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Extension of star-operation

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Abstract. Let D be an integral domain, $*$ a star operation on D and S a multiplicative subset of D. In this paper, we generalize the notion of $*$-ideals (resp, $*$-invertible) of D, by introducing the concept of S-$*$-ideals (resp, S-$*$-invertible) of D. A fractional ideal of D is called S-$*$-ideals (resp, S-$*$-invertible) if there exists an $s \in S$ such that $sI^* \subseteq I \subseteq I^*$ (resp, if there exists an $s \in S$ and a fractional ideal J of D such that $sD \subseteq (IJ)^* \subseteq D$). We investigate many proprieties and characterizations of the notion S-$*$-ideals (resp, S-$*$-invertible).

Key Words: $*$-operation, S-$*$-ideals, S-$*$-invertible.
2010 MSC: 13G05, 13A15.

1 Introduction

Throughout this paper D will be an integral domain with quotient field K. We denote by $\mathcal{F}(D)$, the set of nonzero fractional ideals of D. A $*$-operation on D is a mapping $I \longmapsto I^*$, from $\mathcal{F}(D)$ to $\mathcal{F}(D)$ which satisfies the following conditions for $a \in K\backslash\{0\}$ and $I, J \in \mathcal{F}(D)$:

1. $(a)^* = (a)$ and $(aI)^* = aI^*$,
2. $I \subseteq I^*$; if $I \subseteq J$, then $I^* \subseteq J^*$ and
3. $(I^*)^* = I^*$.

$I \in \mathcal{F}(D)$ is called a $*$-ideal if $I^* = I$. We use the notation $*$-$Max(D)$ for the set of $*$-ideals which are maximal among proper integral $*$-ideals of D. An element I of $\mathcal{F}(D)$ is called to be $*$-invertible if $(IJ)^* = D$ for some $J \in \mathcal{F}(D)$ or equivalently $(II^{-1})^* = D$, where $I^{-1} = \{x \in K \mid xI \subseteq D\}$. We can construct the $*$-operation $*_s$ defined by $I^{*_s} = \bigcup\{(I')^* \mid I' \in \mathcal{F}(D),\ I'$ is finitely generated and $I' \subseteq I\}$. We say $*_s$ that is the finite type $*$-operation induced by $*$. Also, $*$ is said to be of finite type if $* = *_s$ i.e., $I^* = I^{*_s}$ for each $I \in \mathcal{F}(D)$. For the general theory of $*$-operations, the reader is referred to [4, Sects. 32 and 34]. An important $*$-operation is the v-operation given by $I_v = (I^{-1})^{-1}$ for each $I \in \mathcal{F}(D)$. The finite type $*$-operation induced by the v-operation is called the t-operation. For $f = a_0 + \cdots + a_n X^n \in K[X]$, A_f will denote the D-submodule of K generated by $\{a_0, ..., a_n\}$. The set $N_* = \{f \in D[X] \mid (A_f)^* = D\}$ is a multiplicatively closed subset of $D[X]$ by [9, Proposition 2.1], and it is easy to see that, $N_* = N_{*_s}$.

In this paper, we generalize the notion of $*$-ideal (resp, $*$-invertible) by introducing the concept of S-$*$-ideal (resp, S-$*$-invertible). Let I be a fractional ideal of an integral domain D and S a multiplicative subset of D. We say that I is S-$*$-ideal if there exists an $s \in S$ such that $sI^* \subseteq I \subseteq I^*$. We say that I is S-$*$-invertible if there exists an $s \in S$ and a fractional ideal J of D such that $sD \subseteq (IJ)^* \subseteq D$, equivalently there exists an $s \in S$ such that $sD \subseteq (II^{-1})^* \subseteq D$ (Proposition 3.4).

In Section 2, we study basic results of S-$*$-ideal, we give an example of an S-$*$-ideal which is not

$*$-ideal. We also, show that every S-invertible ideal (recall from [6], that a fractional ideal I of D is said to be S-*invertible* if $sD \subseteq IJ \subseteq D$ for some $s \in S$ and some fractional ideal J of D) is S-$*$-ideal (Proposition 2.4). An ideal M of D disjoint with S is called S-$*$-maximal if it is maximal in the set of all integral proper S-$*$-ideals of D. We prove that every S-$*$-maximal ideal of D is a prime ideal of D (Proposition 2.8). Let D be an integral domain and S a multiplicative subset of D. We say that S is anti-Archimedean if $\cap_{n\geq 1} s^n D \cap S \neq \emptyset$ for every $s \in S$. In [2], the authors generalized this notion by introducing the concept of weakly anti-Archimedean multiplicative set. According [2], a multiplicative set S of an integral domain D is called weakly anti-Archimedean if for each family $(s_\alpha)_{\alpha\in\Lambda}$ of elements of S we have $(\cap_{\alpha\in\Lambda} s_\alpha D) \cap S \neq \emptyset$. Note that every weakly anti-Archimedean multiplicative set is anti-Archimedean. The converse is not true as was observed in [3, Example 2.7]. Let D be an integral domain, $*$ a finite type $*$-operation on D and S a weakly anti-Archimedean multiplicative subset of D. We show that every integral proper S-$*$-ideal of D is included in an S-$*$-maximal ideal of D (Theorem 2.9). In the particular case when S consists of units of D, we get every integral proper $*$-ideal of D is included in a $*$-maximal ideal of D (Corollary 2.10). Let D be an integral domain, $*$ a finite type $*$-operation on D and S a weakly anti-Archimedean multiplicative subset of D. We prove that for each S-$*$-ideal I of D, $I = \bigcap_{M\in S\text{-}*\text{-}Max(D)} ID_M$ (Theorem 2.12).

In section 3, we study basic propertis of S-$*$-invertible. It's easy to show that if S consists of units of D the notions $*$-invertible and S-$*$-invertible coincide. Let D be an integral domain, $*$ a finite type $*$-operation on D and S a weakly anti-Archimedean multiplicative subset of D. Let I be a fractional ideal of D. We show that I is an S-$*$-invertible ideal of D if and only if I is S-$*$-finite and for each $M \in S$-$*$-Max(D), ID_M is a principal ideal of D_M (Theorem 3.8). In the particular case when S consists of units of D we recover the folloing known result, I is a $*$-invertible ideal of D if and only if I is of $*$-finite type and it is t-locally principal (Corollary 3.9). Let D be an integral domain and S a multiplicative subset of D. It is well-known that for each finitely generated fractional ideal I of D, $(I_S)^{-1} = (I^{-1})_S$. We extented this result to S-$*$-finite ideal of D. We show that if I is an S-$*$-finite ideal of D, then $(I_S)^{-1} = (I^{-1})_S$ (Proposition 3.10) where $*$ a finite type $*$-operation on D and I a fractional ideal of D.

2 Basic properties of S-$*$-ideals

Definition 2.1. Let D be an integral domain, S a multiplicative subset of D and $*$ a star-operation on D. A fractional ideal I of D is called S-$*$-ideal if there exists an $s \in S$ such that $sI^* \subseteq I \subseteq I^*$.

Example 2.2.
1. Every $*$-ideal is an S-$*$-ideal.
2. Let $D = \mathbb{Z}[X]$ and $I = 2\mathbb{Z} + X\mathbb{Z}[X]$. By [1, Lemma 2.1], it is easy to show that $I^{-1} = (\frac{1}{2}\mathbb{Z}) \cap \mathbb{Z} + X\mathbb{Z}[X]$; so $I_v = \mathbb{Z}[X]$ which implies that I is not a divisorial ideal of D. Now, let $S = \{2^n \mid n \in \mathbb{N} \cup \{0\}\}$. Then S is a multiplicative subset of D. Moreover,
$$2I_v = 2\mathbb{Z}[X] \subseteq I \subseteq \mathbb{Z}[X] = I_v.$$
Hence I is an S-v-ideal of D. This shows that the converse of (1) is not true in general.
3. Let D be an integral domain, S a multiplicative subset of D and $*$ a star-operation on D. If S consists of units of D, then the notions of S-$*$-ideals and $*$-ideals are coincide.

Let D be an integral domain and S a multiplicative subset of D. Recall from [8] that an ideal I of D is called S-*principal*, if $sI \subseteq J \subseteq I$ for some principal ideal J of D and some $s \in S$. The next proposition collects some properties of S-$*$-ideals of an integral domain D.

Proposition 2.3. *Let D be an integral domain, S a multiplicative subset of D and $*$ a star-operation on D.*

1. *Let $S \subseteq T$ be multiplicative subsets of D. If I is an S-$*$-ideal of D, then I is a T-$*$-ideal of D.*

2. *Let $\bar{S}$ be the saturation of S. Then I is an S-$*$-ideal of D if and only if I is an $\bar{S}$-$*$-ideal of D.*

3. *If I is S-principal, then I is an S-$*$-ideal of D.*

Proof. (1). Obvious.

(2). The "only if" part follows from (1). Now, assume that I is an $\bar{S}$-$*$-ideal of D. Then there exists an $s \in \bar{S}$ such that $sI^* \subseteq I \subseteq I^*$. Since $s \in \bar{S}$, there exists a $t \in S$ such that $t = ss'$ for some $s' \in D$. Thus

$$tI^* \subseteq sI^* \subseteq I \subseteq I^*,$$

and hence I is an S-$*$-ideal of D.

(3). Since I is S-principal, there exist an $s \in S$ and $d \in D$ such that $sI \subseteq dD \subseteq I$. This implies that

$$sI^* = (sI)^* \subseteq (dD)^* = dD \subseteq I \subseteq I^*.$$

Hence I an S-$*$-ideal of D. □

Recall from [6], that for a multiplicative set S in D, a fractional ideal I of D is said to be *S-invertible* if $sD \subseteq IJ \subseteq D$ for some $s \in S$ and some fractional ideal J of D. It is shown that I is an S-invertible ideal of D if and only if $sD \subseteq II^{-1} \subseteq D$ for some $s \in S$. It well known that every invertible ideal is a $*$-ideal. Our next Proposition generalize this result.

Proposition 2.4. *Let D be an integral domain, $*$ a star-operation on D and S a multiplicative subset of D. Each S-invertible ideal of D is S-$*$-ideal.*

Proof. Let I be an S-invertible ideal of D. By [6, Remark 2.4], $sJ^{-1} \subseteq I \subseteq J^{-1}$ for some $s \in S$ and some fractional ideal J of D. This implies that

$$sJ^{-1} = (sJ^{-1})^* \subseteq I^* \subseteq (J^{-1})^* = J^{-1}.$$

Thus $sI^* \subseteq sJ^{-1} \subseteq I$, and hence I is an S-$*$-ideal of D. □

Example 2.5. Let D be a Pr*ü*fer domain, $*$ a star-operation on D and S a multiplicative subset of D. Then each nonzero S-finite ideal of D is S-$*$-ideal. Indeed, let I be an S-finite ideal of D. Then there exist an $s \in S$ and a nonzero finitely generated ideal F of D such that $sI \subseteq F \subseteq I$. Thus $sF^{-1} \subseteq I^{-1}$. Since D is a Pr*ü*fer domain, $FF^{-1} = D$; so

$$sD = sFF^{-1} \subseteq FI^{-1} \subseteq II^{-1} \subseteq D$$

which implies that I is an S-invertible ideal of D. Hence by the previous Proposition, I is an S-$*$-ideal of D.

Let D be an integral domain and S a multiplicative subset of D. We say that S is *anti-Archimedean* if $\cap_{n \geq 1} s^n D \cap S \neq \emptyset$ for every $s \in S$. In [2], the authors generalized this notion by introducing the concept of weakly anti-Archimedean multiplicative set. According [2], a multiplicative set S of an integral domain D is called *weakly anti-Archimedean* if for each family $(s_\alpha)_{\alpha \in \Lambda}$ of elements of S we have $(\cap_{\alpha \in \Lambda} s_\alpha D) \cap S \neq \emptyset$. Note that every weakly anti-Archimedean multiplicative set is anti-Archimedean. The converse is not true as was observed in [3, Example 2.7].

Proposition 2.6. *Let D be an integral domain, $*$ a finite type $*$-operation on D and S a weakly anti-Archimedean multiplicative subset of D. Let $(I_\alpha)_{\alpha \in \Lambda}$ be a totally ordered family of fractional ideals of D. If for each $\alpha \in \Lambda$, I_α is S-$*$-ideal, then $\cup_{\alpha \in \Lambda} I_\alpha$ is an S-$*$-ideal of D.*

Proof. For each $\alpha \in \Lambda$, there exists an $s_\alpha \in S$ such that $s_\alpha I_\alpha^* \subseteq I_\alpha$. Since S is weakly anti-Archimedean, $\cap_{\alpha\in\Lambda} s_\alpha D \cap S \neq \emptyset$. Let $t \in \cap_{\alpha\in\Lambda} s_\alpha D \cap S$. Note that for each $\alpha \in \Lambda$, $tI_\alpha^* \subseteq I_\alpha$. We show that $t(\cup_{\alpha\in\Lambda} I_\alpha)^* \subseteq \cup_{\alpha\in\Lambda} I_\alpha$. Let $x \in (\cup_{\alpha\in\Lambda} I_\alpha)^*$. Since $*$ is of finite character, there exists a finitely generated subideal J of $\cup_{\alpha\in\Lambda} I_\alpha$ such that $x \in J^*$. Since J is a finitely generated ideal of D, there exists a $\beta \in \Lambda$ such that $J \subseteq I_\beta$. We have $tx \in tJ^* \subseteq tI_\beta^* \subseteq I_\beta$; so $tx \in I_\beta$ for some $\beta \in \Lambda$ which implies that $t(\cup_{\alpha\in\Lambda} I_\alpha)^* \subseteq \cup_{\alpha\in\Lambda} I_\alpha$, and hence $\cup_{\alpha\in\Lambda} I_\alpha$ is an S-$*$-ideal of D. □

Notation 2.7. Let D be an integral domain, $*$ a star-operation on D and S a multiplicative subset of D. An ideal M of D disjoint with S is called S-$*$-maximal if it is maximal in the set of all integral proper S-$*$-ideal of D. We denote by S-$*$-Max(D) the set of all S-$*$-maximal ideals of D.

Proposition 2.8. *Every S-$*$-maximal ideal of D is a prime ideal of D.*

Proof. Let P be an S-$*$-maximal ideal of D. Assume that P is not prime, there exist $a, b \in D\backslash P$ such that $ab \in P$. Let $I = P + aD$ and $J = P + bD$. Since $P \subsetneq I \subseteq I^* \subseteq D$, by maximality of P in the set of all integral proper S-$*$-ideal of D, $I^* = D$. In the same way we can prove $J^* = D$. This implies that $(IJ)^* = (I^*J^*)^* = D$. But $IJ = P^2 + aP + bP + abP \subseteq P$; so $P^* = D$. Now, since P is an S-$*$-ideal of D, there exists an $s \in S$ such that $sP^* \subseteq P$ which implies that $sD \subseteq P$, a contradiction because $P \cap S = \emptyset$. Hence P is a prime ideal of D. □

Theorem 2.9. Let D be an integral domain, $*$ a finite type $*$-operation on D and S a weakly anti-Archimedean multiplicative subset of D. Then every integral proper S-$*$-ideal of D is included in an S-$*$-maximal ideal of D.

Proof. Let $\mathcal{F}$ be the set of all integral proper S-$*$-ideals of D. Then $\mathcal{F} \neq \emptyset$, since $\mathcal{F}$ contain all integral proper S-principal ideals of D. Now, let $(I_\alpha)_{\alpha\in\Lambda}$ be a totally ordered family of elements of $\mathcal{F}$. By Proposition 2.6, $\cup_{\alpha\in\Lambda} I_\alpha$ is an element of $\mathcal{F}$; so we conclude by Zorn's Lemma our result. □

In the particular case when S consists of units of D, we regain the following well-known result.

Corollary 2.10. *Let D be an integral domain and $*$ a finite type $*$-operation on D. Then every integral proper $*$-ideal of D is included in a $*$-maximal ideal of D.*

Lemma 2.11. *Let D be an integral domain, $*$ a star-operation on D and S a multiplicative subset of D. Let $(I_k)_{1\leq k\leq n}$ be a finite family of fractional ideals of D such that $\cap_{1\leq k\leq n} I_k \neq (0)$. If for each $1 \leq k \leq n$, I_k is S-$*$-ideal, then $\cap_{1\leq k\leq n} I_k$ is an S-$*$-ideal of D.*

Proof. For each $1 \leq k \leq n$, there exists an $s_k \in S$ such that $s_k I_k^* \subseteq I_k$. Let $t = s_1 s_2 \cdots s_n$. Then $t \in S$ and for each $1 \leq k \leq n$, $tI_k^* \subseteq I_k$. For each $1 \leq m \leq n$, $t(\cap_{1\leq k\leq n} I_k)^* \subseteq tI_m^* \subseteq I_m$. This implies that $t(\cap_{1\leq k\leq n} I_k)^* \subseteq \cap_{1\leq k\leq n} I_k$, and hence $\cap_{1\leq k\leq n} I_k$ is an S-$*$-ideal of D. □

Theorem 2.12. Let D be an integral domain, $*$ a finite type $*$-operation on D and S a weakly anti-Archimedean multiplicative subset of D. Then for each S-$*$-ideal I of D,

$$I = \bigcap_{M\in S\text{-}*\text{-}Max(D)} ID_M.$$

Proof. Let x be a nonzero element of $\bigcap_{M\in S\text{-}*\text{-}Max(D)} ID_M$. Then for each S-$*$-maximal ideal M of D, there exists an $s_M \in D\backslash M$ such that $s_M x \in I$. Let $J = D \cap (\frac{1}{x}I)$. Then $s_M \in J$ for each S-$*$-maximal ideal M of D. Moreover, Since I is an S-$*$-ideal of D, $\frac{1}{x}I$ is an S-$*$-ideal of D; so by Lemma 2.11, J is an S-$*$-ideal of D. Assume that $J \neq D$. Then J is an integral proper S-$*$-ideal of D; so by Theorem 2.9, there exists $M \in S$-$*$-$Max(D)$ such that $J \subseteq M$ which implies that $s_M \in J \subseteq M$, a contradiction. Thus $J = D$ which implies that $x \in I$. Hence $I \subseteq \bigcap_{M\in S\text{-}*\text{-}Max(D)} ID_M$. This completed the proof, since other inclusion is obvious. □

Corollary 2.13. *Let D be an integral domain, $*$ a finite type $*$-operation on D and I a $*$-ideal of D. Then*

$$I = \bigcap_{M\in *\text{-}Max(D)} ID_M.$$

Remark 2.14. Let I be an S-$*$-ideal of an integral domain D, where S is a multiplicative subset of D and $*$ a star-operation of finite character on D. Then there exits an $s \in S$ such that $sI^* \subseteq I$. But $I^* = \bigcap_{M\in *\text{-}Max(D)} I^*D_M$; so

$$s(\bigcap_{M\in *\text{-}Max(D)} ID_M) \subseteq s(\bigcap_{M\in *\text{-}Max(D)} I^*D_M) = sI^* \subseteq I \subseteq \bigcap_{M\in *\text{-}Max(D)} ID_M.$$

Hence there exists an $s \in S$ such that

$$s(\bigcap_{M\in *\text{-}Max(D)} ID_M) \subseteq I \subseteq \bigcap_{M\in *\text{-}Max(D)} ID_M.$$

3 S-$*$-invertible ideals

In this section we extended the notion of S-invertible using the $*$-operation and we generalize some classical results concerning the notion of $*$-invertibility. We begin this section by the following definition.

Definition 3.1. Let D be an integral domain, $*$ a star-operation on D and S a multiplicative subset of D. A fractional ideal I of D is called S-$*$-invertible if there exists an $s \in S$ and a fractional ideal J of D such that $sD \subseteq (IJ)^* \subseteq D$.

Example 3.2. Let $D = \mathbb{Z} + X\mathbb{Z}[i][X]$, $S = \{2^n \mid n \in \mathbb{N}\}$ and $I = 2\mathbb{Z} + (1+i)X\mathbb{Z}[i][X]$. Since $2 \in I$, then $2D \subseteq I.D \subseteq D$. Which implies that I is S-invertible. On the other part, by [1, Lemma 2.1], it is easy to show that $I^{-1} = \mathbb{Z} + X\frac{1-i}{2}\mathbb{Z}[i][X]$. Thus if $II^{-1} = D$, then $1 = P_1(0)Q_1(0) + \cdots + P_n(0)Q_n(0)$ for some $P_1, ..., P_n \in I$ and $Q_1, ..., Q_n \in I^{-1}$. But for $1 \leq j \leq n$, $P_j(0) \in 2\mathbb{Z}$ and $Q_j(0) \in \mathbb{Z}$; so $1 = 2m_1 + \cdots + 2m_n$, $m_j \in \mathbb{Z}$. A contradiction. Hence I is not invertible.

Remark 3.3. Let D be an integral domain, $*$ a star-operation on D and S a multiplicative subset of D.

1. Since $I^* \subseteq I_v$ for each fractional ideal I of D, every S-$*$-invertible ideal of D is S-v-invertible.

2. Note that for a fractional ideal I of D, we have I is S-$*$-invertible if and only if I^* is S-$*$-invertible. Indeed, I is S-$*$-invertible if and only if $sD \subseteq (IJ)^* = (I^*J)^* \subseteq D$ for some $s \in S$ and some fractional ideal J of D if and only if I^* is S-$*$-invertible.

3. Let I be a fractional S-$*$-invertible ideal of D, then there exist an $s \in S$ and a fractional ideal J of D such that $sD \subseteq (IJ)^* \subseteq D$. We have

$$sI^{-1} = (I^{-1}sD)^* \subseteq (I^{-1}(IJ)^*)^* = (I^{-1}(IJ))^* \subseteq J^*.$$

   Moreover, since $IJ^* \subseteq (IJ)^* \subseteq D$, $J^* \subseteq I^{-1}$. Thus $sI^{-1} \subseteq J^* \subseteq I^{-1}$. Note that in the same way we can prove that $sJ^{-1} \subseteq I^* \subseteq J^{-1}$.

4. By [6, Proposition 2.7], every S-principal ideal of D is S-invertible. This implies that each S-principal ideal of D is S-$*$-invertible.

Proposition 3.4. *Let I be a fractional ideal of an integral domain D, S a multiplicative subset of D and $*$ a star-operation on D. Then I is S-$*$-invertible if and only of there exists an $s \in S$ such that $sD \subseteq (II^{-1})^* \subseteq D$. In particular, I^{-1} is also an S-$*$-invertible ideal of D.*

Proof. If I is S-$*$-invertible, then there exist an $s \in S$ and a fractional ideal J of D such that $sD \subseteq (IJ)^* \subseteq D$. But by Remark 3.3(3), $J^* \subseteq I^{-1}$; so $sD \subseteq (IJ)^* = (IJ^*)^* \subseteq (II^{-1})^* \subseteq D$. The other implication is obvious. □

Definition 3.5. Let D be an integral domain, S a multiplicative subset of D and $*$ a star-operation on D. A fractional ideal I of D is called of S-$*$-finite type if there exist an $s \in S$ and a fractional finitely generated ideal F of D such that $sI \subseteq F^* \subseteq I^*$.

Let D be an integral domain and S a multiplicative subset of D. According to [5], D is called an S-*Mori domain* if every increasing sequence of integral divisorial ideals of D is S-stationary (an increasing sequence $(I_k)_{k\in\mathbb{N}}$ of ideals of D is called S-*stationary* if there exist a positive integer n and an $s \in S$ such that for each $k \geq n$, $sI_k \subseteq I_n$ [8]). It was shown in [5], that if D is an S-Mori domain, then for each nonzero fractional ideal I of D, $sI \subseteq J_v \subseteq I_v$ for some $s \in S$ and some finitely generated fractional ideal J of D such that $J \subseteq I$. This implies that in an S-Mori domain every nonzero fractional ideal I of D is of S-v-finite type.

Remark 3.6. Let D be an integral domain, $*$ a star-operation on D and S a multiplicative subset of D. Let I be a fractional ideal of D of S-$*$-finite type. Then there exist an $s \in S$ and a fractional finitely generated ideal J of D such that $sI \subseteq J^* \subseteq I^*$. If the star-operation $*$ is of finite character, then we can suppose that $J \subseteq I$. Indeed, let $J = (a_1,...,a_n)$, where $a_i \in I^*$. Then for each $1 \leq i \leq n$, there exist a finitely generated subideal J_i of I. Let $J' = J_1 + \cdots + J_n$. Then J' is a finitely generated subideal of I. Moreover, $J \subseteq J_1^* + \cdots + J_n^* \subseteq (J')^*$; so $sI \subseteq J^* \subseteq (J')^* \subseteq I^*$.

Let D be an integral domain and $*$ a star-operation on D. Let I and J be tow fractional ideals of D. It will known that if $*$ is of finite character, then

$$(IJ)^* = \cup\{(I'J')^* \mid I' \subseteq I, J' \subseteq J, \text{two finitely generated fractional ideals of } D\}.$$

Our next Theorem prove a neccesary and sufficient condition for a fractional ideal to be S-$*$-invertible. This extended a result proved by Kang in [9]. To prove it we need the following Lemma.

Lemma 3.7. *Let D be an integral domain, $*$ a finite type $*$-operation on D and S a multiplicative subset of D. Every S-$*$-invertible ideal of D is an S-$*$-finite ideal of D.*

Proof. Let I be an S-$*$-invertible ideal of D. There exist an $s \in S$ and a fractional ideal J of D such that $sD \subseteq (IJ)^* \subseteq D$. Since $*$ is of finite character, there exist two finitely generated fractional ideals I' and J' of D such that $I' \subseteq I$, $J' \subseteq J$ and $s \in (I'J')^*$. This implies that $sD \subseteq (I'J')^* \subseteq D$. Now by Remark 3.3(3), $s(J')^{-1} \subseteq (I')^* \subseteq (J')^{-1}$ and $sJ^{-1} \subseteq I^* \subseteq J^{-1}$. Since $J' \subseteq J$, $J^{-1} \subseteq (J')^{-1}$; so

$$sI \subseteq sI^* \subseteq s(J')^{-1} \subseteq (I')^* \subseteq I^*.$$

Hence I is of S-$*$-finite type. □

Theorem 3.8. Let D be an integral domain, $*$ a finite type $*$-operation on D and S a weakly anti-Archimedean multiplicative subset of D. Let I be a fractional ideal of D. Then the following statements are equivalent.

1. I is an S-$*$-invertible ideal of D.

2. I is S-$*$-finite and for each $M \in S$-$*$-$\mathrm{Max}(D)$, ID_M is a principal ideal of D_M.

Proof. (1) $\Rightarrow$ (2) By Lemma 3.7, I is of S-$*$-finite type. Let M be an S-$*$-maximal ideal of D. We have $II^{-1} \nsubseteq M$, indeed, if $II^{-1} \subseteq M$, then $sD \subseteq (II^{-1})^* \subseteq M$ for some $s \in S$; so $s \in M$, a contradiction because $S \cap M = \emptyset$. This implies that $(ID_M)(I^{-1}D_M) = II^{-1}D_M = D_M$, and thus ID_M is an invertible ideal of D_M. Hence ID_M is principal since D_M is a local ring.

(2) $\Rightarrow$ (1) By hypothesis, there exist an $s \in S$ and a fractional finitely generated subideal J of I such that $sI \subseteq J^* \subseteq I^*$. Assume that I is not S-$*$-invertible. Then $(II^{-1})^* \subsetneq D$; so by Theorem 2.9, there exist an S-$*$-maximal ideal M of D such that $(II^{-1})^* \subseteq M$. By hypothesis, ID_M is principal, then $ID_M = aD_M$ for some $a \in I$. This implies that $\frac{1}{a}I \subseteq D_M$; so $\frac{1}{a}J \subseteq D_M$. Since J is finitely generated, there exists a $t \in D \backslash M$ such that $\frac{t}{a}J \subseteq D$. We have

$$\frac{st}{a}I \subseteq \frac{st}{a}I^* \subseteq \frac{t}{a}J^* \subseteq D.$$

Thus $\frac{st}{a} \in I^{-1}$ which implies that $st \in aI^{-1} \subseteq II^{-1} \subseteq M$. Since $t \notin M$, $s \in M$ because M is a prime ideal of D by Proposition 2.8. This contradict that $M \cap S = \emptyset$. Hence I is an S-$*$-invertible ideal of D. □

In the particular case when S consists of units of D we regain the following well-known result proved by B.G. Kang ([9]).

Corollary 3.9. *Let D be an integral domain, $*$ a finite type $*$-operation on D and I a fractional ideal of D. Then the following statements are equivalent.*

1. *I is a $*$-invertible ideal of D.*
2. *I is of $*$-finite type and it is t-locally principal.*

Let D be an integral domain and S a multiplicative subset of D. It is well-known that for each finitely generated fractional ideal I of D, $(I_S)^{-1} = (I^{-1})_S$. Our next Proposition improves this result.

Proposition 3.10. *Let S a multiplicative subset of an integral domain D, $*$ a finite type $*$-operation on D and I a fractional ideal of D. If I is an S-$*$-finite ideal of D, then $(I_S)^{-1} = (I^{-1})_S$.*

Proof. We have always that $(I^{-1})_S \subseteq (I_S)^{-1}$, so we must prove the converse in order to conclude. Since I is S-$*$-finite, there exist an $s \in S$ and a finitely generated ideal $J \subseteq I$ such that $sI \subseteq J^* \subseteq I^*$. Thus $J^{-1} \subseteq \frac{1}{s}I^{-1}$, and consequently $(J^{-1})_S \subseteq (I^{-1})_S$. Since J is finitely generated, $(J^{-1})_S = (J_S)^{-1}$. Moreover, $J_S \subseteq I_S$. Thus $(I_S)^{-1} \subseteq (J_S)^{-1} = (J^{-1})_S \subseteq (I^{-1})_S$, and hence $(I^{-1})_S = (I_S)^{-1}$. □

Next, we give a relation between S-t-invertible ideals of D and t-invertible ideals of the localization D_S, where t- is the t-operation.

Proposition 3.11. *Let S a multiplicative subset of an integral domain D and I a fractional ideal of D.*

1. *If I is an S-t-invertible ideal of D, then I_S is a t-invertible ideal of D_S.*
2. *Assume that for each t-finite type ideal J of D, $(J_S)_t \cap D = J_t : s$ for some $s \in S$. Then I is S-t-invertible if and only if I_S is t-invertible and I is an S-$*$-finite ideal of D.*

Proof. (1). Since I is S-t-invertible, $sD \subseteq (II^{-1})_t \subseteq D$ for some $s \in S$. This implies that $D_S = ((II^{-1})_t)_S$. But $((II^{-1})_t)_S \subseteq ((II^{-1})_S)_t$; so $D_S = ((II^{-1})_S)_t$ because $((II^{-1})_S)_t \subseteq D_S$. Thus $D_S = (I_S(I^{-1})_S)_t$, and hence I_S is a t-invertible ideal of D_S.

(2). The "only if" part follows from (1) and Lemma 3.7, since t is a finite type $*$-operation. For the "if" part, let $s \in S$ and J a finitely generated subideal of I such that $sI \subseteq J_t \subseteq I_t$. This implies that

$(I_t)_S = (J_t)_S$. First we show that J_S is t-invertible. Since I_S is t-invertible, $D_S = (I_S(I^{-1})_S)_t$. Thus

$$\begin{aligned} D_S &= (I_S(I^{-1})_S)_t \\ &\subseteq ((I_t)_S(I^{-1})_S)_t \\ &\subseteq ((J_t)_S(J^{-1})_S)_t \\ &= ((J_tJ^{-1})_S)_t \\ &\subseteq ((JJ^{-1})_S)_t \\ &\subseteq D_S. \end{aligned}$$

This implies that $((J_S(J^{-1})_S))_t = ((JJ^{-1})_S)_t = D_S$, hence J_S is t-invertible. Now, since J_S is t-invertible, $(J_S)^{-1}$ is of t-finite type; so there exists a finitely generated subideal F of J^{-1} such that $(J^{-1})_S = (J_S)^{-1} = (F_S)_t$. Thus $D_S = ((JJ^{-1})_S)_t = ((FJ)_S)_t$; so $D = ((FJ)_S)_t \cap D$. By hypothesis, $D = (FJ)_t : s'$ for some $s' \in S$, which implies that $s'D \subseteq (FJ)_t$. But $F \subseteq J^{-1} \subseteq \frac{1}{s}I^{-1}$ and $J \subseteq I$, thus $ss'D \subseteq (sFJ)_t \subseteq (II^{-1})_t \subseteq D$, and hence I is an S-t-invertible ideal of D. □

Proposition 3.12. *Let I be a non zero ideal of an integral domain D. Let T be a multiplicatively closed subset of D and S be a multiplicative subset of D.*

1. *If I is an S-t-ideal of D, then $I_T \bigcap D$ is an S-t-ideal of D.*
2. *If I_T is an S-t-ideal of D_T, then $I_T \bigcap D$ is an S-t-ideal of D.*

Proof. 1. Let I be a S-t-ideal of D. Then $sI_t \subseteq I$ for some $s \in S$. We show that $s(I_T \bigcap D)_t \subseteq I_T \bigcap D$. Let $\alpha \in (I_T \bigcap D)_t$, thus there exists a finitely generated fractional ideal F of D contained in $(I_T \bigcap D)$ such that $\alpha \in F_v$. Since $F \subseteq F_T \subseteq I_T$, then $s\alpha \in s(I_T)_t$ and there exists an $r \in T$ such that $rF \subseteq I$. Then $r\alpha \in rF_v = (rF)_v \subseteq I_t \subseteq \frac{1}{s}I$. Hence $sr\alpha \subseteq I$, so $s\alpha \subseteq I_T$, then $s\alpha \subseteq I_T \bigcap D$. Therefore $s(I_T \bigcap D)_t \subseteq I_T \bigcap D$.

2. Let I_T be an S-t-ideal of D_T. Then $s(I_T)_t \subseteq I_T$ for some $s \in S$. We show that $s(I_T \bigcap D)_t \subseteq I_T \bigcap D$. Let $\alpha \in (I_T \bigcap D)_t$, thus there exists a finitely generated fractional ideal J of D contained in $(I_T \bigcap D)$ such that $\alpha \in J_v$. Since $J \subseteq J_T \subseteq I_T$, then $s\alpha \in s(I_T)_t$. Hence $s\alpha \in s(I_T)_t \bigcap D \subseteq I_T \bigcap D$. Therefore $s(I_T \bigcap D)_t \subseteq I_T \bigcap D$.

□

Let D be an integral domain with quotient field K. Let $*$ be a star operation on D. Let $f = a_0 + \cdots + a_nX^n \in K[X]$, A_f will denote the D-submodule of K generated by $\{a_0, ..., a_n\}$. The set $N_* = \{f \in D[X] \mid (A_f)^* = D\}$ is a multiplicatively closed subset of $D[X]$. We defined the ring $D[X]_{N_*}$ by $D[X]_{N_*} = \{\frac{f}{g} \mid f \in D[X], g \in N_*\}$.

Proposition 3.13. *Let $*$ be a $*$-operation on an integral domain D with quotient field K, S be a multiplicative subset of D. Let I be an ideal of D. Then :*

1. *If I is S-$*$-ideal, then there exist $s \in S$ such that $s(ID[X]_{N_*} \bigcap K) \subseteq I$.*
2. *If I is an S-v-ideal (resp., S-t-ideal) of D, then $I[X]_{N_v}$ is an S-v-ideal (resp., S-t-ideal) of $D[X]_{N_v}$.*

Proof. 1. Let I be S-$*$-ideal. Then $sI^* \subseteq I$, for some $s \in S$. We show that $s(ID[X]_{N_*} \bigcap K) \subseteq I$. Let $a \in (ID[X]_{N_*} \bigcap K)$. Then $ag = f$ for some $g \in N_*$ and $f \in I[X]$. Hence $(a) = (aA_g)^* = (A_{ag})^* = (A_f)^* \subseteq I^* \subseteq \frac{1}{s}I$. So $sa \in I$. Therefore $s(ID[X]_{N_*} \bigcap K) \subseteq I$.

2. Suppose that I is a S-v-ideal, then $sI_v \subseteq I$, for some $s \in S$. Then $s(I[X]_{N_v})_v = sI_v[X]_{N_v}$ by [9, Proposition 2.2]. Hence $s(I[X]_{N_v})_v \subseteq I[X]_{N_v}$. Therefore $I[X]_{N_v}$ is a S-v-ideal of $D[X]_{N_v}$. In the some way we can show that $I[X]_{N_v}$ is an S-t-ideal of $D[X]_{N_v}$.

□

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