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On semi radical ideals of noncommutative rings

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Abstract. Since the introduction of n-ideals and J-ideals in commutative rings many different aspects of these ideals have been investigated. As a generalization the notion of weakly n-ideals and weakly J-ideals was introduced and studied. Recently it was proved that many of the results are also true for noncommutative rings as a special case of a more general situation. In a recent paper Khashan et. al introduced the notion of semi n-ideals as a generalization of n-ideals where n is the prime radical and studied this generalization. In this note we show that these results are special cases of a more general situation. If ρ is a special radical and R a noncommutative ring then the ideal I of R is a semi ρ-ideal if $aRa \subseteq I$, then $a \in \rho(R)$ or $a \in I$. This covers a wide spectrum of semi ideals and if ρ is the prime radical we have the notion of semi n-ideals for noncommutative rings. In this note we prove that most of the results for the semi n-ideals are satisfied for noncommutative rings as a special case.
Key Words: Special radical, semi n-ideal, semi ρ-ideal, semi ρ-submodule.
2010 MSC: 16N20, 16N40, 16N80, 16L30.

1 Introduction

Throughout this paper, all rings are assumed to be noncommutative with nonzero identity. We recall that a proper ideal I of a ring R is called semiprime if whenever $a \in R$ is such that $aRa \subseteq I$, then $a \in I$. In 2017, Tekir, Koc and Oral in [10] introduced the concept of n-ideals of commutative rings. A proper ideal I of a commutative ring R is called an n-ideal if whenever $a, b \in R$ are such that $ab \in I$ and $a \notin \mathcal{P}(R)$, then $b \in I$ where $\mathcal{P}(R)$ is the prime radical of the ring R. Recently, Khashan and Bani-Ata in [8] generalized n-ideals by defining and studying the class of J-ideals. A proper ideal I of R is called a J-ideal if $ab \in I$ and $a \notin J(R)$ imply $b \in I$ for $a, b \in R$, where $J(R)$ denotes the Jacobson radical of R. In [5] Groenewald introduce the notion of ρ-ideals for a noncommutative ring and a special radical ρ. An ideal I of a noncommutative ring R is a ρ-ideal if for $a, b \in R$ such that $aRb \subseteq I$ and $a \notin \rho(R)$, then $b \in I$. In [1] the notion of a semi n-ideal is introduced as a new generalization of the concept of n-ideals by defining a proper ideal I of a commutative ring R to be a semi n-ideal if whenever $a \in R$ is such that $a^2 \in I$, then $a \in \mathcal{P}(R)$ or $a \in I$. Some examples of semi n-ideals are given and semi n-ideals are investicated under various contexts. In this paper we introduce the notion of semi ρ-ideals for a special radical ρ and a noncommutative ring R as new generalization of the concept of ρ-ideals. If I is an ideal of the noncommutative ring R and ρ is a special radical, then I is a semi ρ-ideal if $aRa \subseteq I$ and $a \notin \rho(R)$, then $a \in I$. The class of semi ρ-ideals is a generalization of semiprime and n-ideals. We start Section 2 by giving some examples (see Example 2.4) to show that this generalization is proper. Next, we determine several characterizations of semi ρ-ideals for a special radical ρ. In the rest of the paper ρ will always be a special radical. We investigate semi ρ-ideals under various contexts of constructions such as homomorphic images and idealizations, see Propositions 5.1 and 5.2. Moreover, for a direct product of rings $R = R_1 \times R_2 \times ... \times R_k$, we determine all semi ρ-ideals of R, see Theorems 3.2 and 3.3.

In 1978, the concept of semiprime submodules is presented. A proper submodule is said to be semiprime if whenever $r \in R, m \in M$ and $rRrm \subseteq N$, then $rm \in N$. See [3] for properties of semiprime submodules. Afterwards, the notions of ρ-submodules are introduced and studied in [5]. A proper submodule N is called an ρ-submodule of M if whenever $rRm \subseteq N$ and $r \notin (\rho(R)M : M)$, then $m \in N$. As a new generalization of the above structures, in Section 4, we define a proper submodule N of M to be a semi ρ-submodule if whenever $rRrm \subseteq N$ and $r \notin (\rho(R)M : M)$, then $rm \in N$. We illustrate (see Example 4.7) that this generalization of ρ-submodules is proper.

In what follows, R is a ring (associative, not necessarily commutative and not necessarily with identity) and M is an $R-R$-bimodule. The idealization of M is the ring $R \boxplus M$ with $(R \boxplus M, +) = (R, +) \oplus (M, +)$ and the multiplication is given by $(r, m)(s, n) = (rs, rn + ms)$. $R \boxplus M$ itself is, in a canonical way, an $R-R$-bimodule and $M \simeq 0 \boxplus M$ is a nilpotent ideal of $R \boxplus M$ of index 2. We also have $R \simeq R \boxplus 0$ and the latter is a subring of $R \boxplus M$. If I is an ideal of R and N is an $R-R$-bi-submodule of M, then $I \boxplus N$ is an ideal of $R \boxplus M$ if and only if $IM + MI \subseteq N$. If ρ is a special radical, it follows from [11] that if R is any ring, then $\rho(R \boxplus M) = \rho(R) \boxplus M$ for all $R-R$-bimodules M. In Proposition 5.1, we clarify the relation between semi ρ-ideals of the idealization ring $R \boxplus M$ and those of R. For the following definitions of special radicals and related results we refer the reader to [12].

A class ρ of rings forms a radical class in the sense of Amitsur-Kurosh if ρ has the following three properties

1. The class ρ is closed under homomorphism, that is, if $R \in \rho$, then $R/I \in \rho$ for every $I \lhd R$.
2. Let R be any ring. If we define $\rho(R) = \sum\{I \lhd R : I \in \rho\}$, then $\rho(R) \in \rho$.
3. For any ring R the factor ring $R/\rho(R)$ has no nonzero ideal in ρ i.e. $\rho(R/\rho(R)) = 0$.

A class $\mathcal{M}$ of rings is a **special class** if it is hereditary, consists of prime rings and satisfies the following condition (∗) if $0 \neq I \lhd R$, $I \in \mathcal{M}$ and R a prime ring, then $R \in \mathcal{M}$.

Let $\mathcal{M}$ be any special class of rings. The class $\mathcal{U}(\mathcal{M}) = \{R : R$ has no nonzero homomorphic image in $\mathcal{M}\}$ of rings forms a radical class of rings and the upper radical class $\mathcal{U}(\mathcal{M})$ is called a special radical class.

Let ρ be a special radical with special class $\mathcal{M}$ i.e. $\rho = \mathcal{U}(\mathcal{M})$. Now let $\mathcal{S}_\rho = \{R : \rho(R) = 0\}$. If $\mathcal{P}$ denotes the class of prime rings, then for the special radical ρ it follows from [12] that $\rho = \mathcal{U}(\mathcal{P} \cap \mathcal{S}_\rho)$. For a ring R we have $\rho(R) = \cap\{I \lhd R : R/I \in \mathcal{P} \cap \mathcal{S}_\rho\}$ i.e. ρ has the intersection property relative to the class $\mathcal{P} \cap \mathcal{S}_\rho$.

Let $I \lhd R$, then $\rho(R/I) = \rho^*(I)/I$ for some uniquely determined ideal $\rho^*(I)$ of R with $\rho(I) \subseteq I \subseteq \rho^*(I)$ and $\rho^*(I)$ is called the radical of the ideal I while $\rho(I)$ is the radical of the ring I.

We also have $\rho^*(I) = \rho(R)$ if and only if $I \subseteq \rho(R)$. Also $I = \rho^*(I)$ if and only if $R/I \in \mathcal{S}_\rho$.

In what follows let ρ be a special radical with special class $\mathcal{M}$. Hence $\rho = \mathcal{U}(\mathcal{P} \cap \mathcal{S}_\rho)$.

The following are some of the well known special radicals which are defined in [12], prime radical β, Levitski radical $\mathcal{L}$, Kőthe's nil radical $\mathcal{N}$, Jacobson radical $\mathcal{J}$ and the Brown McCoy radical $\mathcal{G}$.

Definition 1.1. Let ρ be a special radical. A proper ideal I of the ring R is called a ρ-ideal if whenever $a, b \in R$ and $aRb \subseteq I$ and $a \notin \rho(R)$, then $b \in I$.

In [10] and [8] the notions of n-ideals and J-ideals were introduced for commutative rings.

Definition 1.2. [10, Definition 2.1] and [8, Definition 2.1] If ρ is the prime radical or the Jacobson radical of a commutative ring, then a proper ideal I of R is a ρ-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin \rho(R)$, then $b \in I$.

Remark 1.3. Let R be a commutative ring and I a proper ideal of R. I is a ρ-ideal if and only if $a, b \in R$ with $ab \in I$ and $a \notin \rho(R)$, then $b \in I$.

2 Semi-ρ-ideals

Definition 2.1. Let ρ be a special radical. A proper ideal I of the ring R is called a semi ρ-ideal if whenever $a \in R$ and $aRa \subseteq I$, then $a \in I$ or $a \in \rho(R)$.

Proposition 2.2. *If ρ is a special radical, then I is a semi ρ-ideal if $I = \rho^*(I)$ or $\rho(R) = \rho^*(I)$.*

Proof. Since $I = \rho^*(I)$ if and only if $R/I \in \mathcal{S}_\rho$, it is clear that I is a semiprime ideal and hence a semi ρ-ideal. Now, if $\rho(R) = \rho^*(I)$ we have that $I \subseteq \rho(R)$ and if $aRa \subseteq I$, then $aRa \subseteq \rho(R)$ and since $\rho(R)$ is a semiprime ideal, we have $a \in \rho(R)$ and hence I is a semi ρ-ideal. □

It is known that if R is a commutative ring and ρ is the prime radical then if I is a semi ρ-ideal then $I = \rho^*(I)$ or $\rho(R) = \rho^*(I)$ (see [1, Proposition 2.2]). It is not clear if this is also the case for noncommutative rings.

Since for any special radical ρ and a ring R, $\rho(R)$ is a semiprime ideal, the following properties of semi ρ-ideals can be easily observed.

Proposition 2.3. *For a special radical ρ and a ring R, the following statements hold.*

1. Every ρ-ideal is a semi ρ-ideal.

2. Every (weakly) semiprime ideal I is a semi ρ-ideal. The converse also holds if $\rho(R) \subseteq I$.

3. For every proper ideal I of R, $\rho^*(I)$ is a (semiprime) semi ρ-ideal. In particular, $\rho(R)$ is a semi ρ-ideal of R.

4. If I is an ideal such that $I \subseteq \rho(R)$, then I is a semi ρ-ideal.

5. If ρ is a special radical and $R \in \mathcal{S}_\rho$, then an ideal I of R is a semi ρ-ideal if and only if it is a semi-prime ideal.

However, the converses of 1. and 2. in Proposition 2.3 are not true in general.

Example 2.4.

1. Let ρ be a special radical and $R \in \mathcal{S}_\rho$. If I is a nonzero ideal of R then I is a semi ρ-ideal which is not a ρ-ideal. This follows from [5, Proposition 1.5] since $I \neq \rho(R) = \{0\}$.

2. Let $\rho = \mathcal{P}$ and $R = M_2(\mathbb{Z}_{32})$. $I = M_2(\langle\overline{16}\rangle)$ is a semi ρ-ideal which is not a semi prime ideal.

Remark 2.5. If R is an Artinian ring, then since $\beta(R) = \mathcal{L}(R) = \mathcal{N}(R) = \mathcal{J}(R)= \mathcal{G}(R)$ the notions of $\beta, \mathcal{L}, \mathcal{N}, \mathcal{J}$ and semi $\mathcal{G}$-ideals are the same. For a commutative ring R, we have $\beta(R) = \mathcal{L}(R) = \mathcal{N}(R)$. Hence for commutative rings the notions semi β, semi $\mathcal{L}$ and semi $\mathcal{N}$-ideals are the same.

Next, we give some equivalent conditions that characterize semi ρ-ideals for a special radical ρ.

Theorem 2.6. Let ρ be a special radical and let I be a proper ideal of a ring R. The following statements are equivalent.

1. I is a semi ρ-ideal of R.

2. Whenever $a \in R$ with $0 \neq aRa \subseteq I$, then $a \in \rho(R)$ or $a \in I$.

3. Whenever $a \in R$ with $\langle a\rangle^2 \subseteq I$, then $\langle a\rangle \subseteq \rho(R)$ or $\langle a\rangle \subseteq I$.

4. If A is an ideal of R such that $A^2 \subseteq I$, then $A \subseteq \rho(R)$ or $A \subseteq I$.

5. If A is an ideal of R such that $A^n \subseteq I$ for some positive integer n, then $A \subseteq \rho(R)$ or $A \subseteq I$.

6. If A is a left ideal (right ideal) of R such that $A^2 \subseteq I$, then $A \subseteq \rho(R)$ or $A \subseteq I$.

Proof. (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) Let $a \in R$ such that $aRa \subseteq I$. If $aRa = \{0\}$, then $aRa \subseteq \rho(R)$ and since $\rho(R)$ is a semiprime ideal, we have $a \in \rho(R)$. If $0 \neq aRa \subseteq I$ the result follows from (2).

(1) $\Rightarrow$ (3) Let $a \in R$ with $\langle a\rangle^2 \subseteq I$. Now $aRa \subseteq \langle a\rangle^2 \subseteq I$ and we have $a \in I$ or $a \in \rho(R)$ and hence $\langle a\rangle \subseteq I$ or $\langle a\rangle \subseteq \rho(R)$.

(3) $\Rightarrow$ (4) Let A be an ideal of R such that $A^2 \subseteq I$. Suppose $A \not\subseteq \rho(R)$, then $A^2 \not\subseteq \rho(R)$ since $\rho(R)$ is a semiprime ideal of R. We show that $A \subseteq I$. Suppose $a \in A^2$ and $a \notin \rho(R)$. Let b be any element of A. Now $\langle b\rangle^2 \subseteq A^2 \subseteq I$. If $b \notin \rho(R)$, then $b \in I$ from (3). Suppose $b \in \rho(R)$. We have $(\langle a+b\rangle)^2 \subseteq (\langle a\rangle + \langle b\rangle)^2 \subseteq \langle a\rangle\langle a\rangle + \langle a\rangle\langle b\rangle + \langle b\rangle\langle a\rangle + \langle b\rangle\langle b\rangle \subseteq A^2 \subseteq I$. Hence $\langle a+b\rangle \subseteq I$ or $\langle a+b\rangle \subseteq \rho(R)$. $\langle a+b\rangle \not\subseteq \rho(R)$ for if $\langle a+b\rangle \subseteq \rho(R)$, then $a \in \rho(R)$ a contradiction. Hence $\langle a+b\rangle \subseteq I$. Since $a \in I$, we have $b \in I$ and hence $A \subseteq I$.

(4) $\Rightarrow$ (5) Let $A^n \subseteq I$ for some positive integer n. To prove the argument, we use mathematical induction. If $n \leqslant 2$ the result follows from (4). Assume that the claim of (4) holds for all $2 < k < n$. We show that it is also true for n. Suppose n is even, say, $n = 2t$ for some positive integer t. Now, $A^n = (A^t)^2 \subseteq I$. From (4) we have $A^t \subseteq I$ or $A^t \subseteq \rho(R)$. If $A^t \subseteq \rho(R)$, then $A \subseteq \rho(R)$ since $\rho(R)$ is a semi prime ideal of R. If $A^t \subseteq I$, then by the induction hypothesis, we conclude that $A \subseteq I$. Now, suppose n is odd. Then $n+1 = 2s$ for some $s < n$. Similarly, since $(A^s)^2 \subseteq I$, $(A^s) \subseteq I$ or $A^s \subseteq \rho(R)$. If $A^s \subseteq \rho(R)$, then $A \subseteq \rho(R)$ since $\rho(R)$ is a semi prime ideal of R. If $A^t \subseteq I$, then by the induction hypothesis, we conclude that $A \subseteq I$, so we are done.

(5) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (6) Let T be a left ideal of R such that $T^2 \subseteq I$. Now $TRTR \subseteq T^2R \subseteq I$. From (4) $TR \subseteq I$ or $TR \subseteq \rho(R)$. Since R has an identity, we have $T \subseteq I$ or $T \subseteq \rho(R)$ and we are done.

(6) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (1) Let $a \in R$ such that $aRa \subseteq I$. Now $RaRRaR \subseteq I$ and from (4) we have that $a \in RaR \subseteq I$ or $a \in RaR \subseteq \rho(R)$ and we are done. □

Lemma 2.7. *Let ρ be a special radical and I and J be ideals of R with $I, J \not\subseteq \rho(R)$. Then*

1. *If I and J are semi ρ-ideals with $I^2 = J^2$, then $I = J$.*
2. *If I^2 is a semi ρ-ideal, then $I^2 = I$.*

Proof. 1. Since $I^2 \subseteq J$ and $I \not\subseteq \rho(R)$, then by Theorem 2.3, we have $I \subseteq J$. Similarly, since $J^2 \subseteq I$ and $J \not\subseteq \rho(R)$, we have $J \subseteq I$. Thus, we have the equality.

2. Since $I^2 \subseteq I^2$, $I \not\subseteq \rho(R)$ and I^2 is a semi ρ-ideal, we have $I \subseteq I^2$ and so $I^2 = I$. □

Proposition 2.8. *Let ρ_1 and ρ_2 be two special radicals such that $\rho_1 \leq \rho_2$, then every semi ρ_1-ideal is a semi ρ_2-ideal.*

Proof. Let I be a semi ρ_1-ideal of the ring R and suppose $aRa \subseteq I$ and $a \notin \rho_2(R)$. Since $\rho_1 \leq \rho_2$, we have $\rho_1(R) \subseteq \rho_2(R)$ and therefore $a \notin \rho_1(R)$. Since I is a semi ρ_1-ideal, we have $a \in I$ and we are done. □

Remark 2.9. The converse of Proposition 2.8 is not true in general as can be seen from the following example. Consider the local ring $R = \mathbb{Z}_{\langle 2\rangle} = \{\frac{a}{b} : a, b \in \mathbb{Z}, 2 \nmid b\}$ and let $I = \langle 4\rangle_{\langle 2\rangle} = \{\frac{a}{b} : a \in \langle 4\rangle, 2 \nmid b\}$. Since R is a local ring, I is a $\mathcal{J}$-ideal and hence also a semi $\mathcal{J}$-ideal. I is not a semi $\mathcal{P}$-ideal of R. For example, $\left(\frac{2}{3}\right)^2 \in I$ but $\frac{2}{3} \notin \mathcal{P}(R) = \{0\}$ and $\frac{2}{3} \notin I$.

Proposition 2.10. *Let $\{I_i\}_{i\in\Delta}$ be a family of semi ρ-ideals of R, then $\bigcap_{i\in\Delta} I_i$ is a semi ρ-ideal of R.*

Proof. Let $aRa \subseteq \bigcap_{i\in\Delta} I_i$ with $a \notin \rho(R)$ for $a \in R$. Then $aRa \subseteq I_i$ for every $i \in \Delta$. Since I_i is a semi ρ-ideal of R and $a \notin \rho(R)$, we get $a \in I_i$ for every $i \in \Delta$. Hence $a \in \bigcap_{i\in\Delta} I_i$. □

Theorem 2.11. Let R and S be rings and $f: R \to S$ be a surjective ring-homomorphism. If ρ is a special radical, then the following statements hold:

1. If I is a semi ρ-ideal of R and $\ker(f) \subseteq I$, then $f(I)$ is a semi ρ-ideal of S.

2. If J is a semi ρ-ideal of S and $\ker(f) \subseteq \rho(R)$, then $f^{-1}(J)$ is a semi ρ-ideal of R.

Proof. **1**. Let $c \in S$ such that $cSc \subseteq f(I)$ and $c \notin \rho(S)$. Since f is surjective we can choose $a \in R$ such that $f(a) = c$. Now, $cSc = f(a)f(R)f(a) = f(aRa) \subseteq f(I)$ and since $\ker(f) \subseteq I$, we have $aRa \subseteq I$. Because $c \notin \rho(S)$ we have $a \notin \rho(R)$ for if $a \in \rho(R)$, then $c = f(a) \in f(\rho(R) \subseteq \rho(S)$ since ρ is a special radical. Thus $a \notin \rho(R)$ and since $aRa \subseteq I$ and a semi ρ-ideal of R, we get $a \in I$. Hence $c = f(a) \in f(I)$ and therefore $f(I)$ is a semi ρ-ideal of S.

2. Let $a \in R$ such that $aRa \subseteq f^{-1}(J)$ and $a \notin \rho(R)$. Now, $f(a)Sf(a) = f(aRa) \subseteq J$. We show that $f(a) \notin \rho(S)$. Suppose $f(a) \in \rho(S)$ and $M \lhd R$ such that $R/M \in \mathcal{S}_\rho \cap \mathcal{P}$. Since f is a surjective homomorphism and $\ker(f) \subseteq \rho(R) \subseteq M$, we have $f(R)/f(M) \simeq R/\ker(f)/M/\ker(f) \simeq R/M$. Hence $f(R)/f(M) \in \mathcal{S}_\rho \cap \mathcal{P}$ and therefore $f(a) \in f(M)$. Hence $a \in M$ since $\ker(f) \subseteq M$ and therefore $a \in \cap\{I \lhd R : R/I \in \mathcal{P} \cap \mathcal{S}_\rho\} =\rho(R)$ which is a contradiction. Since J is a semi ρ-ideal, we have $f(a) \in J$ and so $a \in f^{-1}(J)$. It follows that $f^{-1}(J)$ is a semi ρ-ideal of R. □

Corollary 2.12. *Let ρ be a special radical and let R be a ring and let I, K be two ideals of R with $K \subseteq I$. Then the following hold.*

1. *If I is a semi ρ-ideal of R, then I/K is a semi ρ-ideal of R/K.*

2. *If I/K is a semi ρ-ideal of R/K and $K \subseteq \rho(R)$, then I is a semi ρ-ideal of R.*

3. *If I/K is a semi ρ-ideal of R/K and K is a semi ρ-ideal of R, then I is a semi ρ-ideal of R.*

Proof. **1.** Assume that I is a semi ρ-ideal of R with $K \subseteq I$. Let $\pi: R \to R/K$ be the natural epimorphism defined by $\pi(R) = r+K$. Note that $\ker(\pi) = K \subseteq I$. Thus, by Theorem 2.11 1., it follows that $\pi(I) = I/K$ is a semi ρ-ideal of R/K.

2. Again consider the natural epimorphism $\pi: R \to R/K$. Since $K \subseteq \rho(R)$, by Theorem 2.11 2., $I = \pi^{-1}(I/K)$ is a semi ρ-ideal of R.

3. This is clear by 2. and Theorem 2.11. □

Proposition 2.13. *Let ρ be a special radical and let I and J be two semi ρ-ideals in a ring R. If $I + J$ is proper in R, then $I + J$ is a semi ρ-ideal of R.*

Proof. By (1) of Corollary 2.12, $I/I \cap J$ is a semi ρ-ideal of $R/I \cap J$. Thus, $(I+J)/J \cong I/I \cap J$ is also a semi ρ-ideal of R/J. Therefore, by (2) of Corollary 2.12, we conclude that $I + J$ is a semi ρ-ideal of R. □

However, if I and J are two semi $\mathcal{P}$-ideals in a ring R, then IJ need not be a semi $\mathcal{P}$-ideal. For example, while $M_2(\langle 2\rangle)$ is a semi $\mathcal{P}$-ideal of $M_2(\mathbb{Z})$, $(M_2(\langle 2\rangle))^2 = M_2(\langle 4\rangle)$ is not so.

Let I be a proper ideal of R, then $Z_I(R)$ denote the set $\{r \in R : sr \in I$ for some $s \in R \backslash I\}$.

Proposition 2.14. *Let ρ be a special radical and R a ring with S a non-empty subset of R where $\langle S\rangle \cap Z_{\rho(R)}(R) = \emptyset$. If I is a semi ρ-ideal of R with $S \not\subseteq I$, then $(I : \langle S\rangle)$ is a semi ρ-ideal of R.*

Proof. Let $a \in R$ such that $aRa \subseteq (I : \langle S\rangle)$ but $a \notin \rho(R)$. Then $asRas \subseteq aRa\langle S\rangle \subseteq I$ for all $s \in \langle S\rangle$. As I is a semi ρ-ideal of R, we have either $as \in \rho(R)$ or $as \in I$ for all $s \in \langle S\rangle$. If $as \in \rho(R)$, then $\langle S\rangle \cap Z_{\rho(R)}(R) \neq \emptyset$, a contradiction. Thus, $as \in I$ for all $s \in \langle S\rangle$ and so $a \in (I : \langle S\rangle)$ as required. □

Theorem 2.15. Let ρ be a special radical and R a commutative ring. If an ideal I of R is a maximal semi ρ-ideal satisfying $Z_{\rho(R)}(R) \subseteq I$, then I is semi prime in R. Additionally, if $I \subseteq \rho(R)$, then $I = \rho(R)$ is a prime ideal.

Proof. The same as [1, Theorem 3.1] by replacing $\mathcal{P}(R)$ with $\rho(R)$. □

3 Product of rings

Suppose that R_1, R_2 are two noncommutative rings with nonzero identities and $R = R_1 \times R_2$. Then R becomes a noncommutative ring with coordinate-wise addition and multiplication. Also, every ideal I of R has the form $I = I_1 \times I_2$, where I_i is an ideal of R_i for $i = 1, 2$. Now, we give the following result.

Proposition 3.1. *Let R_1 and R_2 be two noncommutative rings and let ρ be a special radical such that $\rho(R) = \rho(R_1) \times \rho(R_2)$. Then $R_1 \times R_2$ has no ρ-ideals.*

Proof. Assume that $I = I_1 \times I_2$ is a ρ-ideal of $R_1 \times R_2$, where I_i is an ideal of R_i for $i = 1, 2$. Since $(0,1)R_1 \times R_2(1,0) \subseteq I_1 \times I_2$, $(0,1) \notin \rho(R_1 \times R_2) = \rho(R_1) \times \rho(R_2)$ and $(1,0) \notin \rho(R_1 \times R_2) = \rho(R_1) \times \rho(R_2)$, we conclude that $(0,1), (1,0) \in I$ and so $I = R_1 \times R_2$, a contradiction.

By characterizing semi ρ-ideals of R, the next theorem allows us to build some examples for semi ρ-ideals which are not ρ-ideals. □

Theorem 3.2. Let R_1 and R_2 be two noncommutative rings and let ρ be a special radical such that $\rho(R) = \rho(R_1) \times \rho(R_2)$. Then a proper ideal $I = I_1 \times I_2$ is a semi ρ-ideal of R if and only if one of the following statements holds.

1. I is a semiprime ideal of R.
2. I_1 is a semi ρ-ideal of R_1 and $I_2 = \rho(R_2)$.
3. I_2 is a semi ρ-ideal of R_2 and $I_1 = \rho(R_1)$.

Proof. ⇒Suppose $I = I_1 \times I_2$ is a semi ρ-ideal which is not a semiprime ideal. Hence there exists $(x,y) \in R_1 \times R_2$ such that $(x,y)(R_1 \times R_2)(x,y) \subseteq I_1 \times I_2$ but $(x,y) \notin I_1 \times I_2$. We show that $I_1 = \rho(R_1)$ or $I_2 = \rho(R_2)$. Assume not. If $I_1 \neq \rho(R_1)$ and $I_2 \neq \rho(R_2)$, then there exist $a \in I_1 \backslash \rho(R_1)$ and $b \in I_2 \backslash \rho(R_2)$. Now $(x+a)R_1(x+a) = xR_1x + xR_1a + aR_1x + aR_1a \subseteq I_1$ and also $(y+b)R_2(y+b) \subseteq I_2$. From this it follows that $(x+a, y+b)(R_1 \times R_2)(x+a, y+b) \subseteq I_1 \times I_2 = I$. We have $(x,y) \notin I_1 \times I_2$, so without lost of generality we may suppose $x \notin I_1$. Hence $(x+a) \notin I_1$ and so $(x+a, y+b) \notin I$. Since $I = I_1 \times I_2$ is a semi ρ-ideal, we have $(x+a, y+b) \in \rho(R) = \rho(R_1) \times \rho(R_2)$. Hence $(x+a) \in \rho(R_1)$ and $(y+b) \in \rho(R_2)$ which implies that $(x,y) \notin \rho(R)$ since $a \notin \rho(R_1)$ and $b \notin \rho(R_2)$. This is impossible since I is a semi ρ-ideal.

Suppose without loss of generality that $I_1 \neq \rho(R_1)$ and $I_2 = \rho(R_2)$. Let $aR_1a \subseteq I_1$ and $a \notin I_1$. Now, $(a,0)R(a,0) = (aR_1a, 0) \subseteq I_1 \times I_2 = I$. Since $(a,0) \notin I$ and I a semi ρ-ideal, we have $(a,0) \in \rho(R) = \rho(R_1) \times \rho(R_2)$. Hence $a \in \rho(R_1)$ and I_1 is a semi ρ-ideal of R_1. Similarly if $I_1 = \rho(R_1)$ and $I_2 \neq \rho(R_2)$ we get I_2 is a semi ρ-ideal of R_2

⇐ If I is a semiprime ideal of R then I is a semi ρ-ideal of R by Proposition 2.6. Suppose $I = I_1 \times \rho(R_2)$ with I_1 a semi ρ-ideal of R_1. Let $(a,b) \in R = R_1 \times R_2$ such that $(a,b)(R_1 \times R_2)(a,b) \subseteq I_1 \times \rho(R_2)$ and $(a,b) \notin \rho(R) = \rho(R_1) \times \rho(R_2)$. Now, $bR_2b \subseteq \rho(R_2)$ and since $\rho(R_2)$ is a semiprime ideal, we have $b \in \rho(R_2)$. Since $(a,b) \notin \rho(R_1) \times \rho(R_2)$, it now follows that $a \notin \rho(R_1)$. Since $aR_1a \subseteq I_1$ and $a \notin \rho(R_1)$, it follows that $a \in I_1$ from the fact that I_1 is a semi ρ-ideal. Hence we have $(a,b) \in I = I_1 \times \rho(R_2)$ and therefore I is a semi ρ-ideal of R. □

Generalizing Theorem 3.2 we have the following for a special radical ρ such that $\rho(R_1 \times R_2 \times \cdots \times R_n) = \rho(R_1) \times \rho(R_2) \times \cdots \times \rho(R_n)$.

Theorem 3.3. Let $R_1, R_2, ..., R_n$ be rings and $R = R_1 \times R_2 \times \cdots \times R_n$, where $n \geqslant 2$. Then a proper ideal I of R is a semi ρ-ideal if and only if one of the following statements is satisfied.

1. I is a semiprime ideal of R.

2. $I = I_1 \times I_2 \cdots \times I_n$, where I_k is a semi ρ-ideal of R_k for some $k \in \{1, ..., n\}$ and $I_j = \rho(R_j)$ for all $j \in \{1, ..., n\} \setminus \{k\}$.

Proof. This follows simmilar to the proof of [1, Theorem 3.3]. □

4 Semi ρ-submodules

de la Rosa and Veldsman in [4] defined a weakly special class of modules. We follow the definition in [4] of a weakly special class of modules to define a special class of modules.

Definition 4.1. For a ring R, let $\mathcal{K}_R$ be a (possibly empty) class of R-modules. Let $\mathcal{K} = \cup\{\mathcal{K}_R : R$ a ring$\}$. $\mathcal{K}$ is a special class of modules if it satisfies:

S1 $M \in \mathcal{K}_R$ and $I \lhd R$ with $I \subseteq (0 : M)_R$ implies $M \in \mathcal{K}_{R/I}$.

S2 If $I \lhd R$ and $M \in \mathcal{K}_{R/I}$, then $M \in \mathcal{K}_R$.

S3 $M \in \mathcal{K}_R$and $I \lhd R$ with $IM \neq 0$ implies $M \in \mathcal{K}_I$.

S4 $M \in \mathcal{K}_R$ implies $RM \neq 0$ and $R/(0 : M)_R$ is a prime ring.

S5 If $I \lhd R$ and $M \in \mathcal{K}_I$, then there exists $N \in \mathcal{K}_R$ such that $(0 : N)_I \subseteq (0 : M)_I$.

Following similar techniques of [4], we get the following theorems.

Theorem 4.2. [6, Theoerem 5.1] Let $\mathcal{M} = \cup\mathcal{M}_R$ be a special class of modules. Then,
$\mathcal{J} = \{R$: there exists $M \in \mathcal{M}_R$ with $(0 : M)_R = 0\} \cup \{0\}$ is a special class of rings. If ρ is the corresponding special radical, then, $\rho(R) := \cap\{(0 : M)_R : M \in \mathcal{M}\}$.

Theorem 4.3. [6, Theoerem 5.2] Let $\mathcal{J}$ be a special class of rings and for every ring R, let $\mathcal{M}_R = \{M : M$ is an R-module, $RM \neq 0$ and $R/(0 : M)_R \in \mathcal{J}\}$. If $\mathcal{M} = \cup\mathcal{M}_R$, then $\mathcal{M}$ is a special class of modules. If ρ is the corresponding special radical and M is any R-module, then
$\rho(M) := \cap\{P \leq M : M/P \in \mathcal{M}_R\}$.

Definition 4.4. [5, Definition 2.4] Let ρ be a special radical and let M be an R-module. The proper submodule N of M is a ρ-submodule if for all $a \in R$ and $m \in M$, whenever $aRm \subseteq N$ and $a \notin (\rho(R)M : M)$, then $m \in N$.

Definition 4.5. Let ρ be a special radical and let M be an R–module. The proper submodule N of M is a semi ρ-submodule if for all $a \in R$ and $m \in M$, whenever $aRam \subseteq N$ and $a \notin (\rho(R)M : M)$, then $am \in N$.

Definition 4.6. A submodule N of M is said to be semiprime if $N \neq M$ and whenever $r \in R$ and $m \in M$ are such that $rRrm \subseteq N$, then $rm \in N$. The reader clearly observe that any semi ρ-submodule of an R-module R is a semi ρ-ideal of R. The zero submodule is always a semi ρ-submodule of M. Also, see the implications:

ρ-submodule

$\searrow$

semi ρ-submodule

$\nearrow$

semiprime submodule

However, the next examples show that these arrows are irreversible.

Example 4.7. 1. Consider the submodule $N = 6\mathbb{Z}\times(0)$ of the $\mathbb{Z}$-module $M = \mathbb{Z}\times\mathbb{Z}$. Let the special radical ρ be the prime radical. Now let $r \notin (\mathcal{P}(\mathbb{Z})M : M) = (0)$ and $m = (m_1, m_2) \in M$ such that $r^2\cdot(m_1, m_2) \in N$. Then $r^2 m_1 \in 6\mathbb{Z}$, $r^2 m_2 = 0$. Since $6\mathbb{Z}$ and (0) are semi $\mathcal{P}$-ideals of $\mathbb{Z}$, then $r\cdot(m_1, m_2) \in N$ and so N is a semi $\mathcal{P}$-submodule of M. On the other hand, we have $2\cdot(3,0) \in N$ with $2 \notin (\mathcal{P}(\mathbb{Z})M : M)$ and $(3,0) \notin N$ and so N is not a ρ-submodule of M.

2. Consider the submodule $N = \langle\overline{4}\rangle \times \{0\}$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_8 \times \mathbb{Z}$. Let $r \notin (\mathcal{P}(\mathbb{Z})M : M)$ and $m = (m_1, m_2) \in M$ such that $r^2\cdot(m_1, m_2) \in N$. It is clear to observe that as $\langle\overline{4}\rangle$ is a semi $\mathcal{P}$-ideal of $\mathbb{Z}_8$ and $\{0\}$ is a semi $\mathcal{P}$-ideal of $\mathbb{Z}$ that $r(m_1, m_2) \in N$. Hence N is a semi $\mathcal{P}$-submodule of M However, $2^2\cdot(\overline{1}, 0) \in N$ but $2\cdot(\overline{1}, 0) \notin N$ and so N is not a semiprime submodule of M.

Proposition 4.8. *Let ρ be a special radical and let M be an R-module. For N a submodule of M and I an ideal of R. If N is a semi ρ-submodule of M and $(\rho(R)M : M) = \rho(R)$, then $(N : M) = \{r \in R : rm \in N$ for every $m \in M\}$ is a semi ρ-ideal of R.*

Proof. Let $aRa \subseteq (N : M)$ where $a \in R$ and $a \notin \rho(R)$. Then we have $aRaM \subseteq N$ and so $aRam \subseteq N$ for all $m \in M$. Since N is a semi ρ-submodule of M and $a \notin \rho(R) = (\rho(R)M : M)$, $am \in N$ for all $m \in M$. Thus, $aM \subseteq N$ and so $a \in (N : M)$. Therefore, $(N : M)$ is a semi ρ-ideal of R. □

Remark 4.9. If $(\rho(R)M : M) \not\subseteq \rho(R)$, then Proposition 4.8 need not be true. Let $\mathcal{P}$ be the prime radical. For the $\mathbb{Z}$ module $M = \mathbb{Z}_4$ we have $\mathcal{P}(\mathbb{Z}) = (0)$ and $(\mathcal{P}(\mathbb{Z})\mathbb{Z}_4 : \mathbb{Z}_4) = ((0) : \mathbb{Z}_4) = 4\mathbb{Z}$. Now, $N = (0)$ is clearly a semi $\mathcal{P}$-submodule. $(N : M) = ((0) : \mathbb{Z}_4) = 4\mathbb{Z}$ is not a semi $\mathcal{P}$-ideal of $\mathbb{Z}$. We have $2\mathbb{Z}2 \subseteq 4\mathbb{Z}$ with $2 \notin 4\mathbb{Z}$.

In the following proposition, we give a characterization of ρ-submodules for a special radical ρ.

Proposition 4.10. *Let ρ be a special radical and let M be an R-module where R is a ring with identity. Let N be a proper submodule of M. Then N is a semi ρ-submodule of M if for any $a \in R$ and every submodule K of M, we have that $aRaK \subseteq N$ with $a \notin (\rho(R)M : M)$ implies $aK \subseteq N$.*

Proof. Suppose $aRaK \subseteq N$ and $a \notin (\rho(R)M : M)$. Let $k \in K$. Since $aRak \subseteq N$ and N is a semi ρ-submodule of M, $ak \in N$. It follows that $aK \subseteq N$ as needed. □

Proposition 4.11. *Let $\varphi : M_1 \to M_2$ be an R homomorphism. Then*

1. *If φ is surjective and N is a semi ρ-submodule of $M_1$with $\ker(\varphi) \subseteq N$, then $\varphi(N)$ is a semi ρ-submodule of M_2.*

2. *If φ is one-to-one and K is a semi ρ-submodule of M_2, then $\varphi^{-1}(K)$ is a semi ρ-submodule of M_1.*

Proof. 1. Suppose $\varphi(N) = M_2 = \varphi(M_1)$ and $m_1 \in M_1$. Then $\varphi(m_1) = \varphi(n)$ for some $n \in N$ and so $(m_1 - n) \in \ker(\varphi) \subseteq N$. So $m_1 \in N$ and we have $N = M_1$ a contradiction. Hence $\varphi(N)$ is a proper submodule of M_2. Let $r \in R$ and $m_2 \in M_2$ such that $rRrm_2 \subseteq \varphi(N)$ and $r \notin (\rho(R)M_2 : M_2)$. Choose $m_1 \in M_1$ such that $\varphi(m_1) = m_2$. Then $rRrm_2 = rRr\varphi(m_1) = \varphi(rRrm_1) \subseteq \varphi(N)$ which implies $rRrm_1 \subseteq N$ as $\ker(\varphi) \subseteq N$. If $rM_1 \subseteq \rho(R)M_1$, then $rM_2 = r\varphi(M_1) = \varphi(rM_1) \subseteq \varphi(\rho(R)M_1) = \rho(R)\varphi(M_1) = \rho(R)M_2$. Hence $r \in (\rho(R)M_2 : M_2)$ a contradiction. Thus $r \notin (\rho(R)M_1 : M_1)$. Since N is a semi ρ-submodule, $rm_1 \in N$ and hence $rm_2 = \varphi(rm_1) \in \varphi(N)$ as required.

2. Let $r \in R$ and $m_1 \in M_1$ such that $rRrm_1 \subseteq \varphi^{-1}(K)$ and $r \notin (\rho(R)M_1 : M_1)$. Since $\ker(\varphi) = 0$, we have $\varphi(rRrm_1) = rRr\varphi(m_1) \subseteq K$. Moreover, we have $r \notin (\rho(R)M_2 : M_2)$ for if $rM_2 \subseteq \rho(R)M_2$, then $r\varphi(M_1) \subseteq \rho(R)\varphi(M_1)$ and so $\varphi(rM_1) \subseteq \varphi(\rho(R)M_1)$. Now, if $x \in rM_1$, then $\varphi(x) \in \varphi(\rho(R)M_1)$. Hence $(x - y) \in \ker(\varphi) \subseteq \rho(R)M_1$ for some $y \in \rho(R)M_1$. Hence $x \in \rho(R)M_1$ and we have $rM_1 \subseteq \rho(R)M_1$ a contradiction. Since K is a semi ρ-submodule of M_2, $r\varphi(m_1) = \varphi(rm_1) \in K$ and hence $rm_1 \in \varphi^{-1}(K)$ and we are done. □

Corollary 4.12. *Let N and L be two submodules of an R-module M with $L \subseteq N$.*

1. *If N is a semi ρ-submodule of M, then N/L is a semi ρ-submodule of M/L.*
2. *If L is a semi ρ-submodule of M and N/L is a semi ρ-submodule of M/L, then N is a semi ρ-submodule of M.*
3. *If L is a ρ-submodule of M and N/L is a semi ρ-submodule of M/L, then N is a ρ-submodule of M.*

Proof. 1. Clear by Proposition 4.11.

2. Suppose that $rRrm \subseteq N$ and $r \notin (\rho(R)M : M)$. If $rRrm \subseteq L$, then $rm \in L \subseteq N$ since L is a semi ρ-submodule of M. So assume $rRrm \nsubseteq L$. One can easily observe that $r \notin (\rho(R)M/N : M/N)$. N/L is a semi ρ-submodule of M/L and $rRr(m+L) \subseteq N/L$, then $r(m+L) \in N/L$. Therefore $rm \in N$ and N is a semi ρ-submodule of M.

3. Similar to 2. □

Proposition 4.13. *Let $\{N_i : i \in \Delta\}$ be a nonempty set of semi ρ-submodules of an R-module M. Then $\bigcap_{i\in\Delta} N_i$ is a semi ρ-submodule.*

Proof. Suppose $rRrm \in \bigcap_{i\in\Delta} N_i$ for some $r \in R - (\rho(R)M : M)$, $m \in M$. Since N_i is a semi ρ-submodule of M, for every $i \in \Delta$, we have $rm \in I_i$. Thus $rm \in \bigcap_{i\in\Delta} N_i$. □

5 Idealization

We now show how to construct ρ-ideals using the Method of Idealization. In what follows, R is a ring (associative, not necessarily commutative and not necessarily with identity) and M is an $R-R$-bimodule. The idealization of M is the ring $R \boxplus M$ with $(R \boxplus M, +) = (R, +) \oplus (M, +)$ and the multiplication is given by $(r,m)(s,n) = (rs, rn + ms)$. $R \boxplus M$ itself is, in a canonical way, an $R-R$-bimodule and $M \simeq 0 \boxplus M$ is a nilpotent ideal of $R \boxplus M$ of index 2. We also have $R \simeq R \boxplus 0$ and the latter is a subring of $R \boxplus M$. Note also that $R \boxplus M$ is a subring of the Morita ring $\begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$ via the mapping $(r,m) \mapsto \begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$. We will require some knowledge about the ideal structure of $R \boxplus M$. If I is an ideal of R and N is an $R-R$-bi-submodule of M, then $I \boxplus N$ is an ideal of $R \boxplus M$ if and only if $IM + MI \subseteq N$.

If ρ is a special radical, it follows from [11] that if R is any ring, then $\rho(R \boxplus M) = \rho(R) \boxplus M$ for all $R-R$-bimodules M.

Proposition 5.1. *For the special radical ρ, let I be an ideal of the ring R. I is a semi ρ-ideal of R if and only $I \boxplus M$ is a semi ρ-ideal of $R \boxplus M$.*

Proof. Let $(r_1, m_1) \in R \boxplus M$ such that $(r_1, m_1) R \boxplus M (r_1, m_1) \subseteq I \boxplus M$ and $(r_1, m_1) \notin \rho(R \boxplus M) = \rho(R) \boxplus M$. Hence $r_1 R r_1 \subseteq I$ and $r_1 \notin \rho(R)$. Since I is a semi ρ-ideal of R, we conclude that $r_1 \in I$ and so $(r_1, m_1) \in I \boxplus M$. Consequently $I \boxplus M$ is a semi ρ-ideal of $R \boxplus M$.

Conversely, suppose that $I \boxplus M$ is a semi ρ-ideal of $R \boxplus M$ and let $aRa \subseteq I$ but $a \notin I$. Then $(a,0)R \boxplus M(a,0) \subseteq I \boxplus M$ and $(a,0) \notin I \boxplus M$ imply that $(a,0) \in \rho(R \boxplus M) = \rho(R) \boxplus M$. Thus, $a \in \rho(R)$ and we are done. □

If I is a semi ρ-ideal of a ring R and N is a $R-R$-bi-submodule of M with $IM + MI \subseteq N$, then $I \boxplus N$ need not be a semi ρ-ideal of $R \boxplus M$. For example if ρ is the prime radical, $\langle 2 \rangle$ is a semi ρ-ideal of the ring $\mathbb{Z}$ and $\{\overline{0}\}$ is a submodule of the $\mathbb{Z}$-module $\mathbb{Z}_4$. But $\langle 2 \rangle \boxplus \{\overline{0}\}$ is not a semi ρ-ideal of $\mathbb{Z} \boxplus \mathbb{Z}_4$ since $(2,\overline{1})\mathbb{Z} \boxplus \mathbb{Z}_4(2,\overline{1}) \subseteq \langle 2 \rangle \boxplus \{\overline{0}\}$ but $(2,\overline{1}) \notin \mathcal{P}(\mathbb{Z} \boxplus \mathbb{Z}_4) = \mathcal{P}(\mathbb{Z}) \boxplus \mathbb{Z}_4$ and $(2,\overline{1}) \notin \langle 2 \rangle \boxplus \{\overline{0}\}$.

Proposition 5.2. *Let ρ is a special radical and let I be an ideal of R and N a proper $R-R$-bi-submodule of the $R-R$-bi-module M.*

1. *If $I \boxplus N$ is a semi ρ-ideal of $R \boxplus M$, then I is a semi ρ-ideal of R and N is a semi ρ-submodule of M.*
2. *If $(\rho(R)M : M) = \rho(R)$ and N is a ρ-submodule of M with $IM + MI \subseteq N$ and I a semi ρ-ideal then $I \boxplus N$ is a semi ρ-ideal of $R \boxplus M$.*

Proof. (1) Suppose that $I \boxplus N$ is a semi ρ-ideal of $R \boxplus M$. First we show I is a semi ρ-ideal. Let $aRa \subseteq I$ and $a \notin \rho(R)$.Then we have $(a,0)R\boxplus M(a,0) = (aRa,0) \subseteq I\boxplus N$. Since $I\boxplus N$ is a semi ρ-ideal of $R\boxplus M$, and $(a,0) \notin \rho(R)\boxplus M = \rho(R\boxplus M)$ we have that $(a,0) \in I\boxplus N$. Hence $a \in I$ and it follows that I is a semi ρ-ideal of R. Now, we show that N is a semi ρ-submodule of M. Let $aRam \subseteq N$ with $a \notin (\rho(R)M : M)$. Since $a \notin (\rho(R)M : M)$, we have $a \notin \rho(R)$. Then we have $(a,0_M)R \boxplus M(a,0_M)(0,m) = (0,aRam) \subseteq I \boxplus N$ with $(a,0_M) \notin \rho(R\boxplus M)$. Since $I\boxplus N$ is a semi ρ-ideal of $R\boxplus M$, we conclude that $(a,0_M)(0,m) = (0,am) \in I\boxplus N$ and so $am \in N$, as needed.

(2) Let $(r_1,m_1),(r_1,m_1) \in R \boxplus M$ such that $(r_1,m_1)R \boxplus M(r_1,m_1) \subseteq I \boxplus N$ and $(r_1,m_1) \notin \rho(R \boxplus M) = \rho(R) \boxplus M$. We have $r_1Rr_1 \subseteq I$ and $r_1 \notin \rho(R)$. Since I is a semi ρ-ideal of R and $r_1 \notin \rho(R)$, we have $r_1 \in I$. Now, $(r_1,m_1)R \boxplus M(r_1,m_1) = (r_1Rr_1, r_1Rm_1 + m_1Rr_1) \subseteq I \boxplus N$. Since $r_1Rm_1 + m_1Rr_1 \subseteq N$ and $m_1Rr_1 \subseteq N$, we have $r_1Rm_1 \subseteq N$. Since $r_1 \notin \rho(R)$ and N is a ρ-submodule of M, we have $m_1 \in N$. Hence $(r_1,m_1) \in I \boxplus N$ and $I \boxplus N$ is a semi ρ-ideal of $R \boxplus M$. □

The condition $(\rho(R)M : M) = \rho(R)$ in Proposition 5.2 2. can not be discarded. For example, consider the $\mathbb{Z}$-module $\mathbb{Z}_2$. Put $I = \langle 2\rangle$ and $N = \{\overline{0}\}$. Then I is a semi $\mathcal{P}$-ideal of $\mathbb{Z}$ and N is a $\mathcal{P}$-submodule of $\mathbb{Z}_2$. Also note that $(\mathcal{P}(\mathbb{Z})\mathbb{Z}_2 : \mathbb{Z}_2) = \langle 2\rangle \neq \mathcal{P}(\mathbb{Z}) = \{0\}$. However, $I \boxplus N$ is not a semi $\mathcal{P}$-ideal of $\mathbb{Z} \boxplus \mathbb{Z}_2$ because $(2,\overline{1})\mathbb{Z} \boxplus \mathbb{Z}_2(2,\overline{1}) \subseteq I \boxplus N$, $(2,\overline{1}) \notin \mathcal{P}(\mathbb{Z}) \boxplus \mathbb{Z}_2$ and $(2,\overline{1}) \notin I \boxplus N$.

6 Semi $\mathcal{P}$-ideals (semi n-ideals)

In this section the special radical will be the prime radical. In [1] Khashan et al. introduced the notion of semi n-ideals for commutative rings with identity element. They investigated many properties of semi n-ideals.. We show that for the prime radical many of the results proved by Khashan et al. are also true for noncommutative rings.

In what follows for the noncommutative ring R, $\mathcal{P}(R)$ will denote the prime radical of the ring R.

Throughout this section the rings are noncommutative but not necessarily assumed to have a unity unless indicated.

Definition 6.1. A proper ideal I of a ring R is a semi $\mathcal{P}$-ideal if whenever $a \in R$ such that $aRa \subseteq I$ and $a \notin \mathcal{P}(R)$, then $a \in I$.

If R is a commutative ring, then the notion of a semi $\mathcal{P}$-ideal coincides with a semi n-ideal as been defined by Khashan et al. in [1].

Proposition 6.2. *(see [1, Proposition 2.1]) For a ring R, the following statements hold.*

(1) Every $\mathcal{P}$-ideal is a semi $\mathcal{P}$-ideal.

(2) Every (weakly) semiprime ideal I is a semi $\mathcal{P}$-ideal. The converse also holds if $\mathcal{P}(R) \subseteq I$.

(3) For every proper ideal I of R, $\mathcal{P}^*(I)$ is a (semiprime) semi $\mathcal{P}$-ideal. In particular, $\mathcal{P}(R)$ is a semi $\mathcal{P}$-ideal of R.

(4) Any ideal I such that $I \subseteq \mathcal{P}(R)$ is a semi $\mathcal{P}$-ideal.

(5) If R is a semiprime ring then an ideal I of R is a semi $\mathcal{P}$-ideal if and only if is a semiprime ideal.

Example 6.3. In any semiprime ring R the a nonzero ideal I is a semi $\mathcal{P}$-ideal which is not a $\mathcal{P}$-ideal since $I \not\subseteq \mathcal{P}(R) = (0)$ see [5, Proposition 1.5].

Proposition 6.4. *(See [1, Proposition 3.2]) Let $\{I_i\}_{i\in\Delta}$ be a family of semi $\mathcal{P}$-ideals of R, then $\bigcap_{i\in\Delta} I_i$ is a semi $\mathcal{P}$-ideal of R.*

Proof. This follows from Proposition 2.10 by taking ρ to be the prime radical. □

Proposition 6.5. *Let $\mathcal{P}$ be the prime radical and R a ring with S a non-empty subset of R where $\langle S\rangle \cap Z_{\rho(R)}(R) = \emptyset$. If I is a semi $\mathcal{P}$-ideal of R with $S \not\subseteq I$, then $(I : \langle S\rangle)$ is a semi $\mathcal{P}$-ideal of R.*

Proof. This follows from Proposition 2.14 by taking ρ to be the prime radical. □

Proposition 6.6. *[13, Corollary 4]For any ring R the following are equivalent:*

1. *R has an unique prime ideal.*
2. *R is a local ring and $\mathcal{J}(R) = \mathcal{P}(R)$.*
3. *Every non invertible element is nilpotent.*

Theorem 6.7. The following statements are equivalent for a ring R.

1. $\mathcal{P}(R)$ is the unique prime ideal of R.
2. Every proper ideal of R is an $\mathcal{P}$-ideal.
3. R is a local ring and every proper ideal of R is a semi $\mathcal{P}$-ideal.

Proof. (1) ⇒ (3) Let I be any ideal of R and $a \in R$ such that $aRa \subseteq I$. If $a \in \mathcal{P}(R)$, then we done. If $a \notin \mathcal{P}(R)$ then it follows from Propostion 6.7 that $a \notin \mathcal{J}(R)$ since $\mathcal{P}(R) = \mathcal{J}(R)$. Now, since we also have that R is a local ring, a is an invertible element with inverse b. Now, since $a^2 \in aRa \subseteq I$, we have $a = ba^2 \in I$ and we are done.

(3) ⇒ (1) Let R be a local ring with every proper ideal of R a semi $\mathcal{P}$-ideal. Let M be the unique maximal ideal of R and P a prime ideal of R. Assume that $P \not\subseteq \mathcal{P}(R)$. Since P^2 is a semi $\mathcal{P}$-ideal, it follows from Lemma 2.7 that $P = P^2$. From [7, Corollary 4] $P = \bigcap_{n=1}^{\infty} P^n = \bigcap_{n=1}^{\infty} M^n = (0)$, a contradiction. Hence $P = \mathcal{P}(R)$ and is the unique prime ideal of R.

(2) ⇒ (3) Let M be a maximal ideal right ideal of R and $x \in M$. Since $xR1 \subseteq M$ and M is a $\mathcal{P}$-ideal, then we must have $x \in \mathcal{P}(R)$ and so $M \subseteq \mathcal{P}(R) \subseteq \mathcal{J}(R) \subseteq M$. It follows that $M = \mathcal{J}(R)$ and R is a local ring. The other part of (3) follows directly by Proposition 2.3 (1).

(2) ⇒ (1) Suppose every proper ideal of R is an $\mathcal{P}$-ideal. Let P be any prime ideal. Now, since P is a $\mathcal{P}$-ideal and a prime ideal, it follows from [5, Proposition 1.13] that $P = \mathcal{P}(R)$. Hence $\mathcal{P}(R)$ is the unique prime ideal of R. □

We note that the condition "R is local" in (3) of Theorem 6.7 cannot be omitted. For example, in the ring $M_2(\mathbb{Z}_6)$ every proper ideal is a semi $\mathcal{P}$-ideal but $M_2(\mathbb{Z}_6)$ has no $\mathcal{P}$-ideals. Also it is known that in a local ring every proper ideal is a $\mathcal{J}$-ideal see [5, Theoerem 5.6]. In the following example, we see that we may find a non semi $\mathcal{P}$-ideal in a local ring. Consider the local ring $R = \mathbb{Z}_{\langle 2\rangle} = \{\frac{a}{b} : a, b \in \mathbb{Z}, 2 \nmid b\}$ and let $I = \langle 4\rangle_{\langle 2\rangle} = \{\frac{a}{b} : a \in \langle 4\rangle, 2 \nmid b\}$. R is a local ring but I is not a semi $\mathcal{P}$-ideal of R. For example, $\left(\frac{2}{3}\right)^2 \in I$ but $\frac{2}{3} \notin \mathcal{P}(R) = \{0\}$ and $\frac{2}{3} \notin I$.

Proposition 6.8. *(See [1, Proposition 3.1]) Let R and S be rings and $f : R \to S$ be a surjective ring-homomorphism. Then the following statements hold:*

1. *If I is a semi $\mathcal{P}$-ideal of R and $\ker(f) \subseteq I$, then $f(I)$ is a semi $\mathcal{P}$-ideal of S.*
2. *If J is a semi $\mathcal{P}$-ideal of S and $\ker(f) \subseteq \rho(R)$, then $f^{-1}(J)$ is a semi $\mathcal{P}$-ideal of R.*

Proof. This follows from Theorem 2.11 by taking ρ to be the prime radical. □

Corollary 6.9. *(see [1, Corollary 3.1]) Let R be a ring and let I, K be two ideals of R with $K \subseteq I$. Then the following hold.*

1. *If I is a semi $\mathcal{P}$-ideal of R, then I/K is a semi $\mathcal{P}$-ideal of R/K.*
2. *If I/K is a semi $\mathcal{P}$-ideal of R/K and $K \subseteq \rho(R)$, then I is a semi $\mathcal{P}$-ideal of R.*
3. *If I/K is a semi $\mathcal{P}$-ideal of R/K and K is a semi $\mathcal{P}$-ideal of R, then I is a semi $\mathcal{P}$-ideal of R.*

Proof. Follows from Corollary 2.12by taking ρ to be the prime radical. □

Proposition 6.10. *(see [1, Proposition 3.3]Let ρ be a special radical and let I and J be two semi ρ-ideals in a ring R. If $I+J$ is proper in R, then $I+J$ is a semi ρ-ideal of R.*

Proof. Follows from Proposition 2.13 by taking ρ to be the prime radical. □

Theorem 6.11. (see [1, Theorem 3.2]) Let R_1 and R_2 be two noncommutative rings. Then a proper ideal $I = I_1 \times I_2$ is a semi $\mathcal{P}$-ideal of R if and only if one of the following statements holds.

1. I is a semi prime-ideal of R..
2. I_1 is a semi $\mathcal{P}$-ideal of R_1 and $I_2 = \mathcal{P}(R_2)$.
3. I_2 is a semi $\mathcal{P}$-ideal of R_2 and $I_1 = \mathcal{P}(R_1)$.

Proof. Follows from Theorem 3.2 by taking ρ to be the prime radical. □

Theorem 6.12. (see [1, Theorem 3.3]Let $R_1, R_2, ..., R_n$ be rings and $R = R_1 \times R_2 \times \cdots \times R_n$, where $n \geqslant 2$. Then a proper ideal I of R is a semi $\mathcal{P}$-ideal if and only if one of the following statements is satisfied.

1. I is a semiprime ideal of R.
2. $I = I_1 \times I_2 \cdots \times I_n$, where I_k is a semi $\mathcal{P}$-ideal of R_k for some $k \in \{1, ..., n\}$ and $I_j = \mathcal{P}(R_j)$ for all $j \in \{1, ..., n\} \setminus \{k\}$.

Proposition 6.13. *Let I be a semi $\mathcal{P}$-ideal of R and N an $R-R$-bi-submodule of the $R-R$-bi-module M. Then*

1. *$I \boxplus N$ is a semi $\mathcal{P}$-ideal of $R \boxplus M$.*
2. *If $(\mathcal{P}(R)M : M) = \mathcal{P}(R)$ and N is a semi $\mathcal{P}$-submodule of M with $IM + MI \subseteq N$, then $I \boxplus N$ is a semi $\mathcal{P}$-ideal of $R \boxplus M$.*

Proof. Follows from Proposition 5.1 by taking ρ to be the prime radical. □

Proposition 6.14. *Let I be an ideal of R and N a proper $R-R$-bi-submodule of the $R-R$-bi-module M. If $I \boxplus N$ is a semi $\mathcal{P}$-ideal of $R \boxplus M$, then I is a semi $\mathcal{P}$-ideal of R and N is a semi ρ-submodule of M.*

Proof. Follows from Proposition 5.2 by taking ρ to be the prime radical. □

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