Moroccan Journal of Algebra and Geometry with Applications
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco

Volume 3, Issue 1 (2024), pp 187-196

ISSN: 2820-7114

Title :

On Residually (Completely) Integrally Closed Rings

Author(s):

Ali Tamoussit

Moroccan Journal of Algebra and Geometry with Applications
Vol. 3(1) (2024), 187–196
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco
ISSN: 2820-7114

On Residually (Completely) Integrally Closed Rings

Ali Tamoussit
Department of Mathematics, The Regional Center for Education and Training Professions Souss Massa, Inezgane, Morocco.
e-mail: *a.tamoussit@crmefsm.ac.ma*

Communicated by El Hassan El Kinani
(Received 09 November 2023, Revised 06 January 2024, Accepted 11 January 2024)

Abstract. All rings considered in this paper are commutative rings with identity. A ring R is called a *residually completely integrally closed* (for short, *RCIC*) *ring* if the integral domain R/P is completely integrally closed, for all prime ideals P of R. The main goal of this paper is to study the behavior of the RCIC property in some distinguished constructions such as homomorphic image, finite direct products, Nagata ring, amalgamated duplications of rings and trivial ring extensions. Our study is then extended to the case of residually integrally closed rings.

Key Words: (Completely) integrally closed, Nagata ring, amalgamated duplication, trivial ring extension.
2010 MSC: Primary 13A15, 13B22; Secondary 13B02, 13F99.

Introduction

An integral domain D with quotient field K is said to be *completely integrally closed* (for short, *CIC*) if for any element x of K and for any nonzero element d of D such that $dx^n \in D$ for all $n \geqslant 1$, x necessarily belongs to D; or equivalently, D contains all elements x of K such that $D[x]$ is contained in a finitely generated D-module. It is well known that completely integrally closed domains are integrally closed, and valuation domains are completely integrally closed if and only if they have rank at most one as proved in [9, Theorem 17.5]. Moreover, it is worth mentioning that the intersection of any family of completely integrally closed domains with the same quotient field, or more generally, contained in some large given field, is still completely integrally closed. Consequently, Krull domains form a (proper) subclass of completely integrally closed domains. Furthermore, we note that a quotient ring of a completely integrally closed domain need not be completely integrally closed. Indeed, the ring $\mathbb{Z}[X]$ is known to be completely integrally closed but the quotient ring $\mathbb{Z}[X]/(X^2+3) \simeq \mathbb{Z}[\sqrt{-3}]$ is not (it is, *a fortiori*, not integrally closed). Motivated by this last fact, we introduce the notion of *residually completely integrally closed ring*. We say that a ring R is residually completely integrally closed (for short, RCIC) if the integral domain R/P is CIC for all prime ideals P of R; or equivalently, R/P is a CIC domain for all non-maximal prime ideals P of R. Trivially, any ring of (Krull) dimension 0 is an RCIC ring. Additionally, RCIC domains form a subclass of CIC domains, and one-dimensional CIC domains must be RCIC. On the other hand, the above example leads also to consider the notion of residually integrally closed rings. In a similar way, a ring R is said to be *residually integrally closed* (for short, *RIC*) if R/P is an integrally closed domain for all prime ideals P of R. From the fact that any completely integrally closed domain is integrally closed, it follows that the class of RIC rings includes that of RCIC rings. But, an RIC ring need not be RCIC. As a matter of fact, take any Prüfer domain that is not CIC (for example, $\mathbb{Z}+X\mathbb{Q}[X]$) because any Prüfer domain is always integrally closed and the property of being Prüfer domain is stable under homomorphic image.

In this paper, we investigate the transfer of the residually (completely) integrally closed to various contexts of constructions. Among other things, we show that any homomorphic image of an RCIC ring is always an RCIC ring (Proposition 1.4), and that the finite direct product of RCIC rings is again an RCIC ring (Proposition 1.6). Also, we characterize RCIC rings issued from Nagata ring, amalgamated duplications of rings and trivial ring extensions (Theorems 1.7 and 1.11). Next we establish some necessary and sufficient conditions for an amalgamation of rings to be an RCIC ring (Proposition 1.12). Then, we generalize the results concerning RCIC rings to the case of RIC rings (Proposition 1.15, Theorem 1.17 and Proposition 1.18). Moreover, we show that being RIC is preserved under flat overring extensions (Proposition 1.19).

Throughout this paper all rings are assumed to be commutative with identity, all modules are unitary and also all homomorphisms are unital. The symbol $\subset$ (resp., $\subseteq$) denotes the proper (resp., large) containment.

1 Residually (completely) integrally closed rings

We begin by recalling some well known facts about CIC domains.
— [9, Theorem 13.1(2)] Any CIC domain is integrally closed.
— [9, Theorem 17.5] A valuation domain is CIC if and only if it has rank at most one.
— [9, Theorem 23.4(3)] One-dimensional Prüfer domains are CIC.
— [9, Exercise 11, page 145] For any subfield L of the quotient field of a CIC domain D, the intersection $D \cap L$ is a CIC domain.
— [13, Corollary 7] For any CIC domain D, the Nagata ring $D(X)$ is also CIC.
— [1, Theorem 2.7] For any extension of integral domains $A \subseteq B$, the integral domain $A + XB[X]$ is CIC if and only if so is A and $A = B$.

We next state the principal definition of this paper as stated in the introduction.

Definition 1.1. A ring R is said to be *residually completely integrally closed* (for short, *RCIC*) if the integral domain R/P is CIC for all prime ideals P of R.

Remark 1.2. (1) It is worth noting that RCIC rings can be characterized by replacing prime ideals with non-maximal prime ideals in the previous definition, since the integral domain R/M is a field for any maximal ideal M of R.

(2) Any RCIC domain is always a CIC domain, and the converse holds for one-dimensional domains. Therefore, every one-dimensional Prüfer domain is an RCIC domain. Consequently, the class of RCIC rings contains Dedekind domains, and, moreover, the integral domain $D + XK[X]$ is never RCIC when D is an integral domain different from its quotient field K.

(3) It is clear that finite rings and 0-dimensional rings are RCIC rings but not conversely. Indeed, the ring of integers $\mathbb{Z}$ is an example of an RCIC ring that is neither finite nor 0-dimensional.

(4) As mentioned in the introduction, the ring $\mathbb{Z}[X]$ is a CIC domain that is not RCIC. Moreover, an RCIC ring is not necessarily a CIC domain. For instance, consider any 0-dimensional ring R that is not a field, such as $R := \prod_{n=0}^{\infty} \mathbb{F}_2$ (the infinite product of copies of $\mathbb{F}_2$). In this case, R is an RCIC ring that is not CIC since any 0-dimensional domain must be a field.

(5) A two-dimensional Prüfer domain is RCIC if and only if it is CIC. To see this, it suffices to prove the reverse implication. Let D be a CIC Prüfer domain of (Krull) dimension 2, and let P be a non-maximal prime ideal of D. Since the height of P is at most one, we have either $D/P \simeq D$ or D/P is a one-dimensional Prüfer domain. In both cases, the integral domain D/P is CIC, and thus, D is an RCIC domain.

The previous remarks yield the following examples:

Example 1.3. (1) For any field K, the rings $K[X]$ and $K[[X]]$ are both RCIC and CIC.

(2) Consider the ring $D := \mathrm{Int}(\mathbb{Z})$, which is the ring of integer-valued polynomials. It is well known that D is a two-dimensional Prüfer domain that is also CIC. Hence, from Remark 1.2(5), we conclude that D is an RCIC ring.

In what follows, we show that the class of RCIC rings is closed under homomorphic images.

Proposition 1.4. *Let R be a ring. Then R is an RCIC ring if and only if so is R/I for each ideal I of R.*

Proof. Assume that R is an RCIC ring, and let Q be a prime ideal of R/I. We have Q is of the form P/I, where P is a prime ideal of R containing I. Since R is an RCIC ring, R/P is a CIC domain, and hence $(R/I)/Q = (R/I)/(P/I) \simeq R/P$ is also a CIC domain. Thus, R/I is an RCIC ring. The converse is trivial. □

Let us provide some additional remarks regarding Proposition 1.4.

Remark 1.5. (1) It is important to note that in Proposition 1.4, the statement "each ideal I of R" cannot be replaced by "each nonzero ideal I of R". To illustrate this, consider a two-dimensional valuation domain (V,M) and a nonzero ideal I of V. According to [9, Theorem 17.5], V is not a CIC domain, and therefore it is not an RCIC ring (since RCIC domains are CIC). However, the prime ideals over I can be M or P, where P is the only height-one prime ideal of V. Thus, V/M is a field and V/P is a one-dimensional valuation domain, and therefore V/I is RCIC for all nonzero ideals I of V.

(2) It is worth mentioning that if $R[X]$ is an RCIC ring, Proposition 1.4 ensures that $R \simeq R[X]/XR[X]$ is also an RCIC ring. However, the converse is not true in general. For instance, $\mathbb{Z}$ is an RCIC ring, but $\mathbb{Z}[X]$ is not (as noticed before).

Next, we study the transfer of the RCIC property to finite direct product of rings.

Proposition 1.6. *Let $\{R_k\}_{1\leqslant k\leqslant n}$ be a finite set of rings. Then $\prod_{k=1}^{n} R_k$ is an RCIC ring if and only if so is each R_k.*

Proof. By induction on n it suffices to prove the result for the case of two rings, say R and S.

Assume that $R\times S$ is an RCIC ring, and let P be a prime ideal of R. Since $P\times S$ is a prime ideal of $R\times S$, we have that $(R\times S)/(P\times S)$ is a CIC domain. Then $R/P \simeq (R\times S)/(P\times S)$ is also a CIC domain, which implies that R is an RCIC ring. By a symmetry argument, we can prove that S is an RCIC ring.

Conversely, assume that R and S are both RCIC rings, and let I be a prime ideal of $R\times S$. We have I is of the form $P\times S$ or $R\times Q$, where P is a prime ideal of R and Q is a prime ideal of S. Since R and S are RCIC rings, R/P and S/Q are CIC domains, and then so are $(R\times S)/(P\times S) \simeq R/P$ and $(R\times S)/(R\times Q) \simeq S/Q$. Therefore, $(R\times S)/I$ is a CIC domain, which implies that $R\times S$ is an RCIC ring. □

The following theorem shows that the RCIC property is conserved between a ring and its Nagata ring. Recall that the *Nagata ring* of a ring R, denoted by $R(X)$, is defined as follow

$$R(X) := \left\{\frac{f}{g};\ f,g \in R[X] \text{ and } c(g) = R\right\},$$

where X is an indeterminate over R and $c(g)$ is the ideal of R generated by the coefficients of g.

Theorem 1.7. For a ring R, the Nagata ring $R(X)$ is RCIC if and only if so is R.

To prove this result, we require the following two lemmas.

Lemma 1.8. *Let D be an integral domain. Then $D(X)$ is a CIC domain if and only if so is D.*

Proof. The direct implication follows from the equality $D = D(X) \cap K$, where K is the quotient field of D. The converse is proved in [13, Corollary 7]. □

Lemma 1.9 ([10, Theorem 14.1])**.** *Let R be a ring. Then there is a one-to-one correspondance between the prime ideals of R and the prime ideals of $R(X)$ given by $P \longleftrightarrow PR(X)$. Moreover, for each prime ideal P of R, $R(X)/PR(X) \simeq (R/P)(X)$.*

Proof of Theorem 1.7. Assume that $R(X)$ is an RCIC ring, and let P be a prime ideal of R. By Lemma 1.9, we have $PR(X)$ is a prime ideal of $R(X)$, and then $R(X)/PR(X)$ is a CIC domain because $R(X)$ is an RCIC ring. Again by Lemma 1.9, we have $R(X)/PR(X) \simeq (R/P)(X)$. Thus, $(R/P)(X)$ is a CIC domain, forces that R/P is also a CIC domain. Therefore, R is an RCIC ring.

Conversely, assume that R is an RCIC ring, and let Q be a prime ideal of $R(X)$. By Lemma 1.9, there exists a prime ideal P of R such that $Q = PR(X)$ with $R(X)/Q \simeq (R/P)(X)$. Since R is an RCIC ring, R/P is a CIC domain and then it follows from Lemma 1.8 that $(R/P)(X)$ is also a CIC domain. Hence, $R(X)/Q$ is a CIC domain, and therefore, $R(X)$ is an RCIC ring. □

Now, we will investigate the transfer of the RCIC property to trivial ring extensions and amalgamated duplications of rings. For a ring R, we recall the following two notions:

— *The trivial ring extension* of R by an R-module E is the ring denoted by $R \propto E$, whose underlying group is $R \times E$ with multiplication given by $(r,e)(s,f) = (rs, rf + se)$.
— *The amalgamated duplication* of R along an ideal I of R is a subring of $R \times R$, defined as $R \bowtie I := \{(r, r+i);\ r \in R \text{ and } i \in I\}$.

To prove our next theorem, we need to describe the prime ideals of these two constructions. First, we state the following lemma:

Lemma 1.10. *Let R be a ring, E an R-module, and I an ideal of R. We have the following:*

1. *Each prime ideal $\mathcal{P}$ of $R \propto E$ is of the form $\mathcal{P} = P \propto E$ for some prime ideal P of R. Moreover, $(R \propto E)/\mathcal{P} \simeq (R/P)$.*
2. *Each prime ideal $\mathcal{P}$ of $R \bowtie I$ is of the form $\{(i, i+r);\ i \in I \text{ and } r \in P\}$ or $\{(i+r, i);\ i \in I \text{ and } r \in P\}$, where $P = \mathcal{P} \cap R$. Moreover, in both cases, $(R \bowtie I)/\mathcal{P} \simeq (R/P)$.*

Proof. See [2, Theorem 3.2(2)] and [7, Proposition 2.2]. □

Theorem 1.11. For any ring R, the following statements are equivalent:

1. R is RCIC;
2. The trivial ring extension $R \propto E$ is an RCIC ring, for every R-module E;
3. The amalgamated duplication $R \bowtie I$ is an RCIC ring, for every ideal I of R.

Proof. The implications (2) ⇒ (1), and (3) ⇒ (1) follow directly from Proposition 1.4 and the fact that $(R \propto E)/(\{0\} \propto E) \simeq R$ and $(R \bowtie I)/(\{0\} \times I) \simeq R$.

(1) ⇒ (2) Assume that R is an RCIC ring and let Q be a prime ideal of $R \propto E$. By Lemma 1.10(1), there is a prime ideal P of R such that $Q = P \propto E$. Since R is an RCIC ring, R/P is a CIC domain and then so is $(R \propto E)/Q = (R \propto E)/(P \propto E) \simeq (R/P)$. Thus $R \propto E$ is an RCIC ring.

(1) ⇒ (3) Assume that R is an RCIC ring and let Q be a prime ideal of $R \bowtie I$. Set $P := Q \cap R$. Then it follows from Lemma 1.10(2) that Q is of the form $\{(i, i+r);\ i \in I \text{ and } r \in P\}$ or $\{(i+r, i);\ i \in I \text{ and } r \in P\}$, and so, in both cases, we have: $(R \bowtie I)/Q \simeq (R/P)$. Since R is an RCIC ring, R/P is a CIC domain and then so is $(R \bowtie I)/Q$. Therefore $R \bowtie I$ is an RCIC ring. □

To treat the transfer of the RCIC property to amalgamation construction, we need first to recall the definition of amalgamated algebras along an ideal as presented in [5].

Let R and S be two rings, $f : R \to S$ a ring homomorphism and J an ideal of S. The following subring of $R \times S$:

$$R \bowtie^f J = \{(r, f(r) + j);\ r \in R \text{ and } j \in J\},$$

is called the *amalgamation* of R with S along J with respect to f. Notably, if $R = S$, $f = \iota$ (the identity of R) and $J = I$, then $R \bowtie^f J$ corresponds precisely to the amalgamated duplication of R along I, denoted as $R \bowtie I$. For more details on amalgamated algebra, the reader may consult the survey paper [8].

We now present the following proposition, which provides necessary and sufficient conditions for an amalgamation to be an RCIC ring. Here, $\mathrm{Nil}(R)$ and $\mathrm{Jac}(R)$ denote the nilradical and the Jacobson radical of a ring R, respectively.

Proposition 1.12. *Let R and S be two rings, $f : R \to S$ a ring homomorphism, and J an ideal of S. The following statements hold:*

1. *If $R \bowtie^f J$ is an RCIC ring then R and $f(R) + J$ are RCIC rings.*
2. *If $f^{-1}(J) = \{0\}$, then $R \bowtie^f J$ is an RCIC ring if and only if so is $f(R) + J$.*
3. *If either $J \subseteq \mathrm{Nil}(S)$ or $J \subseteq \mathrm{Jac}(S)$, then $R \bowtie^f J$ is an RCIC ring if and only if so is R.*

Proof. (1) Assume that $R \bowtie^f J$ is an RCIC ring. By Proposition 1.4, we deduce that $(R \bowtie^f J)/(0 \times J) \simeq R$ and $(R \bowtie^f J)/(f^{-1}(J) \times 0) \simeq f(R) + J$ are RCIC rings.

(2) If $f^{-1}(J) = 0$, then it follows from [5, Proposition 5.1(3)] that $R \bowtie^f J \simeq f(R) + J$.

(3) Assume that $J \subseteq \mathrm{Nil}(S)$ or $J \subseteq \mathrm{Jac}(S)$. From [6, Proposition 2.6], we infer that every prime ideal of $R \bowtie^f J$ is of the form $P \bowtie^f J$ for some prime ideal P of R. Since $(R \bowtie^f J)/(P \bowtie^f J) \simeq R/P$, $(R \bowtie^f J)/(P \bowtie^f J)$ is a CIC domain if and only if so is R/P. Thus, we conclude that $R \bowtie^f J$ is an RCIC ring if and only if so is R, as desired. □

We next investigate the transfer of the residually integrally closed property to some remarkable ring extensions. First, we introduce the notion of residually integrally closed rings.

Definition 1.13. A ring R is said to be *residually integrally closed* (for short, *RIC*) if, for each prime ideal P of R, the integral domain R/P is integrally closed.

Remark 1.14. (1) Note that the class of RIC rings includes RCIC rings and Prüfer domains.

(2) It is clear that RIC domains are integrally closed. Conversely, one-dimensional integrally closed domains are RIC.

(3) Any RCIC ring is RIC, but the converse is not true in general. For instance, let $D = \mathbb{Z} + X\mathbb{Q}[X]$ which is a Prüfer domain and then it is RIC. However, D is not RCIC since it is not CIC, as asserted in [1, Theorem 2.7].

In the following, we consider transferring the RIC property to localization and finite direct product.

Proposition 1.15. *For any two rings R and S, we have:*

1. *If R is an RIC ring, then $T^{-1}R$ is an RIC ring for any multiplicative subset T of R.*
2. *The direct product $R \times S$ is an RIC ring if and only if so are R and S.*

Proof. (1) Assume that R is an RIC ring, and let T be multiplicative subset T of R and Q a prime ideal of $T^{-1}R$. Then Q is of the form $T^{-1}P$ for some prime ideal P of R with $P \cap T = \emptyset$. Since R is an RIC ring, R/P is an integrally closed domain. As the property of being integrally closed is stable under localization, $\overline{T}^{-1}(R/P)$ is integrally closed, where $\overline{T}$ denotes the natural image of T in R/P, and hence $T^{-1}R/Q$ is also integrally closed because $T^{-1}R/T^{-1}P \simeq \overline{T}^{-1}(R/P)$. Thus, $T^{-1}R$ is an RIC ring.

(2) The proof of this statement is similar to that of Proposition 1.6. □

As a quick consequence of Proposition 1.15(2), we have the following:

Corollary 1.16. *Let $\{R_k\}_{1\leqslant k\leqslant n}$ be a finite set of rings. Then $\prod_{k=1}^{n} R_k$ is an RIC ring if and only if so is each R_k.*

In the remainder of this section, we examine the transfer of the RIC property in various ring extensions.

Theorem 1.17. For any ring R, the following statements are equivalent:

1. R is an RIC ring;
2. R is an locally RIC ring (that is, R_P is an RIC ring, for every prime ideal P of R);
3. R_M is an RIC ring, for every maximal ideal M of R;
4. R/I is an RIC ring, for every ideal I of R;
5. The Nagata ring $R(X)$ is RIC;
6. The trivial ring extension $R \propto E$ is an RIC ring, for every R-module E;
7. The amalgamated duplication $R \bowtie I$ is an RIC ring, for every ideal I of R.

Proof. The proof of the equivalences (1) ⇔ (4) ⇔ (5) ⇔ (6) ⇔ (7) are similar to the case of RCIC rings.

The implication (1) ⇒ (2) follows from Proposition 1.15(1), and (2) ⇒ (3) is trivial.

To prove (3) ⇒ (1), assume that R_M is an RIC ring for every maximal ideal M of R, and let P be a non-maximal prime ideal of R. Using the fact that the integrally closed property is a local property, we need to check that $(R/P)_M$ is integrally closed for all maximal ideals M of R/P. Let M be a maximal ideal of R/P. Then $M = m/P$ for some maximal ideal m of R containing P. Since R_m is RIC, we have $(R/P)_M \simeq R_m/P_m$ is an integrally closed domain, which implies that R/P is also integrally closed. Thus, R/I is an RIC ring, and this completes the proof. □

Proposition 1.18. *Let R and S be two rings, $f : R \to S$ a ring homomorphism, and J an ideal of S. The following statements hold:*

1. *If $R \bowtie^f J$ is an RIC ring then so are the rings R and $f(R)+J$.*
2. *If $f^{-1}(J) = \{0\}$, then $R \bowtie^f J$ is an RIC ring if and only if so is $f(R)+J$.*
3. *If either $J \subseteq \mathrm{Nil}(S)$ or $J \subseteq \mathrm{Jac}(S)$, then $R \bowtie^f J$ is an RIC ring if and only if so is R.*
4. *If J is a maximal ideal and $R \bowtie^f J$ is an RIC ring, then R and S are RIC rings.*

Proof. The statements (1), (2) and (3) are similar to the case of RCIC rings.

(4) Assume that J is a maximal ideal and $R \bowtie^f J$ is an RIC ring. From statement (1), we conclude that R is RIC. Furthermore, since J is maximal, we have $\overline{Q}^f := \{(r, f(r)+j);\ r \in R,\ j \in J \text{ and } f(r)+j \in Q\}$ is a prime ideal of $R \bowtie^f J$ for each prime ideal Q of S. Using Proposition 1.15(1), we deduce that $(R \bowtie^f J)_{\overline{Q}^f} \simeq S_Q$ is an RIC ring, and therefore by Theorem 1.17, S is an RIC ring. □

Inspired by [12], we can establish that the RIC property is preserved under flat overrings. To prove this, we need to recall the notion of generalized transform of a ring with respect to a generalized multiplicative system.

Let R be a ring with the total quotient ring K, I an ideal of R, and $\mathcal{S}$ a generalized multiplicative system of R, i.e., $\mathcal{S}$ is a multiplicative set of ideals of R. The $\mathcal{S}$-*transform* of R (or the *generalized transform* of R with respect to $\mathcal{S}$) is an overring $R_{\mathcal{S}} := \{x \in K;\ xA \subseteq R \text{ for some } A \in \mathcal{S}\}$. Moreover, $I_{\mathcal{S}} := \{x \in K;\ xA \subseteq I \text{ for some } A \in \mathcal{S}\}$ is an ideal of $R_{\mathcal{S}}$ containing I. Here, an overring of a ring R refers to a subring of K that contains R.

Proposition 1.19. *Let R be a ring. Then R is an RIC ring if and only if any flat overring T of R is RIC.*

Proof. We will only prove the direct implication. Assume that R is an RIC ring, and let T be a flat overring of R and M a maximal ideal of T. Set $P := M \cap R$. By [3, Theorems 1.1 and 1.3], there exists a generalized multiplicative system $\mathcal{S}$ of R such that $T = R_{\mathcal{S}}$ and $M = P_{\mathcal{S}}$. Furthermore, we can suppose that any generalized multiplicative system of R is saturated, as stated in [4, Proposition 4.6]. Then it follows from [4, Theorem 4.12] that $T_M = (R_{\mathcal{S}})_{P_{\mathcal{S}}} \simeq R_P$. Since R is an RIC ring, R_P is also an RIC ring, and hence T_M is an RIC ring. Thus by Theorem 1.17, we conclude that T is an RIC ring, as desired. □

Remark 1.20. It is worth noting that the previous proposition cannot be extended to the general case of a flat ring extension. Indeed, consider the ring $\mathbb{Z}$, which is an RIC domain. As mentioned in the introduction, $\mathbb{Z}[X]$ is not an RIC. However, the ring extension $\mathbb{Z} \hookrightarrow \mathbb{Z}[X]$ is flat.

2 Further generalization

In this section, we aim to show that the previously established results hold in a more general context by considering a specific property $\mathcal{X}$ of integral domains instead of the notion of a (completely) integrally closed domain. To do this, we need to introduce the following definitions:

— A ring R is said to be *residually* $\mathcal{X}$ if the integral domain R/P has the property $\mathcal{X}$, for all prime ideals P of R.

— A ring R is said to be *totally* $\mathcal{X}$ if R_P is a residually $\mathcal{X}$ ring, for all prime ideals P of R.

— We say that $\mathcal{X}$ *has a good behavior under integral extensions* if, for any integral extension of domains $R \subseteq S$, we have R is an $\mathcal{X}$ domain if and only if so is S.

— We say that $\mathcal{X}$ *has a good behavior under Nagata ring*, if the integral domains D and $D(X)$ simultaneously have the same property $\mathcal{X}$.

Theorem 2.1. Let $\mathcal{X}$ denote a property of integral domains, and let R and S be two rings. We have:

1. If R is a residually $\mathcal{X}$ ring then so is R/I, for each ideal I of R.
2. The direct product $R \times S$ is a residually $\mathcal{X}$ ring if and only if so are R and S.
3. The following statements are equivalent:
   (a) R is a residually $\mathcal{X}$ ring;
   (b) The trivial ring extension $R \propto E$ is a residually $\mathcal{X}$ ring, for every R-module E;
   (c) The amalgamated duplication $R \bowtie I$ is a residually $\mathcal{X}$ ring, for every ideal I of R.
4. Assume that $\mathcal{X}$ is stable under localization. If R is a residually $\mathcal{X}$ ring then so is $T^{-1}R$ for each multiplicative subset T of R.

5. Assume that $\mathcal{X}$ has a good behavior under integral extensions. If $R \subseteq S$ is an integral extension of rings, then R is a residually $\mathcal{X}$ ring if and only if so is S.

6. Let $f : R \to S$ be a ring homomorphism and J an ideal of S.

   (a) If $R \bowtie^f J$ is a residually $\mathcal{X}$ ring then so are the rings R and $f(R)+J$.

   (b) If either $J \subseteq \mathrm{Nil}(S)$ or $J \subseteq \mathrm{Jac}(S)$, then $R \bowtie^f J$ is a residually $\mathcal{X}$ ring if and only if so is R.

   (c) If $\mathcal{X}$ has a good behavior under integral extensions, then $R \bowtie^f J$ is a residually $\mathcal{X}$ ring if and only if so are the rings R and $f(R)+J$.

7. Assume that $\mathcal{X}$ has a good behavior under Nagata ring. Then R is a residually $\mathcal{X}$ ring if and only if so is $R(X)$.

Proof. The proofs of (1), (2), and (3) are similar to those of Propositions 1.4 and 1.6, and Theorem 1.11.

The proof of (4) is similar to the case of RIC rings.

(5) Assume that R is a residually $\mathcal{X}$ ring and let Q be a prime ideal of S. Since $R/(Q\cap R) \subseteq S/Q$ is an integral extension and $R/(Q\cap R)$ is an $\mathcal{X}$ domain, it follows that S/Q is also an $\mathcal{X}$ domain. Thus, S is a residually $\mathcal{X}$ ring.

Conversely, assume that S is a residually $\mathcal{X}$ ring and let P be a prime ideal of R. By Lying-Over, there is a prime ideal Q of S such that $P = Q \cap R$, and so S/Q is an $\mathcal{X}$ domain. Moreover, since $R/P \subseteq S/Q$ is an integral extension, R/P is an $\mathcal{X}$ domain, and therefore R is a residually $\mathcal{X}$ ring.

(6) The statements (a) and (b) are similar to those of Proposition 1.12.

(c) It is well known that the ring $A \times (f(A)+J)$ is integral over $A \bowtie^f J$, as asserted in [6, Lemma 3.3]. Then by statement (5), $A \bowtie^f J$ is a residually $\mathcal{X}$ ring if and only if so is $A \times (f(A)+J)$, and thus the conclusion follows from statement (2).

(7) The proof of this statement is similar to that of Theorem 1.7. □

Theorem 2.2. Let $\mathcal{X}$ denote a property of integral domains which is stable under localization. For any ring R, the following statements are equivalent:

1. R is a totally $\mathcal{X}$ ring;

2. R_P is a totally $\mathcal{X}$ ring, for every prime ideal P of R;

3. R_M is a totally $\mathcal{X}$ ring, for every maximal ideal M of R;

4. Any flat overring T of R is totally $\mathcal{X}$ (in particular, every localization of a totally $\mathcal{X}$ ring is totally $\mathcal{X}$);

5. R/I is a totally $\mathcal{X}$ ring, for every ideal I of R;

6. The trivial ring extension $R \propto E$ is a totally $\mathcal{X}$ ring, for every R-module E;

7. The amalgamated duplication $R \bowtie I$ is a totally $\mathcal{X}$ ring, for every ideal I of R.

Proof. The proof of (1) $\Rightarrow$ (2) is similar to that of Proposition 1.15(1).

The implications (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (1), and (5) $\Rightarrow$ (1) are straightforward.

(3) $\Rightarrow$ (1) Assume that R_M is a totally $\mathcal{X}$ ring for every maximal ideal M of R and let P be a nonmaximal prime ideal of R. Then there exists a maximal ideal M of R such that $P \subset M$. So, from the fact that the localization of a residually $\mathcal{X}$ ring is still residually $\mathcal{X}$ (see Theorem 2.1(4)), it follows that $(R_M)_{PR_M} = R_P$ is a residually $\mathcal{X}$ ring, and hence R is a totally $\mathcal{X}$ ring.

(1) ⇒ (4) Assume that R is a totally $\mathcal{X}$ ring, and let T be a flat overring of R and M a maximal ideal of T and set $P := M \cap R$. Then by [3, Theorem 1.3], there exists a generalized multiplicative system $\mathcal{S}$ of R such that $T = R_{\mathcal{S}}$ and $AT = T$ for all $A \in \mathcal{S}$. Also by [3, Theorem 1.1] we have $M = P_{\mathcal{S}}$. It follows from [4, Proposition 4.6] that we may assume that any generalized multiplicative system of R is saturated in this situation. Thus by [4, Theorem 4.12] there is an isomorphism $(R_{\mathcal{S}})_{P_{\mathcal{S}}} \simeq R_P$. Since R is totally $\mathcal{X}$, R_P is a residually $\mathcal{X}$ ring and then so is T_M. Therefore $T = R_{\mathcal{S}}$ is totally $\mathcal{X}$ by the implication (3) ⇒ (1).

(1) ⇒ (5) Assume that R is a totally $\mathcal{X}$ ring and let M be a maximal ideal of R/I. Then $M = m/I$ for some maximal ideal m of R containing I. Since R is totally $\mathcal{X}$ and m is a maximal ideal of R, R_m is residually $\mathcal{X}$, and then it follows from Theorem 2.1(1) that $(R/I)_M \simeq R_m/I_m$ is also a residually $\mathcal{X}$ ring. Therefore by the implication (3) ⇒ (1), R/I is a totally $\mathcal{X}$ ring.

The implications (6) ⇒ (1) and (7) ⇒ (1) follow from Theorem 2.1(1) and the fact that $(R \propto E)/(\{0\} \propto E) \simeq R$ and $(R \bowtie I)/(\{0\} \times I) \simeq R$.

(1) ⇒ (6) Assume that R is a totally $\mathcal{X}$ ring and let M be a maximal ideal of $R \propto E$. By [2, Theorem 3.2], there is a maximal ideal m of R such that $M = m \propto E$. Since R is a totally $\mathcal{X}$ ring, R_m is a residually $\mathcal{X}$ ring and then so is $(R \propto E)_M = (R \propto E)_{(m \propto E)} \simeq R_m \propto E_m$ by Theorem 2.1(4). Thus, $R \propto E$ is a totally $\mathcal{X}$ ring.

(1) ⇒ (7) Assume that R is a totally $\mathcal{X}$ ring and let M be a maximal ideal of $R \bowtie I$. Then by [7, Proposition 2.2], M is of the form $\{(i, i+r);\ i \in I \text{ and } r \in m\}$ or $\{(i+r, i);\ i \in I \text{ and } r \in m\}$ with $m = M \cap R$. So, we discuss the following two cases:

Case 1: $I \subseteq m$. Since R is totally $\mathcal{X}$, R_m is residually $\mathcal{X}$ and then it follows from Theorem 2.1(3) and [7, Proposition 2.2] that $(R \bowtie I)_M \simeq R_m \bowtie I_m$ is residually $\mathcal{X}$.

Case 2: $I \not\subseteq m$. Then by Theorem 2.1(3) and [7, Proposition 2.2], $(R \bowtie I)_M \simeq R_m$ is residually $\mathcal{X}$ because R_m is residually $\mathcal{X}$.

Therefore, $R \bowtie I$ is a totally $\mathcal{X}$ ring. □

References

[1] D.D. Anderson, D.F. Anderson and M. Zafrullah, Rings between $D[X]$ and $K[X]$, Houston J. Math. **17** (1991), 109–129.

[2] D.D. Anderson and M. Winderes, Idealization of a module, J. Comm. Algebra **1** (2009), 3–56.

[3] J.T. Arnold and J.W. Brewer, On flat overrings, ideal transforms and generalized transforms of a commutative ring, J. Algebra **18** (1971), 254–263.

[4] H.S. Butts and C.G. Spaht, Generalized quotient rings, Math. Nachr. **53** (1972), 181–210.

[5] M. D'Anna, C.A. Finocchiaro and M. Fontana, Amalgamated algebras along an ideal, in: *Commutative Algebra and Applications*, Walter de Gruyter, Berlin, (2009), 155–172.

[6] M. D'Anna, C.A. Finocchiaro and M. Fontana, Properties of chains of prime ideals in an amalgamated algebra along an ideal, J. Pure Appl. Algebra **214** (2010), 1633–1641.

[7] M. D'Anna and M. Fontana, The amalgamated duplication of a ring along a multiplicative-canonical ideal, Ark. Mat. **45** (2007), 241–252.

[8] A. El Khalfi, H. Kim and N. Mahdou, Amalgamation extension in commutative ring theory: a survey, Moroccan Journal of Algebra and Geometry with Applications **1** (1) (2022), 139–182.

[9] R. Gilmer, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Appl. Math., vol. 90, Queen's University, Kingston, Ontario, (1992).

[10] J.A. Huckaba, *Commutative Rings with Zero Divisors*, Marcel Dekker, 1988.

[11] S. Kabbaj, Matlis' semi-regularity and semi-coherence in trivial ring extensions: a survey, Moroccan Journal of Algebra and Geometry with Applications **1** (1) (2022), 1–17.

[12] H. Kim, O. Ouzzaouit and A. Tamoussit, Noetherian-like properties and zero-dimensionality in some extensions of rings, Afr. Mat. **34** (3) (2023), https://doi.org/10.1007/s13370-023-01083-3

[13] T.G. Lucas, Characterizing when $R(X)$ is completely integrally closed, in: *Factorization in Integral Domains*, Lect. Notes in Pure Appl. Math. **189**, Marcel Dekker, New York (1997), pp. 401–415.