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On weakly prime ideals and weak Krull dimension

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Abstract. Let R be a commutative ring with identity and let P be a proper ideal of R. The notion of weakly prime (resp., weakly semiprime) ideals are introduced by Anderson-Smith (resp., by Badawi), and considered a generalization of prime (resp., semiprime) ideals. An ideal P is called weakly prime (resp., weakly semiprime) if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$ (resp., $0 \neq a^2 \in P$ implies $a \in P$). The aim of this paper is to describe the weakly prime and weakly semiprime ideals in trivial ring extensions. Also, we introduce and study the weak Krull dimension of ring.

Key Words: Weakly prime, weak Krull dimension, trivial ring extension, decomposable ring.
2010 MSC: 13F05, 13A15, 13E05, 13F20, 13C10, 13C11, 13F30, 13D05, 16D40, 16E10, 16E60.

1 Introduction

Throughout this paper, all rings are commutative with identity elements and all modules are unital.

Weakly prime ideals derived from the study of factorization in commutative rings with zero-divisors. They were introduced by A.G Agargun, D.D. Anderson and Valdes-Leon in [1] and after that studied by D.D. Anderson and E. Smith in [2].

Authors in [2], defined a proper ideal (i.e., different from R) P to be a weakly prime ideal of R if $0 \neq ab \in P$ where $a, b \in R$ implies $a \in P$ or $b \in P$. Every prime ideal of R is weakly prime. However, the converse is not true. For example, 0 is always a weakly prime ideal of R, but it is prime if and only if R is an integral domain. They showed that every proper ideal in a ring R is weakly prime ideal if and only if either R is a quasilocal ring (possibly a field) whose maximal ideal is square 0, or R is a product of two fields [2, Theorem 8]. They also give the form of weakly prime ideals in a decomposable ring R (i.e, there exists nontrivial rings R_1 and R_2 such that $R = R_1 \times R_2$) and showed that if P is a weakly prime ideal of R, either $P = 0$ or P is prime. On the other hand, they show how to construct examples of weakly prime ideals using trivial ring extensions. We also introduce and study a notion we call "weakly Krull dimension".

In [5], A. Badawi introduced and studied the notion of a weakly semiprime ideal in a commutative ring. A proper ideal P of R is said to be weakly semiprime if $a \in R$ and $0 \neq a^2 \in P$ imply $a \in P$. As every prime ideal is weakly prime and every weakly prime ideal is weakly semiprime, so weakly prime ideals and weakly semiprime ideals are both generalizations of prime ideals.

Let R be ring, E be an A-module and $R = A \propto E$ be the set of pairs (a,e) with pairwise addition and multiplication given by:

$$(a,e)(b,f) = (ab, af + be).$$

A ring R is called a trivial ring extension of A by E. Trivial ring extension have been studied extensively, the basic properties of the trivial ring extensions are summarized in [3], Glaz's book [11] and Huckaba's book [12]. See for instance [3, 4, 8, 9, 11, 12, 13].

The purpose of the first section in this paper is to study the form of weakly prime ideals and weakly semiprime ideals in trivial ring extension. In the second section, we introduce and study the weakly Krull dimension denoted by $wk\text{-}dim(R)$. Some properties of the $wk\text{-}dim(R)$ are given. Among other results, the weakly Krull dimension of product of two commutative rings is given in Theorem 3.3. Also, in particular case, we characterise the WP-ring(i.e, ring in which every nonzero weakly prime ideals is prime) A by $wk\text{-}dim(A) = 1 + dim(A)$ (see Corollary 3.7). In the last part of this section, we provide upper bounds for $wk\text{-}dim(A \propto B)$ in Theorem 3.9.

2 Weakly prime and weakly semiprime ideals in trivial ring extension

A ring R is called Noetherian if every ideal of R is finitely generated; or equivalently, if every prime ideal of R is finitely generated. We next give a new characterization of Noetherian rings.

Proposition 2.1. *Let R be a commutative ring. Then R is a Noetherian ring if and only if every weakly prime ideal of R is finitely generated.*

Proof. Assume that every weakly prime ideal of R is finitely generated and let I be a prime ideal of R. Then I is a weakly prime ideal and so I is finitely generated. Hence, R is a Noetherian ring.
The converse is clear and this completes the proof. □

Recall the following result.

Theorem 2.2. [2, Theorem 17] Let A be a commutative ring, E be an A-module, and let I be a proper ideal of A. Then $I \propto E$ is a weakly prime ideal of $R := A \propto E$ if and only if I is a weakly prime ideal of A and for $a, b \in A$ with $ab = 0$, but $a \notin I$ and $b \notin I$, $a \in Ann(E)$ and $b \in Ann(E)$.

It is clear that a weakly prime ideal in trivial ring extension $R := A \propto E$ have need not be of the form $P \propto E$ where P is a weakly prime ideal of A. Consider, for example, K be a field and E be a K-vector space. Then, $K \propto E$ is a local ring with a maximal ideal $0 \propto E$ and $(0 \propto E)^2 = 0$. Hence, every proper ideal of $K \propto E$ is weakly prime.

Our first main result is to generalize Theorem 2.2, that is to give a general form of weakly prime ideals in a particular trivial ring extension of A by E, where E is an A-module.

Theorem 2.3. Let A be a ring such that $a^2 = 0$ implies $a = 0$ for every $a \in A$, E be an A-module, $R = A \propto E$ be the trivial ring extension of A by E, and let J be a weakly prime ideal of R. Then:

1. If $J \subseteq 0 \propto E$, then $J = 0 \propto F$, where F is an A-submodule of E.

2. If $J \nsubseteq 0 \propto E$, then $J = I \propto E$, where I is weakly prime ideal of A.

Before proving Theorem 2.3, we establish the following lemma with independent interest.

Lemma 2.4. *Let A be a ring, E be an A-module, $R = A \propto E$ be the trivial ring extension of A by E, and let $J = 0 \propto F$ be an ideal of R. Then:*
1) *If A is not an integral domain and $Ann(F) = 0$, then $0 \propto F$ is not a weakly prime ideal of R.*
2) *Assume that A is an integral domain. Then the following conditions are equivalent:*
i) *$0 \propto F$ is a weakly prime ideal.*
ii) *For every $a \in A \setminus \{0\}$ and for every $f \in E \setminus F$, we have $af \notin F$ or $af = 0$.*
3) *Assume that A is an integral domain with quotient field K and E be a K-vector space. Then the following conditions are equivalent:*
i) *$0 \propto F$ is a weakly prime ideal.*
ii) *For every $a \in A \setminus \{0\}$ and for every $f \in E \setminus F$, we have $af \notin F$.*
iii) *F is a K-vector subspace of E.*

Proof. 1) Assume that A is a ring with zero-divisors and let $a \in Z(A) \setminus \{0\}$ and $b \in A \setminus \{0\}$ such that $ab = 0$. Then $(a,e)(b,f) = (ab, af + be) \in 0 \propto F$ where, $e, f \in F$, but $(a,e) \notin 0 \propto F$ and $(b,f) \notin 0 \propto F$. Hence, $0 \propto F$ is not a weakly prime ideal of R.
2) Now, assume that A is an integral domain.
$i \Rightarrow ii$) Assume that $0 \propto F$ is a weakly prime ideal and we claim ii). Deny. Then there exists $a \in A \setminus \{0\}$ and $f \in E \backslash F$ such that $af \in F$. Hence, $(a,e)(0,f) \in 0 \propto F$, where $e \in E$ but $(a,e) \notin 0 \propto F$ and $(0,f) \notin 0 \propto F$, a desired contradiction. Therefore, we obtain ii).
$ii \Rightarrow i$) By the contrapositive of definition of a weakly prime ideal (i.e., if $a \notin I$ and $b \notin I$, then $ab \notin I$ or $ab = 0$), let $(a,e) \in R - (0 \propto F)$, $(b,f) \in R - (0 \propto F)$. Three cases are then possible:
Case 1: $a = b = 0$.
In this case, we have $e \notin F$ and $f \notin F$, and so $(0,e)(0,f) = (0,0)$, as desired.
Case 2: $a \neq 0$ and $b = 0$.
In this case, $f \notin F$. Hence, $(a,e)(0,f) = (0,af) \notin 0 \propto F$ or $(a,e)(0,f) = (0,0)$ by ii), as desired.
Case 3: $a \neq 0$ and $b \neq 0$.
In this case, $(a,e)(b,f) = (ab, af + be) \notin 0 \propto F$ since $ab \neq 0$ (since A is an integral domain), as desired. Therefore, $0 \propto F$ is a weakly prime ideal of R.
3) Assume that A is an integral domain with quotient field K and E be a K-vector space. By 2), it remains to show that $ii) \Longleftrightarrow iii)$. Also, remark that ii) is equivalent to say that for all $a \in A \setminus \{0\}$ and for all $f \in E$ such that $af \in F$, we have $f \in F$. Hence, $iii) \Rightarrow ii)$ is clear.
Conversely, assume that we have ii) and we must to show that F is a K-vector subspace of E. Let $f \in F$ and $a, b \in A$ such that $b \neq 0$. Then $(1/b)f \in F$ by ii) since $b(1/b)f = f \in F$ and so $(a/b)f = a(1/b)f \in F$. Hence, F is a K-vector subspace of E and this completes the proof of Lemma 2.4. □

Proof of Theorem 2.3
1) If $J \subseteq 0 \propto E$, the result is clear.
2) Now, assume that $J \not\subseteq 0 \propto E$. Then there exists $(a,e') \in J$ such that $a \neq 0$. Let $I = \{\alpha \in A/(\alpha, e) \in J$ for some $e \in E\}$. It is clear that $J \subseteq I \propto E$. Now we claim that $J = I \propto E$. Let $(\alpha$,e)$\in I \propto E$. Two cases are then possible:
Case 1: $\alpha = 0$.
We claim that $(0,e) \in J$. Deny. Hence, $(0,e)^2 = 0$ and $(a,e') \in J$ and so $[(a,e') + (0,e)]^2 = 0$ by [5, Theorem 2.3]. Therefore, $a^2 = 0$ and so $a = 0$ by hypothesis, a desired contradiction.
Case 2: $\alpha \neq 0$.
Hence, there exists $f \in E$ such that $(\alpha$,$f) \in J$. We claim that $(0, f - e) \in J$. Deny. Then $(0, f - e) \notin J$ and $(0, f - e)^2 = 0$. Hence, by the same argument, we have $[(\alpha, f) - (0, f - e)]^2 = 0$ and so $\alpha^2 = 0$, then $\alpha = 0$, a desired contradiction.
Hence, $(0, f - e) \in J$ and $(\alpha, e) \in J$, then $(\alpha, e) \in J$.
Therefore, $(\alpha, e) \in J$ in all cases. Hence, $J = I \propto E$. It remains to show that I is a weakly prime ideal of A.

Let $0 \neq ab \in I$ where $a, b \in A$. Then $(0,0) \neq (a,e)(b,f) \in I \propto E$ implies $(a,e) \in I \propto E$ or $(b,f) \in I \propto E$ since $I \propto E$ is a weakly prime ideal of R. Therefore, $a \in I$ or $b \in I$ and so I is a weakly prime ideal of A which completes the proof of Theorem 2.3. □

Let A be an integral domain with quotient field K, E be a K-vector space, and let $R := A \propto E$ be the trivial ring extension of A by E. by Theorem 2.3 and Lemma 2.4(3), we have:

Corollary 2.5. *Let A be an integral domain with quotient field K, E be a K-vector space, and let $R := A \propto E$ be the trivial ring extension of A by E. Then, an ideal J of R is weakly prime if and only if $J = 0 \propto F$, where F is a K-vector subspace of E or $J = I \propto E$ where, I is a weakly prime ideal of A.*

Remark 2.6.

1. The condition "$a^2 = 0$ implies $a = 0$ for every $a \in A$" in Theorem 2.3 is necessary. Indeed, let R be the trivial ring extension of $\mathbb{Z}/8\mathbb{Z}$ by the $\mathbb{Z}/8\mathbb{Z}$-module $4\mathbb{Z}/8\mathbb{Z}$, that is R=$\mathbb{Z}/8\mathbb{Z} \propto 4\mathbb{Z}/8\mathbb{Z}$. Let I=$4\mathbb{Z}/8\mathbb{Z} \propto 4\mathbb{Z}/8\mathbb{Z}$ be an ideal of R. Then I is not weakly prime ideal of R since $(0,0) \neq (\bar{2},\bar{0})(\bar{2},\bar{0}) \in I$, but $(\bar{2},\bar{0}) \notin I$.
2. Note that, an ideal of the form $0 \propto F$ need not be a weakly prime ideal. Indeed, let (B, M) an integral domain local which is not a field. Consider $A = B \propto B/M$ be the trivial ring extension of B by B/M and $R = A \propto A/Nil(A)$ with $Nil(A) = 0 \propto B/M$ is a prime ideal of A. Note that $Ann(F) = Nil(A)$. Now, we take $F = I/Nil(A)$ for some ideal I of A which contain properly $Nil(A)$. Let $a \in Z(A) \setminus Nil(A)$ and $f \in F$ such that $af \neq 0$. Since $a \in Z(A) \setminus Nil(A)$, then there exists $0 \neq b \in A$ such that $ab = 0$. So, $(0,0) \neq (a,0)(b,f) = (0,af) \in 0 \propto F$ but neither $(a,0) \in 0 \propto F$ nor $(b,f) \in 0 \propto F$.

Now, we are able to give a general form of a weakly semiprime ideals in a particular trivial ring extension of A by E, where E is an A-module.

Theorem 2.7. Let A be a ring such that $a^2 = 0$ implies $a = 0$ for every $a \in A$, E be an A-module, and $R := A \propto E$ be the trivial ring extension of A by E. Then, an ideal J of R is weakly semiprime if and only if $J = 0 \propto F$, where F is a submodule of E or $J = I \propto E$ where, I is a weakly semiprime ideal of A.

Proof. Let A be a ring such that $a^2 = 0$ implies $a = 0$ for every $a \in A$, E be an A-module, $R := A \propto E$ be the trivial ring extension of A by E, and let J be an ideal of R.
If J is a weakly semiprime ideal of R, then $J = 0 \propto F$ or $J = I \propto E$ by the same argument given in Theorem 2.3.

Conversely, assume that $J = 0 \propto F$. Observe that if $y \in R$ and $y^2 \in J$, then $y^2 = (0,0)$. Hence J is a weakly semiprime ideal by definition.
Now, assume that $J = I \propto E$, where I is a weakly semiprime proper ideal of A. Let $0 \neq (a,e)^2 \in J$, then $(0,0) \neq (a^2, 2ae) \in J$. We claim that $a^2 \neq 0$. Deny. Then, $a = 0$ by hypothesis and hence $(a,e)^2 = (0,0)$, a desired contradiction. Hence, $a^2 (\neq 0) \in I$ and so $a \in I$ since I is a weakly semiprime ideal of A. Therefore, $(a,e) \in J$ and so J is a weakly semiprime ideal of R which completes the proof of Theorem 2.6. □

The next result show that the property for an ideal being weakly prime preserved under contraction.

Proposition 2.8. *Let $f : A \to B$ be a ring homomorphism injective such that $f(1) = 1$. If I is a weakly prime ideal of B, then $f^{-1}(I)$ is a weakly prime ideal of A.*

Proof. Let $0 \neq ab \in f^{-1}(I)$, then $0 \neq f(ab) \in I$ and $f(ab) = f(a)f(b)$. Hence, I is a weakly prime ideal gives $f(a) \in I$ or $f(b) \in I$ and so $a \in f^{-1}(I)$ or $b \in f^{-1}(I)$. Therefore, $f^{-1}(I)$ is a weakly prime ideal of A. □

3 Weak Krull dimension

In this section we introduce and study the notion of a weak Krull dimension considered as a generalization of a Krull dimension of commutative ring. Consider a chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of n+1 prime ideals of R. The lenght of such a chain is the integer n. Recall that the Krull dimension of R denoted by $\dim(R)$, is the supremum of the lenghts of all chains of distinct prime ideals of R.

Definition 3.1. Let R be a commutative ring. The weak Krull dimension of R denoted by $wk\text{-}dim(R)$, is the supremum of the lenghts of all chains of distincts weakly prime ideals of R.

Remarks 3.2.
1. Let $P_0 \subset P_1 \subset ... \subset P_n$ be a maximal chain of prime ideals of R. Since every prime ideal is weakly prime, then $wk\text{-}dim(R) \geq dim(R)$.
2. Note that, if R is an integral domain, then $wk\text{-}dim(R) = dim(R)$ since every weakly prime ideal in an integral domain is prime ideal.
3. Let R be a commutative ring with zero-divisors and let $0 \neq P_0 \subset P_1 \subset ... \subset P_n$ be a chain of prime ideals realizing the $\dim(R)$. Hence, $0 = P_{-1} \subset P_0 \subset P_1 \subset ... \subset P_n$ is a chain of weakly prime ideals of R and so $wk\text{-}dim(R) \geq n+1$. In this case, we have $wk\text{-}dim(R) > dim(R)$.

Theorem 3.3. Let R be a decomposable ring. Then:

$$wk - dim(R) = dim(R) + 1.$$

Proof. Assume that $R = A \times B$. By [2, Theorem 7], a weakly prime ideals of $A \times B$ are 0 or prime ideals . On the other hand, 0 can not be a prime ideal of $A \times B$. Hence, we deduce immediately that $wk\text{-}dim(A \times B) = dim(A \times B) + 1$. □

Proposition 3.4. *Let A be a ring and I be a weakly prime ideal of A. Then:*

$$1 + wk - dim(A/I) \leq wk - dim(A).$$

Proof. Let $Q_0 \subset Q_1 \subset \cdots \subset Q_n$ be a chain of weakly prime ideals realizing $wk\text{-}dim(A/I)$. Note that $Q_0 = 0$. Then, Q_i has the form P_i/I where P_i is proper ideal of A contain I. We claim that P_i is a weakly prime ideal of A. Deny. Let $0 \neq ab \in P_i$, then $\bar{a}\bar{b} \in P_i/I$. Two case are then possible:
Case 1: $\bar{a}\bar{b} = \bar{0}$.
Then $0 \neq ab \in I$. So, I is a weakly prime ideal gives $a \in I \subseteq P_i$ or $b \in I \subseteq P_i$.
Case 2: $\bar{a}\bar{b} \neq \bar{0}$.
Then P_i/I is a weakly prime ideal gives $\bar{a} \in P_i/I$ or $\bar{b} \in P_i/I$ and so $a \in P_i$ or $b \in P_i$. Therefore, P_i is a weakly prime ideal of A.
Hence, we obtain that $P_0 \subset P_1 \subset \cdots \subset P_n$ is a chain of weakly prime ideals of A. Since $Q_0 = 0$, we have $0 \subset P_0 = I \subset \cdots \subset P_n$. Therefore, $wk\text{-}dim(A) \geq n+1$, as desired. □

Remark 3.5. The hypothesis "I is a weakly prime ideal" is necessary condition in Proposition 3.4 since P/I can be weakly prime without P being weakly prime.
Indeed, in [2] Anderson and Smith consider the following example, $R = K[X,Y]$, K field and $P = (X, Y^2)$ and $I = (X,Y)^2$, then $I \subset P$ with P/I weakly prime, but P is not weakly prime.

Proposition 3.6. *Let R be a commutative ring and suppose that $Spec(R) = \{P_0 \subsetneq P_1 \cdots \subsetneq P_r = M\}$. Then for every nonzero weakly prime ideal Q of R, either $Q \subsetneq P_0$ or $Q = P_i$ for some $i \in \{0, \cdots, r\}$.*

Proof. Let $(0) \neq Q$ be a weakly prime ideal of R and let P_i be a minimal prime over Q. Clearly $\sqrt{Q} = P_i$. Now, let $x \in P_i \setminus Q$ and let $n \geq 2$ be the smallest positive integer such that $x^n \in Q$. Then $x^n = 0$. Deny, then $0 \neq x.x^{n-1} = x^n \in Q$ and so $x^{n-1} \in Q$, which is absurd (by the choice of n). Thus $x^n = 0 \in P_0$ and so $x \in P_0$. Hence $P_i \subseteq Q \cup P_0 \subseteq P_i$, and so $Q \cup P_0 = P_i$. Therefore Q and P_0 are comparable. So either $Q \subseteq P_0$ or $P_0 \subseteq Q$. In the last case, $P_i = Q \cup P_0 = Q$ as desired. □

Recall from [10], that a ring R is called WP-ring if for every nonzero weakly prime ideal of R is prime.

Corollary 3.7. *Let R be a commutative ring which is not an integral domain and suppose that $Spec(R) = \{P_0 \subsetneq P_1 \cdots \subsetneq P_r = M\}$. Then $wk\text{-}dim(R) = 1 + dim(R)$ if and only if R is a WP-ring.*

Proof. Assume that $wk\text{-}dim(R) = 1 + dim(R)$ and let Q be a nonzero weakly prime ideal of R. By Proposition 3.6, either $Q \subsetneq P_0$ or $Q = P_i$ for some i. If $Q \subsetneq P_0$, then $(0) \subsetneq Q \subsetneq P_0 \subsetneq P_1 \cdots \subsetneq P_r = M$ is a chain of weakly prime ideals of R, and so $wk\text{-}dim(R) \geq r + 2 = 2 + dim(R)$, which is a contradiction. Thus $Q = P_i$ for some i, as desired.
The converse is trivial. □

Example 3.8. Let $\mathbb{Z}$ be the ring of integers, p a positive prime integer and $r \geq 2$ an integer and $R = \mathbb{Z}/p^r$. Then R is zero-dimensional commutative ring with $wk\text{-}dim(R) = 1$. Moreover, the only weakly prime ideals of R are (0) and $p\mathbb{Z}/p^r$.

Recall that in [12], the Krull dimension of trivial ring extension of A by an A-module E coincide with the Krull dimension of A. Now, we want to study the weakly Krull dimension of $A \propto E$. By Theorem 2.3, we know under some hypothesis that the weakly prime ideals of $A \propto E$ has the forms $0 \propto F$ where, F is A-submodule of E or has the form $P \propto E$ where, P is weakly prime ideal of A. Therefore, we proceed our investigation looking for upper bounds of the weakly Krull dimension of $A \propto B$, where $A \subseteq B$ is an extension of rings.

Theorem 3.9. Let $A \subseteq B$ be an extension of rings such that $a^2 = 0$ implies $a = 0$ for each $a \in A$, and let $A \propto B$ be the trivial ring extension of A by B. Then:

1. $wk\text{-}dim(A \propto B) \leq wk\text{-}dim(A) + wk\text{-}dim(B)$.

2. Assume that A is an integral domain and $K := qf(A) \subseteq B$. Then:

$$wk-dim(A \propto B) = wk-dim(A) + wk-dim(B).$$

Proof. 1. Let $H_0 \subset H_1 \subset ... \subset H_n$ be a maximal chain of weakly prime ideals in $A \propto B$. Then, by Theorem 2.3, H_i has the form $0 \propto I_i$ where, I_i is weakly prime ideal of B or $P_i \propto B$ where, P_i is weakly prime ideal of A. According to the form of H_i, rewrite the given chain as follows:

$$0 \propto I_0 \subset 0 \propto I_1 \subset ... \subset 0 \propto I_k \subset P_{k+1} \propto B \subset ... \subset P_n \propto B.$$

Thus, the chain induce a chain of weakly prime ideals of B of length k and a chain of weakly prime ideals of A of length $n-k$. Therefore, $wk\text{-}dim(A \propto B) = n \leq wk\text{-}dim(A) + wk\text{-}dim(B)$.

2. By 1), it suffices to show that $wk\text{-}dim(A) + wk\text{-}dim(B) \leq wk\text{-}dim(A \propto B)$. Let $I_0 \subset I_1 \subset ... \subset I_k$ be a chain of weakly prime ideals realizing $wk\text{-}dim(B)$ and $P_0 \subset P_1 \subset ... \subset P_l$ be a chain of weakly prime ideals realizing $wk\text{-}dim(A)$. It is clear that the condition $2(ii)$ of Lemma 2.4 hold. Then, $0 \propto I_i$ is a weakly prime ideal of $A \propto B$. Moreover, $P_i \propto B$ is also weakly prime ideal of $A \propto B$ by [2, Theorem 17]. Consequently, we obtain that

$$0 \propto I_0 \subset 0 \propto I_1 \subset ... \subset 0 \propto I_k \subset P_0 \propto B \subset ... \subset P_l \propto B$$

is a chain of weakly prime ideals of $A \propto B$. Therefore, $wk\text{-}dim(A) + wk\text{-}dim(B) \leq wk\text{-}dim(A \propto B)$. □

The following Corollaries are immediate consequences of Theorem 3.9.

Corollary 3.10. *Let $A \subseteq B$ be an extension of domains such that $K := qf(A) \subseteq B$, and let $A \propto B$ be the trivial ring extension of A by B. Then :*

$$wk - dim(A \propto B) = wk - dim(A) + wk - dim(B) = dim(A) + dim(B).$$

Corollary 3.11. *For every positive integer n, there is a commutative ring R such that $wk\text{-}dim(R) = n = dim(R)$.*

Proof. Let A be an integral domain with quotient field K, $X_1 \cdots, X_n$ indeterminate over A and $R = A \propto K[X_1, \cdots, X_n]$. By Corollary 3.10, $wk\text{-}dim(R) = dim(A) + dim(K[X_1, \cdots, X_n]) = n + dim(A)$. □

Corollary 3.12. *Let A be a ring such that $a^2 = 0$ implies $a = 0$ for each $a \in A$, and let $A \propto A$ be the trivial ring extension of A by A. Then :*

$$wk - dim(A \propto A) \leq wk - dim(A).$$

Proof. Let $H_0 \subset H_1 \subset \cdots \subset H_n$ be a chain realizing $wk\text{-}dim(A \propto A)$. All H_i can not have the form $0 \propto C$, where C is a propre ideal of A. Indeed, let $b \in C \setminus \{0\}$. Then $(0,0) \neq (b,0)(0,1) \in 0 \propto C$ but neither $(b,0) \in 0 \propto C$ nor $(0,1) \in 0 \propto C$. Then, by Theorem 2.3, all H_i has the form $P_i \propto B$ where P_i is a weakly prime ideal of A. So, the first chain induce a chain of weakly prime ideals of A of length n. Therefore, $wk\text{-}dim(A \propto A) \leq wk\text{-}dim(A)$. □

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