submodule. A proper submodule N of M is called a weakly 1-absorbing primary submodule if whenever nonunit elements $a,b \in R$, $m \in M$ and $0 \neq abm \in N$, then $ab \in (N : M)$ or $m \in M\text{-}rad(N)$. They gave the relations between this new concept and the other classical types of submodules by using illustrative examples.

Motivated and inspired by the above works, the purposes of this paper are to introduce generalizations of 1-absorbing primary submodule to the context of ϕ-1-absorbing primary submodule. A proper submodule N of M is called a ϕ-1-absorbing primary submodule if whenever nonunit elements $a,b \in R$, $m \in M$ and $abm \in N - \phi(N)$, then $ab \in (N :_R M)$ or $m \in M\text{-}rad(N)$. Besides giving several characterizations of ϕ-1-absorbing primary submodules,we investigate the relations between this new concept and the other classical types of submodules by using illustrative examples. We also investigate the behaviour of ϕ-1-absorbing primary submodules in factor modules, modules of fractions, in cartesian product of modules, in trivial extension (See, Theorem 2.17, Theorem 2.19, Theorem 2.27, Theorem 3.1, Theorem 3.3, Theorem 3.4). Furthermore, we investigate ϕ-1-absorbing primary submodules of multiplication modules (See, Theorem 2.6, Theorem 2.8, Theorem 2.9).

2 Characterization of ϕ-1-absorbing primary submodules

Let R be a commutative ring, M be an R-module. Define a function $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$. This function maps a submodule of M to a submodule of M or $\emptyset$.

Definition 2.1. Let R be a ring, M be an R-module and N a proper submodule of M.

1. N is called a ϕ-1-absorbing primary submodule of M if whenever $abm \in N - \phi(N)$ for some nonunits $a,b \in R$, $m \in M$ then $ab \in (N :_R M)$ or $m \in M\text{-}rad(N)$.
2. N is called a ϕ-pseudo primary submodule of M if whenever $am \in N - \phi(N)$ for some $a \in R, m \in M$, then $a \in (N :_R M)$ or $m \in M\text{-}rad(N)$. In particular, if $\phi(N) = \emptyset$, then a ϕ-pseudo primary submodule N is called a pseudo primary submodule of M.

The following notations are used for the rest of the paper.

Example 2.2. Let R be a commutative ring, M be an R-module and $\phi_\alpha : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be a function. The following gives types of 1-absorbing primary submodules corresponding to ϕ_α.

$\phi_\emptyset$	$\phi(N) = \emptyset$	1-absorbing primary submodule
ϕ_0	$\phi(N) = 0$	weakly-1-absorbing primary submodule
ϕ_2	$\phi(N) = (N :_R M)N$	almost-1-absorbing primary submodule
ϕ_n	$\phi(N) = (N :_R M)^{n-1}N$	n-almost-1-absorbing primary submodule
ϕ_ω	$\phi(N) = \bigcap_{n=0}^{\infty}(N :_R M)^n N$	ω-1-absorbing primary submodule
ϕ_1	$\phi(N) = N$	any submodule

Consider two functions $\phi, \psi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$. We use the notation $\phi \leq \psi$ when $\phi(N) \subseteq \psi(N)$ for all submodules of M. Moreover, note that $\phi_\emptyset \leq \phi_0 \leq \phi_w \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$. We will assume that $\phi(N) \subseteq N$ throughout the paper as $N - \phi(N) = N - (N \cap \phi(N))$. The following proposition follows immediately from the definitions above, so no required proof.

Proposition 2.3. *Let N be a proper submodule of an R-module M. Then we have the following.*

1. Every pseudo primary submodule is a ϕ-1-absorbing primary.
2. Every weakly 1-absorbing primary submodule N of M is ϕ-1-absorbing primary, where $\phi(N) \neq \emptyset$.
3. Every 1-absorbing primary submodule of M is ϕ-1-absorbing primary.
4. Every ϕ-1-absorbing primary submodule of M is ϕ-2-absorbing primary.
5. Every ϕ-prime submodule of M is ϕ-1-absorbing primary.
6. Every idempotent submodule of M is ω-1-absorbing primary.
7. If N is a ϕ-primary submodule of M and $(N :_R M)$ is a radical ideal, then N is a ϕ-1-absorbing primary submodule of M.
8. If N is a ϕ-1-absorbing primary submodule of M and $\phi(N)$ is a 1-absorbing primary submodule of M, then N is a 1-absorbing primary submodule of M.
9. Let (R, Q) be a quasi-local ring. Then every proper submodule N of M satisfying $Q^2 \subseteq (N :_R M)$ is a ϕ-1-absorbing primary submodule.

Proposition 2.4. *Let N be a proper submodule of M.*

1. Let $\phi, \psi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ be two functions with $\phi \leq \psi$. If N is a ϕ-1-absorbing primary submodule, then N is a ψ-1-absorbing primary submodule.
2. N is a 1-absorbing primary submodule $\Rightarrow$ N is a weakly 1-absorbing primary submodule $\Rightarrow$ N is a ω-1-absorbing primary submodule $\Rightarrow$ N is an n-almost 1-absorbing primary submodule for each $n \geq 2$ $\Rightarrow$ N is an almost 1-absorbing primary submodule.
3. N is an n-almost 1-absorbing primary submodule for each $n \geq 2$ if and only if N is a ω-1-absorbing primary submodule.

Proof. (1) It is obvious.

(2) Follows from the order $\phi_\emptyset \leq \phi_0 \leq \phi_w \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \phi_2$ and (1).

(3) By (2), we know that if N is a w-1-absorbing primary submodule, then N is an n-almost 1-absorbing primary submodule for each $n \geq 2$. Now, assume that N is an n-almost 1-absorbing primary submodule for each $n \geq 2$. Let $abm \in N - \bigcap_{n=0}^{\infty}(N : M)^n N$ for some nonunits $a, b \in R, m \in M$. Then there exists $k \geq 2$ such that $abm \notin (N : M)^k N$. Since N is an $(k+1)$-almost 1-absorbing primary submodule of M and $abm \in N - (N : M)^k N$, then either we have $ab \in (N :_R M)$ or $m \in M\text{-}rad(N)$. □

Let M be an R-module. For a subset A of R, by $(N :_M A)$, we mean the set of all $m \in M$ satisfying $Am \subseteq N$. We obtain several equivalent statements to characterize ϕ-1-absorbing primary submodules of an R-module M by the following theorem.

Theorem 2.5. Let N be a proper submodule of an R-module M. Then the following are equivalent.

1. N is a ϕ-1-absorbing primary submodule of M.
2. For all nonunit elements $a, b \in R$ with $ab \notin (N :_R M)$, $(N :_M ab) \subseteq (\phi(N) :_M ab) \cup M\text{-}rad(N)$.
3. For all nonunit elements $a, b \in R$ with $ab \notin (N :_R M)$, $(N :_M ab) = (\phi(N) :_M ab)$ or $(N :_M ab) \subseteq M\text{-}rad(N)$.
4. For all nonunit elements $a, b \in R$ and a submodule K with $abK \subseteq N, abK \nsubseteq \phi(N)$, either $ab \in (N :_R M)$ or $K \subseteq M\text{-}rad(N)$.
5. For all nonunit element $a \in R$, a proper ideal J of R and a submodule K of M with $aJK \subseteq N, aJK \nsubseteq \phi(N)$, either $aJ \subseteq (N :_R M)$ or $K \subseteq M\text{-}rad(N)$.
6. For all nonunit element $a \in R$, a proper ideal J of R if $(N :_M aJ)$ is proper, then $(N :_M aJ) = (\phi(N) :_M aJ)$ or $(N :_M aJ) \subseteq M\text{-}rad(N)$.
7. For proper ideals I, J of R and a submodule K of M with $IJK \subseteq N, IJK \nsubseteq \phi(N)$, either $IJ \subseteq (N :_R M)$ or $K \subseteq M\text{-}rad(N)$.

Proof. $(1) \Rightarrow (2)$: Choose $m \in (N :_M ab)$ implying $abm \in N$. If $abm \in \phi(N)$, then $m \in (\phi(N) :_M ab)$. So, assume that $abm \notin \phi(N)$. Since N is ϕ-1-absorbing primary submodule and $ab \notin (N :_R M)$, we get $m \in M\text{-}rad(N)$. Thus, $(N :_M ab) \subseteq (\phi(N) :_M ab) \cup M\text{-}rad(N)$.

$(2) \Rightarrow (3)$: It is immediate.

$(3) \Rightarrow (4)$: Suppose that $abK \subseteq N$, $abK \nsubseteq \phi(N)$ and $ab \notin (N :_R M)$. Then $(N :_M ab) \neq (\phi(N) :_M ab)$ and we get $K \subseteq (N :_M ab) \subseteq M\text{-}rad(N)$.

$(4) \Rightarrow (5)$: Suppose $aJK \subseteq N$ and $aJK \nsubseteq \phi(N)$ where a is nonunit, J is a proper ideal of R and K is a submodule of M. Assume that $K \nsubseteq M\text{-}rad(N)$. Then there exists $j \in J$ such that $ajK \subseteq N, ajK \nsubseteq \phi(N)$ and this gives $aj \in (N :_R M)$. Take $x \in J$. If $axK \nsubseteq \phi(N)$, then $ax \in (N :_R M)$. Now assume that $axK \subseteq \phi(N)$. Then we have $a(x+j)K \subseteq N$ and $a(x+j)K \nsubseteq \phi(N)$ giving $a(x+j) \in (N :_R M)$. This gives $ax \in (N :_R M)$, that is, $aJ \subseteq (N :_R M)$, as desired.

$(5) \Rightarrow (6)$: Let $m \in (N :_M aJ)$. Then $aJm \subseteq N$. If $aJm \subseteq \phi(N)$, then $m \in (\phi(N) :_M aJ)$. Now assume that $aJm \nsubseteq \phi(N)$. As $(N :_M aJ)$ is proper, we have $aJ \nsubseteq (N : M)$. Now take $K = Rm$. Since $aJK \subseteq N$ and $aJK \nsubseteq \phi(N)$, we get $K = Rm \subseteq M\text{-}rad(N)$. So $(N :_M aJ) \subseteq (\phi(N) :_M aJ) \cup M\text{-}rad(N)$ which implies that $(N :_M aJ) = (\phi(N) :_M aJ)$ or $(N :_M aJ) \subseteq M\text{-}rad(N)$.

$(6) \Rightarrow (7)$: Suppose $IJK \subseteq N$ and $IJK \nsubseteq \phi(N)$ for proper ideals I, J of R and a submodule K of M with $IJ \nsubseteq (N :_R M)$. Then there exists a nonunit $a \in I$ such that $aJ \nsubseteq (N :_R M)$. As $(N :_M aJ) \neq M$ and $aJK \subseteq N$, we have $K \subseteq (N :_M aJ) \subseteq (\phi(N) :_M aJ)$ or $K \subseteq (N :_M aJ) \subseteq M\text{-}rad(N)$. If the latter case holds, we are done. So assume that $K \nsubseteq M\text{-}rad(N)$ and $aJK \subseteq \phi(N)$. As $IJK \nsubseteq \phi(N)$, there exists $x \in I$ such that $xJK \nsubseteq \phi(N)$. This gives $(a+x)JK \subseteq N$ and $(a+x)JK \nsubseteq \phi(N)$ implying $(N :_M (a+x)J) \neq (\phi(N) :_M (a+x)J)$. If $(N :_M (a+x)J)$ is proper, then $K \subseteq (N :_M (a+x)J) \subseteq M\text{-}rad(N)$, a contradiction. So suppose $(N :_M (a+x)J)$ is not proper. Then $(N :_M (a+x)J) = M$ implying $(a+x)J \subseteq (N :_R M)$. Hence $xJ \nsubseteq (N :_R M)$ and so $(N : xJ) \neq M$. Since $xJK \nsubseteq \phi(N)$ we get $K \subseteq (N :_M xJ) \subseteq M\text{-}rad(N)$ by (6) which is a contradiction.

$(7) \Rightarrow (1)$: Let $abm \in N - \phi(N)$. Put $I = aR$, $J = bR$ and $K = Rm$. Then $IJK \subseteq N$ and $IJK \nsubseteq \phi(N)$. By (7), we get $IJ \subseteq (N :_R M)$ or $m \in K \subseteq M\text{-}rad(N)$, as needed. □

Recall from [12] that an R-module M is said to be a multiplication module if every submodule N of M has the form IM for some ideal I of R. It is easy to see that M is a multiplication R-module if and only if $N = (N : M)M$. R. Ameri in his paper [4] defined the product of two submodules as follows: let M be a multiplication module, $N = IM$ and $K = JM$ for some ideals I, J of R. Then NK is defined as $(IJ)M$. This product is independent of the presentations of N and K. In particular, for every $m, m' \in M$, mm' denotes the product of the submodules Rm and Rm' of M. For more details on multiplication modules, we refer [6] and [16] to the reader. The following theorem gives a characterization of ϕ-1 absorbing primary submodules of finitely generated faithful multiplication modules.

Theorem 2.6. Let R be a ring, M be a finitely generated faithful multiplication R-module. Then the following are equivalent.

1. N is a ϕ-1 absorbing primary submodule of M.
2. For proper submodules K_1, K_2, K_3 of M, $K_1K_2K_3 \subseteq N$ and $K_1K_2K_3 \nsubseteq \phi(N)$ imply that either $K_1K_2 \subseteq N$ or $K_3 \subseteq M\text{-}rad(N)$.

Proof. $(1) \Rightarrow (2)$: Let N be a ϕ-1-absorbing primary submodule and $K_1 = I_1M$, $K_2 = I_2M$. Since K_1, K_2 are proper, I_1, I_2 are proper ideals. Then $K_1K_2K_3 = I_1I_2K_3 \subseteq N$ and $K_1K_2K_3 = I_1I_2K_3 \nsubseteq \phi(N)$ which implies that either $I_1I_2 \subseteq (N :_R M)$ or $K_3 \subseteq M\text{-}rad(N)$ by Theorem 2.5. This gives $K_1K_2 = I_1I_2M \subseteq (N :_R M)M = N$ or $K_3 \subseteq M\text{-}rad(N)$. $(2) \Rightarrow (1)$: Suppose $IJK \subseteq N$ and $IJK \nsubseteq \phi(N)$ with $K \nsubseteq M\text{-}rad(N)$ for proper ideals I, J of R and a submodule K of M. Put $K_1 = IM$, $K_2 = JM$. Then by the assumption $K_1K_2 = IJM \subseteq N$ implying $IJ \subseteq (N :_R M)$. Thus N is a ϕ-1-absorbing primary submodule of M by Theorem 2.5. □

Lemma 2.7. *Let R be a ring, I be an ideal of R and M be a finitely generated faithful multiplication R-module. For any submodule of N of M,*

1. $(IN :_R M) = I(N :_R M)$,[17, Lemma 3.4] .
2. $\sqrt{(N :_R M)} = (M\text{-}rad(N) :_R M)$, [22, Lemma 2.4].

Theorem 2.8. Let M be a finitely generated multiplication R-module and $\phi_{res} : S(R) \to S(R) \cup \{\emptyset\}$ be a function such that $\phi_{res}((N : M)) = (\phi(N) : M)$ for every $N \in S(M)$. Then N is a ϕ-1-absorbing primary submodule of M if and only if $(N : M)$ is a ϕ_{res}-1-absorbing primary ideal of R.

Proof. $(\Rightarrow)$: Let $abc \in (N : M) - \phi_{res}((N : M))$ for some nonunits $a, b, c \in R$. Then we have $ab(cM) \subseteq N$ and $ab(cM) \nsubseteq \phi(N)$. If $cM = M$, then we are done. So assume that cM is a proper submodule of M. Since N is a ϕ-1-absorbing primary submodule of M, we conclude that $ab \in (N : M)$ or $cM \subseteq M\text{-}rad(N)$ by Theorem 2.5. This gives $ab \in (N : M)$ or $c \in (M\text{-}rad(N) : M) = \sqrt{(N : M)}$. Therefore, $(N : M)$ is a ϕ_{res}-1-absorbing primary ideal of R.

$(\Leftarrow)$: Let $IJK \subseteq N$ and $IJK \nsubseteq \phi(N)$ for some proper ideals I, J of R and some submodule K of M. Then note that $IJ(K : M) \subseteq (N : M)$ by Lemma 2.7 (1) and $IJ(K : M) \nsubseteq \phi_{res}((N : M))$. Since $(N : M)$ is a ϕ_{res}-1-absorbing primary ideal of R, it is a ϕ_{res}-1-absorbing primary submodule of the R-module R. Then by Theorem 2.5, we conclude that $IJ \subseteq (N : M)$ or $(K : M) \subseteq \sqrt{(N : M)}$. Since M is a finitely generated multiplication module, we have $IJ \subseteq (N : M)$ or $K = (K : M)M \subseteq \sqrt{(N : M)}M = M\text{-}rad(N)$. Therefore, by Theorem 2.5, N is a ϕ-1-absorbing primary submodule of M. □

Next, we give some useful characterizations for n-almost 1-absorbing primary submodules of a finitely generated faithful multiplication R-module.

Theorem 2.9. Let R be a ring, M be a finitely generated faithful multiplication R-module and $n \geq 2$. For a proper submodule N of M, the following statements are equivalent.

1. N is an n-almost 1-absorbing primary submodule of M.
2. $(N :_R M)$ is an n-almost 1-absorbing primary ideal of R.
3. $N = IM$ for some n-almost 1-absorbing primary ideal I of R.

Proof. $(1) \Leftrightarrow (2)$: Suppose that $\phi_{res} = \varphi_n$, where $\varphi_n : S(R) \to S(R) \cup \{\emptyset\}$ is a function defined by $\varphi_n(I) = I^n$ for every $I \in S(R)$. Since M is a finitely generated faithful multiplication module, by Lemma 2.7 (1), $((N : M)^{n-1}N : M) = (N : M)^n$. Now, put $\phi_n(N) = (N : M)^{n-1}N$. Thus, we have $(\phi(N) : M) = \varphi_{res}(N : M)$. The rest follows from Theorem 2.8.

$(2) \Leftrightarrow (3)$: Since M is finitely generated faithful multiplication, M is cancellation. Thus for any ideals I, J of R, $IM = JM$ if and only if $I = J$. This completes the proof. □

Lemma 2.10. *[2]Let N be a submodule of a faithful multiplication R-module M and I be a finitely generated faithful multiplication ideal of R. Then,*

1. $N = (IN :_M I)$.
2. If $N \subseteq IM$, then $(JN :_M I) = J(N :_M I)$ for any ideal J of R.

Proposition 2.11. *Let N be a submodule of a finitely generated faithful multiplication R-module M and I be a finitely generated faithful multiplication ideal of R. If N is an n-almost 1-absorbing primary submodule of IM, then $(N :_M I)$ is an n-almost 1-absorbing primary submodule of M. The converse also holds if $M\text{-}rad(N :_M I) = IM\text{-}rad(N)$.*

Proof. Suppose that N is an n-almost 1-absorbing primary submodule of IM. Let $a,b\in R$ be nonunits and $m\in M$ with $abm\in(N:_M I)\backslash((N:_M I):_R M)^{n-1}(N:_M I)$. Since $((N:_M I):_R M)^{n-1}(N:_M I)=((N:_R IM)^{n-1}N:_M I)$ by Lemma 2.10, we have $abIm\subseteq N$, $abIm\nsubseteq(N:_R IM)^{n-1}N$ and Theorem 2.5 implies that either $ab\in(N:_R IM)=((N:_M I):_R M)$ or $Im\subseteq IM\text{-}rad(N)=\sqrt{(N:_R IM)}IM$. If $ab\in(N:_R IM)=((N:_M I):_R M)$, then there is nothing to prove. So assume that $Im\subseteq IM\text{-}rad(N)=\sqrt{(N:_R IM)}IM$. This gives $m\in\left(\sqrt{(N:IM)}IM:_M I\right)=\sqrt{(N:IM)}(IM:_M I)\subseteq\sqrt{(N:IM)}M=\sqrt{((N:_M I):M)}M=M\text{-}rad(N:_M I)$ by Lemma 2.10. Therefore, $(N:_M I)$ is an n-almost 1-absorbing primary submodule of M. Conversely, let J_1, J_2 be proper ideals of R and K a submodule of IM with $J_1J_2K\subseteq N$ and $J_1J_2K\nsubseteq(N:_R IM)^{n-1}N$. Then it is easy to show that $J_1J_2(K:_M I)\subseteq(N:_M I)$ and $J_1J_2(K:_M I)\nsubseteq((N:_M I):_R M)^{n-1}(N:_M I)$. Since $(N:_M I)$ is an n-almost 1-absorbing primary submodule of M, we conclude either $J_1J_2\subseteq((N:_M I):_R M)=(N:_R IM)$ or $K\subseteq IM\text{-}rad(N)=M\text{-}rad(N:_M I)$. Consequently, N is an n-almost 1-absorbing primary submodule of IM. □

Let N be a ϕ-1-absorbing primary submodule of M, $m\in M$, and a,b be nonunits in R. Then (a,b,m) is called ϕ-1-triple zero of N if $abm\in\phi(N)$, $ab\notin(N:_R M)$ and $m\notin M\text{-}rad(N)$.

Proposition 2.12. *Let R be a ring, M be an R-module, N be a ϕ-1-absorbing primary submodule of M and (a,b,m) be ϕ-1-triple zero of N. Then the following statements hold.*

1. $abN\subseteq\phi(N)$.
2. If $a,b\notin(N:_R m)$, then $a(N:_R M)m, b(N:_R M)m, a(N:_R M)N, b(N:_R M)N, (N:_R M)^2m\subseteq\phi(N)$.
3. If $a,b\notin(N:_R m)$, then $(N:_R M)^2N\subseteq\phi(N)$.

Proof. (1) Suppose $abN\nsubseteq\phi(N)$. Then there exists $n\in N$ such that $abn\notin\phi(N)$. This implies that $ab(m+n)\in N-\phi(N)$. Since N is ϕ-1-absorbing primary submodule, $ab\in(N:_R M)$ or $m+n\in M\text{-}rad(N)$, which are contradictions.

(2) Suppose $a(N:_R M)m\nsubseteq\phi(N)$. Then there exists $r\in(N:_R M)$ such that $arm\notin\phi(N)$. This implies that $a(b+r)m\in N-\phi(N)$ where $a,b+r$ are nonunits in R. As N is ϕ-1-absorbing primary submodule we have either $a(b+r)\in(N:_R M)$ or $m\in M\text{-}rad(N)$ implying $ab\in(N:_R M)$ or $m\in M\text{-}rad(N)$, contradictions. Similar argument shows that $b(N:_R M)m\subseteq\phi(N)$.

Now suppose $a(N:_R M)N\nsubseteq\phi(N)$. Then there exists $r\in(N:_R M)$ and $n\in N$ such that $arn\notin\phi(N)$. We already know that $abN, a(N:_R M)m, b(N:_R M)m\subseteq\phi(N)$. Also, note that $a(b+r)(m+n)\in N-\phi(N)$. Since $a\notin(N:_R m)$, $b+r$ is nonunit. Thus we have $a(b+r)\in(N:_R M)$ or $m+n\in M\text{-}rad(N)$, which implies that $ab\in(N:_R M)$ or $m\in M\text{-}rad(N)$, a contradiction. Thus $a(N:_R M)N\subseteq\phi(N)$. Likewise, $b(N:_R M)N\subseteq\phi(N)$.

Next, we will show that $(N:_R M)^2m\subseteq\phi(N)$. Suppose to the contrary. Then we can find some elements $r,s\in(N:_R M)$ such that $rsm\notin\phi(N)$. As $a(N:_R M)m, b(N:_R M)m\subseteq\phi(N)$, we get $(a+r)(b+s)m\in N-\phi(N)$. Since $a,b\notin(N:_R M)$, by the assumption, $a+r$ and $b+s$ are nonunit elements of R. Then we have either $(a+r)(b+s)\in(N:_R M)$ or $m\in M\text{-}rad(N)$ implying $ab\in(N:_R M)$ or $m\in M\text{-}rad(N)$ which gives a contradiction.

(3) Assume that $(N:_R M)^2N\nsubseteq\phi(N)$. Then there exists $r,s\in(N:_R M)$ and $n\in N$ such that $rsn\notin\phi(N)$. So we have $(a+r)(b+s)(m+n)\in N-\phi(N)$ by (1) and (2). Since $a,b\notin(N:_R M)$, $a+r$ and $b+r$ are nonunit elements of R. Then by the assumption $(a+r)(b+s)\in(N:_R M)$ or $m+n\in M\text{-}rad(N)$. It gives either $ab\in(N:_R M)$ or $m\in M\text{-}rad(N)$. These contradictions shows that $(N:_R M)^2N\subseteq\phi(N)$, as desired. □

Corollary 2.13. *Let N be a submodule of an R-module M. Then we have the following.*

(1) *If N is a weakly 1-absorbing primary submodule of M that is not 1-absorbing primary and there exists a ϕ-1-triple zero (a,b,m) of N with $am,bm\notin N$, then $(N:_R M)^2N=0$.*

(2) *If N is a ϕ-1-absorbing primary submodule of M that is not 1-absorbing primary, and there exists a ϕ-1-triple zero (a,b,m) of N with $am,bm\notin N$ and $\phi\leq\phi_4$, then $(N:_R M)^2N=(N:_R M)^3N$.*

(3) *Suppose that N is a ϕ-1-absorbing primary submodule of M with $\phi\leq\phi_4$. Further, assume that if (a,b,m) is a ϕ-1-triple zero of N, then $am,bm\notin N$. In this case, N is a ϕ_ω-1-absorbing primary submodule of M.*

Proof. (1) and (2) are straightforward.

(3) Assume that N is a 1-absorbing primary submodule of M. Then it is also a ϕ_w-1-absorbing primary submodule of M. Now suppose that N is not a 1-absorbing primary submodule of M. From (2), we have $(N:_R M)^2N=(N:_R M)^3N$. As N is a ϕ-1-absorbing primary submodule of M with $\phi\leq\phi_4$, N is also ϕ_4-1-absorbing primary. Since $\phi_w(N)=(N:_R M)^3N=\phi_4(N)$, N is a ϕ_ω-1-absorbing primary submodule, as desired. □

Proposition 2.14. *Let R be a ring, M be an R-module and $x\in M$ with $Rx\neq M$. If $(0:_R x)\subseteq(Rx:_R M)$ then Rx is ϕ-1-absorbing primary submodule with $\phi\leq\phi_2$ for some ϕ if and only if Rx is 1-absorbing primary submodule.*

Proof. Assume that Rx is a ϕ-1-absorbing primary submodule with $\phi\leq\phi_2$. Then it is also a ϕ_2-1-absorbing primary submodule. Let $abm\in Rx$ for some nonunits $a,b\in R$ and $m\in M$. If $abm\notin(Rx:_R M)Rx$, then we get $ab\in(Rx:_R M)$ or $m\in M\text{-}rad(Rx)$. Now suppose $abm\in(Rx:_R M)Rx$. We also have $ab(m+x)\in Rx$. If $ab(m+x)\notin(Rx:_R M)Rx$, then either $ab\in(Rx:_R M)$ or $m+x\in M\text{-}rad(Rx)$. So assume that $ab(m+x)\in(Rx:_R M)Rx$. This implies that $abx\in(Rx:_R M)Rx$. Then there exists $r\in(Rx:_R M)$ such that $abx=rx$ and it gives $(ab-r)x=0$. Hence, we get $ab-r\in(0:_R Rx)\subseteq(Rx:_R M)$ and it means $ab\in(Rx:_R M)$, as needed. The other direction is immediate. □

Theorem 2.15. Let M_1, M_2 be two R-modules, $f : M_1 \to M_2$ be a module epimorphism and $\phi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be functions for $i = 1,2$.

1. If N_2 is a ϕ_2-1-absorbing primary submodule of M_2 and $\phi_1(f^{-1}(N_2)) = f^{-1}(\phi_2(N_2))$, then $f^{-1}(N_2)$ is a ϕ_1-1-absorbing primary submodule of M_1.
2. If N_1 is a ϕ_1-1-absorbing primary submodule of M_1 containing $Ker(f)$ and $\phi_2(f(N_1)) = f(\phi_1(N_1))$, then $f(N_1)$ is a ϕ_2-1-absorbing primary submodule of M_2.

Proof. (1) Suppose N_2 is a ϕ_2-1-absorbing primary submodule of M_2. Then $f^{-1}(N_2)$ is a proper submodule of M_1 as f is onto. Let a,b be nonunit elements of R and $m_1 \in M_1$ with $abm_1 \in f^{-1}(N_2)\backslash\phi_1(f^{-1}(N_2))$. Then clearly we have $abf(m_1) \in N_2\backslash\phi_2(N_2)$ as $\phi_1(f^{-1}(N_2)) = f^{-1}(\phi_2(N_2))$. It follows $ab \in (N_2 :_R M_2)$ or $f(m_1) \in M_2$-rad(N_2). Since $f^{-1}(M_2\text{-rad}(N_2)) \subseteq M_1\text{-rad}(f^{-1}(N_2))$ by [20, Corollary 1.3], we conclude either $ab \in (f^{-1}(N_2) :_R M_1)$ or $m_1 \in M_1$-rad$(f^{-1}(N_2))$ and so we are done.

(2) Let $abm_2 \in f(N_1)\backslash\phi_2(f(N_1))$ where $a,b \in R$ are nonunits and $m_2 \in M_2$. Choose $m_1 \in M_1$ with $m_2 = f(m_1)$. Hence $f(abm_1) \in f(N_1)$ and since Ker$(f) \subseteq N_1$, that is $abm_1 \in N_1$. Clearly $abm_1 \notin \phi_1(N_1)$ as $\phi_2(f(N_1)) = f(\phi_1(N_1))$ which implies either $ab \in (N_1 :_R M_1)$ or $m_1 \in M_1$-rad(N_1). Thus $ab \in (f(N_1) :_R M_2)$ or $m_2 \in f(M_1\text{-rad}(N_1)) = M_2\text{-}rad(f(N_1))$ by [20, Corollary 1.3]. □

Now, the following result follows immediately from the previous theorem.

Corollary 2.16. *Let K and N be submodules of a multiplication R-module M with $N \subseteq K$ and $n \geq 2$.*

1. If K is a ϕ_n-1-absorbing primary submodule of M, then K/N is a ϕ_n-1-absorbing primary submodule of M/N.
2. If K is a ϕ_ω-1-absorbing primary submodule of M, then K/N is a ϕ_ω-1-absorbing primary submodule of M/N.

Proof. Observe that $\phi_n(K) = (K :_R M)^{n-1}K = K^n$ since M is multiplication. Consider the canonical epimorphism $\pi : M \to M/N$ satisfies the equalities $\phi_n(\pi(K)) = \phi_n(K/N) = (K/N)^n = K^n/N = \phi_n(K)/N = \pi(\phi_n(K))$, and $\phi_\omega(\pi(K)) = \cap_{n=1}^{\infty}(K/N)^n = (\cap_{n=1}^{\infty}K^n)/N = \pi(\phi_\omega(K))$, so the claim is obtained. □

For any submodule N of M define a function $\phi_N : S(M/N) \to S(M/N) \cup \{\emptyset\}$ by $\phi_N(K/N) = (\phi(K) + N)/N$ where $N \subseteq K$ and $\phi_N(K/N) = \emptyset$ if $\phi(K) = \emptyset$. Moreover, $\phi_N(K/N) \subseteq K/N$. Next, we indicate the relation between ϕ-1-absorbing primary submodules and weakly 1-absorbing primary submodules.

Theorem 2.17. Let K and N be two submodules of an R-module M with $N \subseteq K$. The following statements are satisfied.

1. Suppose that $N \subseteq \phi(K)$. Then K is a ϕ-1-absorbing primary submodule of M if and only if K/N is a ϕ_N-1-absorbing primary submodule of M/N.
2. K is a ϕ-1-absorbing primary submodule of M if and only if $K/\phi(K)$ is a weakly 1-absorbing primary submodule of $M/\phi(K)$.
3. If N is a ϕ-1-absorbing primary submodule of M with $\phi(K) \subseteq N$, then K/N is a weakly 1-absorbing primary submodule of M/N.
4. If N is a ϕ-1-absorbing primary submodule of M with $\phi(N) \subseteq \phi(K)$ and K/N is a weakly 1-absorbing primary submodule of M/N, then K is a ϕ-1-absorbing primary submodule of M.

Proof. (1) Let $ab\overline{m} \in K/N - \phi_N(K/N)$ for nonunit elements $a,b \in R$ and $\overline{m} = m + N \in M/N$. Then $abm \in K - \phi(K)$. As K is ϕ-1-absorbing primary submodule, $ab \in (K :_R M) = (K/N :_R M/N)$ or $m \in M\text{-}rad(K)$. This implies that $ab \in (K/N :_R M/N)$ and $\overline{m} = m + N \in M/N\text{-}rad(K/N)$. Conversely, suppose K/N is a ϕ_N-1-absorbing primary submodule of M/N and let $abm \in K\backslash\phi(K)$ for some nonunits $a,b \in R$ and $m \in M$. If $ab(m + K) \in \phi_N(K/N) = (\phi(K) + N)/N$, then $abm \in \phi(K)$ since $N \subseteq \phi(K)$, a contradiction. Thus $ab(m + K) \in (K/N)\backslash\phi_N(K/N)$ implies that either $ab \in (K/N :_R M/N) = (K :_R M)$ or $m + N \in M/N\text{-}rad(K/N)$. Thus $ab \in (K :_R M)$ or $m \in M\text{-}rad(N)$.

(2) Put $N = \phi(K)$ in (1). Since $\phi_{\phi(K)}(K\backslash\phi(K)) = 0_{M/\phi(K)}$, the result is a particular case of (1).

(3) Suppose $0_{M/N} \neq ab(m + N) \in K/N$ for some nonunits $a,b \in R$ and $m \in M$. Since $\phi(K) \subseteq N$, we have $abm \in K\backslash\phi(K)$ which implies either $ab \in (K :_R M)$ or $m \in M\text{-}rad(K)$. Thus, we have either $ab \in (K/N :_R M/N)$ or $m + N \in M/N\text{-}rad(K/N)$, as needed.

(4) Let $abm \in K\backslash\phi(K)$ for some nonunits $a,b \in R$ and $m \in M$. If $abm \in N$, then we have $abm \in N\backslash\phi(N)$ as $\phi(N) \subseteq \phi(K)$. This implies either $ab \in (N :_R M) \subseteq (K :_R M)$ or $m \in M\text{-}rad(N) \subseteq M\text{-}rad(K)$. If $abm \notin N$, then $0_{M/N} \neq ab(m + N) \in K/N$. Since K/N is a weakly 1-absorbing primary submodule of M/N, it follows $ab \in ((K/N) :_R (M/N))$ or $(m + N) \in M/N\text{-}rad(K/N)$. Thus $ab \in (K :_R M)$ or $m \in M\text{-}rad(K)$, so we are done. □

Proposition 2.18. *Let K and L be two ϕ-1-absorbing primary submodules of an R-module M that are not 1-absorbing primary submodules. If $N = K + L$ is proper, $\phi(K) \subseteq L$ and $\phi(L) \subseteq \phi(N)$, then N is a ϕ-1-absorbing primary submodule of M.*

Proof. Suppose that $\phi(K) \subseteq L$ and $\phi(L) \subseteq \phi(N)$. Since $\phi(K) \subseteq L$, clearly $\phi(K) \subseteq K \cap L$ and $K/K \cap L$ is a weakly 1-absorbing primary submodule of $M/K \cap L$ by Theorem 2.17 (3). Since $N/L \cong K/K \cap L$, we conclude that N/L is a weakly 1-absorbing primary submodule of M/L. Now, $\phi(L) \subseteq \phi(N)$ implies N is a ϕ-1-absorbing primary submodule of M by Theorem 2.17 (4). □

Consider $\phi : S(N) \to S(N) \cup \{\emptyset\}$. Define $\phi_S : S(S^{-1}M) \to S(S^{-1}M) \cup \{\emptyset\}$ by $\phi_S(S^{-1}N) = S^{-1}\phi(N)$ and if $\phi(N) = \emptyset$ then $\phi_S(S^{-1}N) = \emptyset$. It is clear that $\phi_S(N) \subseteq N$.

Theorem 2.19. Let R be a ring, M be an R-module and N be a submodule of M. If N is a ϕ-1-absorbing primary submodule of M, S is a multiplicatively closed subset of R with $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is ϕ_S-1-absorbing primary submodule of $S^{-1}M$. Moreover, if $S^{-1}N \neq S^{-1}\phi(N)$ then $S^{-1}N \cap M \subseteq M\text{-}rad(N)$.

Proof. Let $\frac{a}{s_1}\frac{b}{s_2}\frac{m}{s_3} \in S^{-1}N - \phi_S(S^{-1}N)$ for some nonunits $\frac{a}{s_1}, \frac{b}{s_2} \in S^{-1}R$ and $\frac{m}{s_3} \in S^{-1}M$. Then there exists $u \in S$ such that $uabm \in N$ but $uabm \notin \phi(N)$. If $uabm \in \phi(N)$, then $\frac{a}{s_1}\frac{b}{s_2}\frac{m}{s_3} \in S^{-1}\phi(N) = \phi_S(S^{-1}N)$, a contradiction. So $uabm \in N - \phi(N)$. As ua, b are nonunits in R, $m \in M$ and N is ϕ-1-absorbing primary submodule of M, either $uab \in (N :_R M)$ or $m \in M\text{-}rad(N)$. This implies that $\frac{a}{s_1}\frac{b}{s_2} = \frac{uab}{us_1s_2} \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$ or $\frac{m}{s_3} \in S^{-1}rad(N) \subseteq S^{-1}M\text{-}rad(S^{-1}N)$, as desired. Now we will show that $S^{-1}N \cap M \subseteq M\text{-}rad(N)$. Let $x \in S^{-1}N$. Then there exists $s \in S$ such that $sx \in N$. If s is unit, then $x \in N \subseteq M\text{-}rad(N)$. So assume that s is nonunit. If $s^2x = ssx \notin \phi(N)$, then $s^2 \in (N :_R M)$ or $x \in M\text{-}rad(N)$ that the former case gives a contradiction. If $s^2x \in \phi(N)$, then $x \in S^{-1}\phi(N) \cap M$. This means that $S^{-1}N \cap M \subseteq (M\text{-}rad(N)) \cup (S^{-1}\phi(N) \cap M)$. Thus, $S^{-1}N \cap M \subseteq M\text{-}rad(N)$ or $S^{-1}N \cap M = S^{-1}\phi(N) \cap M$ that the latter gives a contradiction. □

To characterize ϕ-1-absorbing primary submodules in cartesian product of modules, we will use the concept of pseudo primary submodules. In fact pseudo primary submodules is an extension of primary ideals in commutative rings to nonzero unital modules such as primary submodules. However, these concepts are different (See the following examples). Before showing the differences between pseudo primary submodules and primary submodules, we give the following lemma which characterizes pseudo primary submodules.

Lemma 2.20. *Let N be a proper submodule of an R-module M. Then N is a pseudo primary submodule of M if and only if for all $a \in R$ and a submodule K of M with $aK \subseteq N$, $a \in (N :_R M)$ or $K \subseteq M\text{-}rad(N)$.*

Proof. Suppose that N is a pseudo primary submodule of M and $aK \subseteq N$ with $a \notin (N :_R M)$ for some $a \in R$, a submodule K of M. Let $k \in K$. Then $ak \in N$ and $a \notin (N :_R M)$ imply $k \in M\text{-}rad(N)$, as needed. Conversely, let $a \in R$, $m \in M$ with $am \in N$. Taking $K = Rm$ in (2), we are done. □

Example 2.21. (A primary submodule which is not pseudo primary) Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}_2$ and take $N = 4\mathbb{Z} \times (\bar{0})$. Then it is easy to see that $(N : M) = 4\mathbb{Z}$. Let $a(m, \bar{n}) = (am, \overline{an}) \in N$ for some $a, m, n \in \mathbb{Z}$. Assume that $a \notin \sqrt{(N : M)} = 2\mathbb{Z}$. Since $\gcd(2, a) = 1$, $4|am$ and $2|an$, we have $4|m$ and $2|n$. Thus we have $(m, \bar{n}) \in N$. Hence, N is a primary submodule of M. On the other hand, note that $P = 2\mathbb{Z} \times (\bar{0})$ is a prime submodule containing N. Then $M\text{-}rad(N) \subsetneq P$. Since $2(2, \bar{1}) = (4, \bar{0}) \in N$, $2 \notin (N : M)$ and $(2, \bar{1}) \notin M\text{-}rad(N)$, it follows that N is not a pseudo primary submodule of M.

Example 2.22. (A pseudo primary submodule which is not primary) Consider the $\mathbb{Z}$-module $E(p) = \{\alpha = \frac{r}{p^n} + \mathbb{Z} : r \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$ where p is a prime number. Then by [24], we know that all submodules of $E(p)$ has the form $G_t = \{\alpha = \frac{r}{p^t} + \mathbb{Z} : r \in \mathbb{Z}\}$ for a fixed $t \in \mathbb{N} \cup \{0\}$. Also $E(p)$ has not any prime submodule. Then $E(p)\text{-}rad(G_t) = E(p)$ and so G_t is a pseudo primary submodule for each $t \in \mathbb{N} \cup \{0\}$. Since $p(\frac{1}{p^{t+1}} + \mathbb{Z}) = \frac{1}{p^t} + \mathbb{Z} \in G_t$, $p \notin \sqrt{(G_t : E(p))} = (0)$ and $\frac{1}{p^{t+1}} + \mathbb{Z} \notin G_t$, it follows that G_t is not a primary submodule of $E(p)$.

Proposition 2.23.
1. *Every primary submodule of a multiplication module is a pseudo primary submodule.*
2. *Every pseudo primary submodule a finitely generated multiplication module is a primary submodule.*
3. *In a finitely generated multiplication module, pseudo primary submodules and primary submodules coincide.*

Proof. (1) Let N be a primary submodule of a multiplication R-module M. First note that $(N : M)$ is a primary ideal since N is a primary submodule of M. Let $am \in N$ for some $a \in R$ and $m \in M$. As M is a multiplication module, $a(Rm : M) \subseteq (N : M)$ which implies that $a \in (N : M)$ or $(Rm : M) \subseteq \sqrt{(N : M)}$. By Lemma 2.7 (2), we know that $\sqrt{(N : M)} \subseteq (M\text{-}rad(N) : M)$. Since M is multiplication, we have $a \in (N : M)$ or $m \in Rm = (Rm : M)M \subseteq (M\text{-}rad(N) : M)M = M\text{-}rad(N)$. Hence, N is a pseudo primary submodule of M.

(2) Let N be a pseudo primary submodule of a finitely generated multiplication R-module M. First we will show that $(N : M)$ is a primary ideal of R. To see this, take $ab \in (N : M)$ for some $a, b \in R$. Then we have $a(bM) \subseteq N$. Then by Lemma 2.20, $a \in (N : M)$ or $bM \subseteq M\text{-}rad(N)$. This gives $a \in (N : M)$ or $b \in (M\text{-}rad(N) : M) = \sqrt{(N : M)}$. Hence, $(N : M)$ is a primary ideal of R. Then one can easily see that N is a primary submodule.

(3) Follows from (1) and (2). □

Let R_1, R_2 be commutative rings, M_1, M_2 be R_1, R_2-modules, N_1, N_2 be submodules of M_1, M_2, respectively and $\phi_1 : \mathcal{S}(M_1) \to \mathcal{S}(M_1) \cup \{\emptyset\}$, $\phi_2 : \mathcal{S}(M_2) \to \mathcal{S}(M_2) \cup \{\emptyset\}$ be two functions. Suppose that $M = M_1 \times M_2$ and $\phi : \mathcal{S}(M) \to \mathcal{S}(M) \cup \{\emptyset\}$ is a function defined by $\phi(N_1 \times N_2) = \phi_1(N_1) \times \phi_2(N_2)$ for each submodule N_k of M_k. Then ϕ is denoted by $\phi = \phi_1 \times \phi_2$. The rest of this section, we discuss ϕ-1-absorbing primary submodules of $M_1 \times M_2$. First, we need the following lemmas.

Lemma 2.24. *Let M be an R-module and $K \subseteq N$ be two submodules of M. Then N is a pseudo primary submodule of M if and only if N/K is a pseudo primary submodule of M/K.*

Proof. Suppose that N is a pseudo primary submodule of M and $a(m+K) \in N/K$ for some $a \in R, m \in M$. Then we have $am \in N$ which implies that $a \in (N :_R M)$ or $m \in M\text{-}rad(N)$. Then we conclude that $a \in (N/K :_R M/K)$ or $m + K \in M/K\text{-}rad(N/K)$, as needed. For the converse, let N/K be a pseudo primary submodule of M/K and $am \in N$ for some $a \in R, m \in M$. Then we have $a(m+K) \in N/K$. As N/K is a pseudo primary submodule of M/K, we get $a \in (N/K :_R M/K) = (N :_R M)$ or $m+K \in N/K$ which implies that $a \in (N :_R M)$ or $m \in N$. □

Lemma 2.25. *Let M_i be an R_i-module, N_i be a submodule of M_i for each $i = 1,2$. Suppose that $R = R_1 \times R_2$, $M = M_1 \times M_2$ and $N = N_1 \times N_2$ is a proper submodule of M. Then N is a pseudo primary submodule of M if and only if one of the following statements holds.*

1. $M_1\text{-}rad(N_1) = M_1$ and $M_2\text{-}rad(N_2) = M_2$.
2. $N_1 = M_1$ and N_2 is a pseudo primary submodule of M_2.
3. $N_2 = M_2$ and N_1 is a pseudo primary submodule of M_1.

Proof. $(\Rightarrow)$: Suppose that N is a pseudo primary submodule of M. If $M_1\text{-}rad(N_1) = M_1$ and $M_2\text{-}rad(N_2) = M_2$, then we are done. So without loss of generality, we may assume that $M_1 \neq M_1\text{-}rad(N_1)$. Now, we will show that $N_2 = M_2$ and N_1 is a pseudo primary submodule of M_1. First, choose $m_1 \in M_1 \backslash M_1\text{-}rad(N_1)$. Then note that $(0,1)(m_1,0) = (0,0) \in N$. Since N is a pseduo primary submodule of M, we have $(0,1) \in (N :_R M)$ or $(m_1,0) \in M\text{-}rad(N)$. This gives $1 \in (N_2 :_{R_2} M_2)$ or $m_1 \in M_1\text{-}rad(N_1)$. The latter case is impossible. Thus we have $N_2 = M_2$. Let $am \in N_1$ for some $a \in R_1, m \in M_1$. Then we have $(a,0)(m,0) = (am,0) \in N$. Since N is a pseduo primary submodule, we conclude that $(a,0) \in (N :_R M)$ or $(m,0) \in M\text{-}rad(N)$. Then we get $a \in (N_1 :_{R_1} M_1)$ or $m_1 \in M_1\text{-}rad(N_1)$. Thus, N_1 is a pseudo primary submodule of M_1. If $M_2 \neq M_2\text{-}rad(N_2)$, then one can similarly show that $N_1 = M_1$ and N_2 is a pseudo primary submodule of M_2.

$(\Leftarrow)$: Suppose (1) holds. Then it is clear that $M\text{-}rad(N) = M$ so N is a pseudo primary submodule of M. Let $N_1 = M_1$ and N_2 is a pseudo primary submodule of M_2. Let $(a_1,a_2)(m_1,m_2) \in N$ for some $a_i \in R_i$ and $m_i \in M_i$. Then we have $a_2m_2 \in N_2$. Since N_2 is a pseudo primary submodule of M_2, we get $a_2 \in (N_2 :_{R_2} M_2)$ or $m_2 \in M_2\text{-}rad(N_2)$. This implies that $(a_1,a_2) \in (N :_R M)$ or $(m_1,m_2) \in M\text{-}rad(N)$. Therefore N is a pseudo primary submodule of M. If $N_2 = M_2$ and N_1 is a pseudo primary submodule of M_1, then one can similarly show that N is a pseudo primary submodule of M. □

Corollary 2.26. *Let M_i be a multiplication R_i-module, N_i be a submodule of M_i for each $i = 1,2$. Suppose that $R = R_1 \times R_2$, $M = M_1 \times M_2$ and $N = N_1 \times N_2$ is a proper submodule of M. Then N is a pseudo primary submodule of M if and only if either $N_1 = M_1$ and N_2 is a pseudo primary submodule of M_2 or $N_2 = M_2$ and N_1 is a pseudo primary submodule of M_1.*

Proof. First, note that M is a multiplication module. For any submodule N of M, we know that $N \neq M$ if and only if $M\text{-}rad(N) \neq M$. The rest follows from the previous lemma. □

Theorem 2.27. Let R_1,R_2 be commutative rings, M_1,M_2 be multiplication R_1,R_2-modules, respectively and $\phi_1 : \mathcal{S}(M_1) \to \mathcal{S}(M_1) \cup \{\emptyset\}$, $\phi_2 : \mathcal{S}(M_2) \to \mathcal{S}(M_2) \cup \{\emptyset\}$ be two functions. Suppose that $N = N_1 \times N_2$ is a proper submodule of M, where N_i is a submodule of M_i for each $i = 1,2$, and $\phi = \phi_1 \times \phi_2$. Further assume that $(\phi_i(N_i) :_{R_i} M_i)$ is not unique maximal ideal of R_i for each $i = 1,2$. Then N is a ϕ-1-absorbing primary submodule of M if and only if one of the following conditions must be hold.

1. $\phi(N) = N$.
2. $N_1 = M_1$ and N_2 is a pseudo primary submodule of M_2.
3. $N_2 = M_2$ and N_1 is a pseudo primary submodule of M_1.

Proof. $(\Rightarrow)$: Suppose that N is a ϕ-1-absorbing primary submodule of M. By Theorem 2.17, $N/\phi(N)$ is a weakly 1-absorbing primary submodule of $M/\phi(N)$. If $\phi(N) = N$, then we are done. So assume that $\phi(N) \neq N$, that is, $N/\phi(N)$ is a nonzero submodule of $M/\phi(N)$. On the other hand, note that $M/\phi(N) \simeq M_1/\phi_1(N_1) \times M_2/\phi_2(N_2)$ and $N/\phi(N) \simeq N_1/\phi_1(N_1) \times N_2/\phi_2(N_2)$. Also, note that $ann_R(M_i/\phi_i(N_i)) = (\phi_i(N_i) :_{R_i} M_i)$ is not unique maximal ideal of R_i. Then by [28, Theorem 5], $N_1/\phi_1(N_1) = M_1/\phi_1(N_1)$ and $N_2/\phi_2(N_2)$ is a pseudo primary submodule of $M_2/\phi_2(N_2)$ or $N_2/\phi_2(N_2) = M_2/\phi(N_2)$ and $N_1/\phi_1(N_1)$ is a pseudo primary submodule of $M_1/\phi_1(N_1)$. By Lemma 2.24, $N_1 = M_1$ and N_2 is a pseudo primary submodule of M_2 or $N_2 = M_2$ and N_1 is a pseudo primary submodule of M_1. This completes the proof.

$(\Leftarrow)$: Suppose that (1) holds. Then N is trivially a ϕ-1-absorbing primary submodule of M. Now, without loss of generality, we may assume that (2) holds, that is, $N_1 = M_1$ and N_2 is a pseudo primary submodule of M_2. Then by Lemma 2.25, N is a pseudo primary submodule of M. Again by Proposition 2.3, N is a ϕ-1-absorbing primary submodule of M. □

Theorem 2.28. Let R_1,R_2 be commutative rings, M_1,M_2 be multiplication R_1,R_2-modules, respectively and $\phi_1 : \mathcal{S}(M_1) \to \mathcal{S}(M_1) \cup \{\emptyset\}$, $\phi_2 : \mathcal{S}(M_2) \to \mathcal{S}(M_2) \cup \{\emptyset\}$ be two functions. Suppose that $N = N_1 \times N_2$ is a nonzero proper submodule of M where N_i is a submodule of M_i for each $i = 1,2$ and $\phi = \phi_1 \times \phi_2$. Further assume that $(\phi_i(N_i) :_{R_i} M_i)$ is not unique maximal ideal of R_i for each $i = 1,2$ and $\phi(N) \neq N$. The following statements are equivalent.

1. N is a ϕ-1-absorbing primary submodule of M.

2. $N_1 = M_1$ and N_2 is a pseudo primary submodule of M_2 or $N_2 = M_2$ and N_1 is a pseudo primary submodule of M_1.
3. N is a pseudo primary submodule of M.
4. N is a 1-absorbing primary submodule of M.
5. N is a weakly 1-absorbing primary submodule of M.

Proof. Follows from [28, Theorem 5], Theorem 2.27, Proposition 2.3. □

3 1-absorbing primary like ideals in trivial extension

This section is dedicated to the study of 1-absorbing primary ideals and their generalizations in trivial extension of an R-module M. Let M be an R-module. The trivial extension of an R-module M, denoted by $R \ltimes M = R \oplus M$, is a commutative ring with componentwise addition and the following multiplication: $(a,x)(b,y) = (ab, ay+bx)$ for every $a,b \in R$ and $x,y \in M$ [7]. If I is an ideal of R and N is a submodule of M, then $I \ltimes N$ is an ideal of $R \ltimes M$ if and only if $IM \subseteq N$ [7, Theorem 3.1]. In that case, $I \ltimes N$ is called a homogeneous ideal of $R \ltimes M$. Also, by [7, Theorem 3.2], we know that $\sqrt{I \ltimes N} = \sqrt{I} \ltimes M$.

Theorem 3.1. Let M be an R-module, I be a proper ideal of R and N be a submodule of M such that $IM \subseteq N$.

1. If $I \ltimes N$ is a 1-absorbing primary ideal of $R \ltimes M$, then I is a 1-absorbing primary ideal of R.
2. If I is a 1-absorbing primary ideal of R and $(N :_M x) = N$ for every $x \in R - \sqrt{I}$, then $I \ltimes N$ is a 1-absorbing primary ideal of $R \ltimes M$.
3. Suppose that $(N :_M x) = N$ for every $x \in R - \sqrt{I}$. Then $I \ltimes N$ is a 1-absorbing primary ideal of $R \ltimes M$ if and only if I is a 1-absorbing primary ideal of R.

Proof. (1): Follows from [26, Proposition 6].

(2): Let $(a,m_1)(b,m_2)(c,m_3) = (abc, abm_3+acm_2+bcm_1) \in I \ltimes N$ for some nonunits $(a,m_1),(b,m_2),(c,m_3) \in R \ltimes M$. Then $abc \in I$ and a,b,c are nonunits in R. Since I is a 1-absorbing primary ideal of R, we have $ab \in I$ or $c \in \sqrt{I}$. If $c \in \sqrt{I}$, then we have $(c,m_3) \in \sqrt{I \ltimes N} = \sqrt{I} \ltimes M$ which completes the proof. So assume that $c \notin \sqrt{I}$. Then by assumption, $(N :_M c) = N$. As $ab \in I$, we conclude that $abm_3 \in IM \subseteq N$. Since $abm_3 + acm_2 + bcm_1 \in N$, we have $acm_2 + bcm_1 \in N$. This implies that $am_2 + bm_1 \in (N :_M c) = N$. Then we have $(a,m_1)(b,m_2) = (ab, am_2 + bm_1) \in I \ltimes N$. Hence, $I \ltimes N$ is a 1-absorbing primary ideal of $R \ltimes M$.

(3): Follows from (1) and (2). □

The converse of Theorem 3.1 (1) is not true in general. See the following example.

Example 3.2. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{18}$ and the trivial extension $\mathbb{Z} \ltimes \mathbb{Z}_{18}$. It is clear that $I = (0)$ is a 1-absorbing primary ideal of $\mathbb{Z}$. However, $(0,\overline{1})(2,\overline{1})(3,\overline{1}) = (0,\overline{6}) \in (0) \ltimes (\overline{6})$, $(0,\overline{1})(2,\overline{1}) = (0,\overline{2}) \notin (0) \ltimes (\overline{6})$ and $(3,\overline{1}) \notin \sqrt{(0) \ltimes (\overline{6})} = (0) \ltimes \mathbb{Z}_{18}$. Therefore, $(0) \ltimes (\overline{6})$ is not a 1-absorbing primary ideal of $\mathbb{Z} \ltimes \mathbb{Z}_{18}$.

Theorem 3.3. Let M be an R-module, I be a proper ideal of R and N a submodule of M such that $IM \subseteq N$. The following statements are satisfied.

1. If $I \ltimes N$ is a weakly 1-absorbing primary ideal of $R \ltimes M$, then I is a weakly 1-absorbing primary ideal of R.
2. $I \ltimes M$ is a weakly 1-absorbing primary ideal of $R \ltimes M$ if and only if I is a weakly 1-absorbing primary ideal of R and whenever $abc = 0$, $ab \notin I$ and $c \notin \sqrt{I}$ for some nonunits $a,b,c \in R$ then $ab, ac, bc \in ann(M)$.

Proof. (1) Suppose that $I \ltimes N$ is a weakly 1-absorbing primary ideal of $R \ltimes M$. Let $0 \neq abc \in I$ for some nonunits $a,b,c \in R$. Then $(0,0) \neq (a,0)(b,0)(c,0) = (abc,0) \in I \ltimes N$ and $(a,0),(b,0),(c,0)$ are nonunits in $R \ltimes M$. This gives $(ab,0) \in I \ltimes N$ or $(c,0) \in \sqrt{I \ltimes N} = \sqrt{I} \ltimes M$. Hence, $ab \in I$ or $c \in \sqrt{I}$ which completes the proof.

(2) Suppose that $I \ltimes M$ is a weakly 1-absorbing primary ideal of $R \ltimes M$. Then by (1), I is a weakly 1-absorbing primary ideal of R. Now, let $abc = 0$, $ab \notin I$ and $c \notin \sqrt{I}$ for some nonunits $a,b,c \in R$. Now, we will show that $ab, ac, bc \in ann(M)$. Let $ab \notin ann(M)$. Then there exists $m \in M$ such that $abm \neq 0$. This gives $(0,0) \neq (a,0)(b,0)(c,m) = (0, abm) \in I \ltimes M$. As $I \ltimes M$ is a weakly 1-absorbing primary ideal of $R \ltimes M$, we have either $(a,0)(b,0) \in I \ltimes M$ or $(c,m) \in \sqrt{I \ltimes M} = \sqrt{I} \ltimes M$. Then we get $ab \in I$ or $c \in \sqrt{I}$ that both of them are contradictions. Similarly, one can show that $ac, bc \in ann(M)$. For the converse, let $(0,0) \neq (a,m_1)(b,m_2)(c,m_3) = (abc, abm_3 + acm_2 + bcm_1) \in I \ltimes M$ for some nonunits $(a,m_1),(b,m_2),(c,m_3)$ in $R \ltimes M$. Then $abc \in I$ and a,b,c are nonunits in R. If $abc \neq 0$, then $ab \in I$ or $c \in \sqrt{I}$ which yields that $(a,m_1)(b,m_2) \in I \ltimes M$ or $(c,m_3) \in \sqrt{I} \ltimes M = \sqrt{I \ltimes M}$. This completes the proof. Now, assume that $abc = 0$. **Case 1:** let $ab \notin I$ and $c \notin \sqrt{I}$. Then by assumption $ab, ac, bc \in ann(M)$ which gives $abm_3 + acm_2 + bcm_1 = 0$. Then we have $(a,m_1)(b,m_2)(c,m_3) = (abc, abm_3 + acm_2 + bcm_1) = (0,0)$ which is a contradiction. **Case 2:** let $ab \in I$ or $c \in \sqrt{I}$. In this case, $(a,m_1)(b,m_2) \in I \ltimes M$ or $(c,m_3) \in \sqrt{I} \ltimes M = \sqrt{I \ltimes M}$. Therefore, $I \ltimes M$ is a weakly 1-absorbing primary ideal of $R \ltimes M$. □

Suppose that $\phi : S(R) \to S(R) \cup \{\emptyset\}$ is a function, $\psi : S(M) \to S(M) \cup \{0\}$ is a function and $\lambda : S(R \ltimes M) \to S(R \ltimes M) \cup \{\emptyset\}$ is a function defined by $\lambda_{\phi \ltimes \psi}(I \ltimes N) = \phi(I) \ltimes \psi(N)$ for every ideal I of R and every submodule N of M. Now, we investigate the homogeneous $\lambda_{\phi \ltimes \varphi}$-1-absorbing primary ideals of $R \ltimes M$.

Theorem 3.4. Let M be an R-module, I be a proper ideal of R and N a submodule of M such that $IM \subseteq N$. The following statements are satisfied.

1. If $I \ltimes N$ is a $\lambda_{\phi\ltimes\psi}$-1-absorbing primary ideal of $R \ltimes M$, then I is a ϕ-1-absorbing primary ideal of R.
2. $I \ltimes M$ is a $\lambda_{\phi\ltimes\psi}$-1-absorbing primary ideal of $R \ltimes M$ if and only if I is a ϕ-1-absorbing primary ideal of R and whenever $abc \in \phi(I)$, $ab \notin I$ and $c \notin \sqrt{I}$ for some nonunits $a, b, c \in R$ then $ab, ac, bc \in (\psi(M) : M)$.

Proof. (1) It is similar to Theorem 3.3 (1).

(2) Suppose that $I \ltimes M$ is a $\lambda_{\phi\ltimes\psi}$-1-absorbing primary ideal of $R \ltimes M$. Then by (1), I is a ϕ-1-absorbing primary ideal of R. Assume that $abc \in \phi(I)$, $ab \notin I$ and $c \notin \sqrt{I}$ for some nonunits $a, b, c \in R$. Now, we will show that $ab, ac, bc \in (\psi(M) : M)$. Assume that $ab \notin (\psi(M) : M)$. Then there exists $m \in M$ such that $abm \notin \psi(M)$. Then we have $(a,0)(b,0)(c,m) = (abc, abm) \in I \ltimes M - \lambda_{\phi\ltimes\psi}(I \ltimes M)$. Since $I \ltimes M$ is a $\lambda_{\phi\ltimes\psi}$-1-absorbing primary ideal of $R \ltimes M$, we conclude that $(a,0)(b,0) \in I \ltimes M$ or $(c,m) \in \sqrt{I \ltimes M} = \sqrt{I} \ltimes M$. However, both cases are impossible. Thus, $ab \in (\psi(M) : M)$. Likewise, $ac, bc \in (\psi(M) : M)$. For the converse, assume that $(a,m_1)(b,m_2)(c,m_3) = (abc, abm_3 + acm_2 + bcm_1) \in I \ltimes M - \lambda_{\phi\ltimes\psi}(I \ltimes M)$ for some nonunits $(a,m_1), (b,m_2), (c,m_3) \in R \ltimes M$. Then we have $abc \in I$. If $abc \notin \phi(I)$, then we are done. So assume that $abc \in \phi(I)$. Now, we will show that $ab \in I$ or $c \in \sqrt{I}$. Suppose not. Then by assumption, we have $ab, ac, bc \in (\psi(M) : M)$. This gives $abm_3, acm_2, bcm_1 \in \psi(M)$. Thus we have $(abc, abm_3 + acm_2 + bcm_1) = (a,m_1)(b,m_2)(c,m_3) \in \lambda_{\phi\ltimes\psi}(I \ltimes M)$ which is a contradiction. Therefore, $I \ltimes M$ is a $\lambda_{\phi\ltimes\psi}$-1-absorbing primary ideal of $R \ltimes M$. □

Remark 3.5. By Example 3.2, one can see that the converses of Theorem 3.3 (1) and Theorems 3.4 (1) are not true in general. Indeed, take $R = \mathbb{Z}$, $M = \mathbb{Z}_{18}$ and $I = (0)$, $N = (\bar{6})$. Suppose that $\phi(I) = (0)$ and $\psi(N) = (\bar{0})$. Then I is a ϕ-1-absorbing primary (weakly 1-absorbing primary) ideal of R. However, one can easily show that $I \ltimes N$ is not a $\lambda_{\phi\ltimes\psi}$-1-absorbing primary (weakly 1-absorbing primary) ideal of $R \ltimes M$.

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