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Direct limits and minimal ring extensions

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Abstract. If $\{A_i \hookrightarrow B_i\}$ is a directed system of minimal (unital) ring extensions (involving associative unital rings that need not be commutative) and the canonical injection $A := \varinjlim_i A_i \to B := \varinjlim_i B_i$ is used to view A as a subring of B, then either $A = B$ or $A \subset B$ is a minimal ring extension. The preceding assertion is the case $n = 1$ of a more general result which assumes that there exists an integer $n \geq 0$ such that for each i, each chain of rings contained between A_i and B_i has length at most n. For commutative rings, an (upward-)directed union of ramified (resp., decomposed) minimal ring extensions $A_i \hookrightarrow B_i$ for which each (A_i, M_i) is quasi-local, $M_j \cap B_i = M_i$ whenever $i \leq j$ in I, and each transition map $A_i \hookrightarrow A_j$ is an integral extension produces a minimal ring extension $A := \varinjlim_i A_i \to \varinjlim_i B_i =: B$ (that is, $\cup_i A_i \hookrightarrow \cup_i B_i$) such that if $M := \cup_i M_i$, then the minimal ring extension $A/M \subset B/M$ is ramified (resp., decomposed) and $A \subset B$ is a minimal ring extension. Applications involving denumerable (upward-)directed unions of fields whose "steps" are algebraic are given to algebraically closed fields and to perfect closures (in the sense of Bourbaki), by using the μ-field extensions of Gilbert and Quigley.

Key Words: Unital associative ring, minimal ring extension, field, noncommutative, field extension, λ-field extension, μ-field extension, algebraically closed field, perfect field, perfect closure, direct limit, length of a chain
2010 MSC: Primary 16B99; Secondary 12F05, 12F15, 13B99.

1 Introduction

All rings considered below are assumed to be unital and associative, but not necessarily commutative. All inclusions of rings, ring extensions, subrings, algebras, and ring/algebra homomorphisms will be assumed unital. Proper inclusions will be denoted by $\subset$ or $\supset$. Recall that if $A \subset B$ are (distinct) rings, then $A \subset B$ is called a *minimal ring extension* (and B is called a *minimal ring extension of* A) if there does not exist a ring C such that $A \subset C \subset B$. Although the original definition of "minimal ring extension" (in [15]) also required (A and) B to be commutative and minimal ring extensions involving commutative rings have been extensively investigated, the "minimal ring extension" concept has been fruitfully studied for arbitrary rings in recent years, perhaps most notably, in chronological order, in [14], [1], [7] and [8]. To be more complete, one should note that a special class of commutative minimal ring extensions was introduced by Gilmer and Heinzer [19] a few years before the appearance of [15], and it is noteworthy that some of the reasoning in [19] does carry over to the more general setting in [15].

As explained more fully later in this Introduction, our recent work in [7] was necessarily confined to the study of certain ring extensions that involved only finite rings. The impetus for the present work was the naïve thought that one way to build "larger" minimal ring extensions of "larger" rings may be to use the union of an upward-directed strictly increasing infinite chain of rings, or more generally, direct limits that would generalize such unions. Thus, the two titular topics were critical to the motivation for this work. However, the work will be pursued in greater generality. So, before summarizing that work, we next devote two paragraphs to some background for that more general study.

Let $(I, \leq)$ be a nonempty poset; that is, I is a nonempty set and $\leq$ is a partial order on I. Recall that for any positive integer n, a (*finite*) *chain* of *length* $\ell(\mathcal{C}) = n$ *in* I is a finite subset $\mathcal{C} = \{i_1, \ldots, i_n\}$ of I such that $i_1 < \ldots < i_n$; and that a (*finite*) *chain* of *length* $\ell(\mathcal{C}) = 0$ *in* I is a singleton subset of I. The length of I, denoted by $\ell(I)$, is defined to be a nonnegative integer m if the supremum of the lengths of finite chains in I is m; and we take $\ell(I) = \infty$ if no such finite supremum exists. These concepts can be applied to any ring extension $A \subseteq B$, as follows. As usual, let $[A, B]$ denote the set of intermediary rings, that is, $\{C \mid C$ is a ring such that $A \subseteq C \subseteq B\}$. Since $[A, B]$ is a poset under inclusion, it makes sense to discuss $\ell([A, B])$. Note that $A \subset B$ is a minimal ring extension if and only if $\ell([A, B]) = 1$; and that $A = B$ if and only if $\ell([A, B]) = 0$. The relationship hinted at in this paper's title concerns how the ℓ operator behaves in regard to direct limits. Before stating the main results along those lines, we pause to make some relevant ideas more precise.

Recall that $(I, \leq)$ is said to be a *directed set* if $\leq$ is a reflexive and transitive binary relation on the (nonempty) set I such that for all $\alpha, \beta, \gamma \in I$ with $\alpha \leq \beta$ and $\alpha \leq \gamma$, there exists $\delta \in I$ such that $\beta \leq \delta$ and $\gamma \leq \delta$. We assume that the reader is somewhat familiar with the usual construction of a direct limit of rings indexed by a directed set (cf. [2, Exercise 21, page 34], noting that the riding assumption in [2] that all rings are commutative plays no role in the just-cited exercise). Nevertheless, the statements of our first main result (Theorem 2.1) and its first application (Corollary 2.2) carefully specify all the technical aspects of the ambient direct limit structures (including their transition maps, their maps from the domains and codomains into the respective direct limits, and the compatibility equations among these various ring homomorphisms). After those two results, subsequent results, applications and examples feature statements that are slightly more relaxed in regard to the technical aspects involving direct limits. Readers seeking additional facts about direct limits (indexed by directed sets) may consult standard sources such as [2, pages 32-34].

A "relaxed" version of the statement of Theorem 2.1 is the following: if $0 \leq n < \infty$ and $\{A_i \subseteq B_i \mid i \in I\}$ is a directed system of ring extensions (in the obvious sense) such that $\ell([A_i, B_i]) \leq n$ for each i, then $\ell([\varinjlim_i A_i, \varinjlim_i B_i]) \leq n$. The most important application of Theorem 2.1 for us here is its case $n = 1$ (Corollary 2.2), which states that any direct limit of minimal ring extensions is either an identity map or a minimal ring extension. As Example 2.3 illustrates, the inequality asserted in Theorem 2.1 (and implicitly in Corollary 2.2) can be strict. Indeed, by using classic information from multiplicative ideal theory, Example 2.3 builds, for each positive integer n, a denumerable upward-directed system of ring extensions $\{A_i \subseteq B_i\}$ such that $\ell([A_i, B_i]) = n$ for each i but $\ell([\varinjlim_i A_i, \varinjlim_i B_i]) = 0$. In other words, although each "level/layer" of the directed system has length $n \geq 1$, that system "collapses" in the (direct) limit, in the sense that taking the direct limit of the inclusion maps $A_i \subseteq B_i$ leads to $\varinjlim_i A_i = \varinjlim_i B_i$. The basic idea behind the construction is best understood in case $n = 1$, where one starts with a minimal ring extension $A_1 \subset B_1$, uses the fact that every nonzero ring R has a standard minimal ring extension, let us denote it here by $\mathcal{E}(R)$, that is constructed using an idealization (for commutative rings in [5] and, by tweaking that method, for arbitrary rings, in [14]), and builds the subsequent levels of the directed system in the following "crisscross" manner: $A_2 := B_1$, $B_2 := \mathcal{E}(A_2)$, $A_3 := B_2$, $B_3 = \mathcal{E}(A_3)$, ... Notice that this construction satisfies $\varinjlim_i A_i = \cup_i A_i = \cup_i B_i = \varinjlim_i B_i$. When confronted with the drama of this "collapse", one's first response may be to despair of finding a nontrivial sufficient condition for the union of a strictly increasing upward-directed sequence of minimal ring extensions to be a minimal ring extension (that is, to fail to collapse). Nevertheless, we present such a sufficient condition in our second main result, Theorem 2.5. Its proof uses information about what has come to be called the "crucial maximal ideal" of any minimal ring extension $A \subset B$ involving commutative rings. (For such an extension, the important roles played by that maximal ideal of A were discovered by Ferrand and Olivier and presented in [15, Théorème 2.2]; as with all necessary background, details are provided at opportune points in the paper.) Proposition 2.6 shows how to begin with a suitable strictly ascending denumerable chain of commutative ring extensions

$A_1 \subset A_2 \subset A_3 \subset \dots$ and then construct (warning: I am about to use some standard terminology for two of the three kinds of integral minimal ring extensions of commutative rings) “ramified" or “decomposed" layers $A_n \to B_n$ forming an upward-directed strictly increasing directed system that satisfies the sufficient condition from Theorem 2.5, thus avoiding the “crisscross" pitfall, and hence producing, in principle, a ramified example and a decomposed example where $\varinjlim_n A_n \hookrightarrow \varinjlim_n B_n$ is a minimal ring extension. With this methology in place and background summarized in Lemma 2.7, we then produce some concrete examples of nontrivial directed unions of minimal ring extensions that are (either ramified or decomposed, as desired) commutative minimal ring extensions. These examples are, of course, constructed by beginning with a suitable chain $A_1 \subset A_2 \subset A_3 \subset \dots$ of fields. We choose to highlight two such chains: one due to Quigley [31], who found such a chain going from any field that is not real closed or algebraically closed to an algebraically closed field; and the other chain, due to M. S. Gilbert [17] who found a suitable chain going from any non-perfect field K to the perfect closure of K (in the sense of [3]) in any algebraic closure of K.

Several questions are raised but not fully pursued here, and we hope that some of these will be of interest to some readers. For instance, one could consider non-collapsing analogues of Example 2.3 for the ℓ operator going beyond the context of minimal ring extensions; generalizations of Theorem 2.5 for more complicated direct limits; attention to the behavior of inert extensions, which form the third kind of commutative integral minimal ring extension of commutative rings; attention to the behavior of integrally closed minimal ring extensions in a possible variant of Theorem 2.5; search for sufficient conditions for a directed union of noncommutative minimal ring extensions to fail to collapse; .... Other possible avenues for study may occur to readers of Remarks 2.9 and 2.10. As a final tribute to the work of Ferrand and Olivier, we devote the final remark to a detailed proof of a minor result from [15], a result whose proof was (perhaps wisely) left to the reader in [15] although Remark 2.11 argues that this result can now be seen as a thematic precursor for the present Corollary 2.2.

We think/hope that it is significant that the setting for (Theorem 2.1 and) Corollary 2.2 is broad enough to accommodate the topic of noncommutative minimal ring extensions. Note that the recent paper [7] was able to study minimal ring extensions, especially for finite noncommutative rings, in some detail, because of the fact that any minimal ring extension of a finite ring is itself a finite ring. This important ring-theoretic fact was easily obtained by Jarboui and the author in [11, Lemma 2.1 (c)] by using a deep rng-theoretic result that is due independently to Klein [26] and Laffey [27]. Beyond the universe of finite rings, it seems that much specific information remains to be learned about noncommutative minimal ring extensions. In Remark 2.10 (a), we do give an analogue of part of Theorem 2.5 for certain noncommutative minimal ring extensions. Remark 2.10 (c) mentions the theme of a paper that is in preparation in this general area and also comments on some other relevant open matters.

As usual, if R is a commutative ring, then Spec(R) denotes the set of prime ideals of R, viewed as a poset under inclusion. Any unexplained material is in standard references, such as [2], [18], [25].

2 Results

We move at once to our first main result. In its proof, it will be convenient, for rings $C \subseteq D$ and an element $w \in D$, to let $C\langle w\rangle$ denote the subring of D that is generated by $C \cup \{w\}$.

Theorem 2.1. Let n be a nonnegative integer. Let $(I, \leq)$ be a directed set. For each $i \in I$, let $h_i : A_i \hookrightarrow B_i$ be a ring extension such that $\ell([A_i, B_i]) \leq n$. If $i \leq j$ in I, let the transition maps be ring homomorphisms denoted by $f_{ij} : A_i \to A_j$ and $g_{ij} : B_i \to B_j$, such that f_{ii} and g_{ii} are identity maps. If $i \leq j \leq k$ in I, suppose that $f_{jk}f_{ij} = f_{ik} : A_i \to A_k$ and $g_{jk}g_{ij} = g_{ik} : B_i \to B_k$. Consider the direct limits $A := \varinjlim_{i\in I} A_i$ and $B := \varinjlim_{i\in I} B_i$. For all $i \in I$, let $\alpha_i : A_i \to A$ and $\beta_i : B_i \to B$ denote the canonical ring homomorphisms (such that, for all $j \leq k$ in I, $\alpha_k f_{jk} = \alpha_j$ and $\beta_k g_{jk} = \beta_j$). View the induced injective ring homomorphism $h : A \to B$ (such that, for all $i \in I$, $h\alpha_i = \beta_i h_i : A_i \to B$) as an inclusion map. Then $\ell([A,B]) \leq n$.

Proof. As the assertion is clear if $n = 0$, we can suppose, without loss of generality, that $n \geq 1$, that is, $A \subset B$. It will be convenient to let

$$\mathbb{N}_n := \{\nu \mid \nu \text{ is a nonnegative integer such that } \nu \leq n\}.$$

We next suppose that the assertion fails (and it will suffice to produce a contradiction). Then there exists a chain of rings

$$A = R_0 \subset R_1 \subset R_2 \subset \ldots \subset R_{n+1} = B.$$

For the rest of this paragraph, fix $\nu \in \mathbb{N}_n$. Since $R_\nu \subset R_{\nu+1}$, we can choose $x_\nu \in R_{\nu+1} \setminus R_\nu$. By a standard construction of direct limits (cf. [2, Exercise 15, page 33]), there exist $k_\nu \in I$ and $x_{\nu,k_\nu} \in B_{k_\nu}$ such that $\beta_{k_\nu}(x_{\nu,k_\nu}) = x_\nu$.

As I is a directed set, there exists $\sigma \in I$ such that $k_\nu \leq \sigma$ for all $\nu \in \mathbb{N}_n$. Put

$$y_{\nu,\sigma} := g_{k_\nu,\sigma}(x_{\nu,k_\nu}) (\in B_\sigma), \text{ for all } \nu \in \mathbb{N}_n.$$

Then $y_{\nu,\sigma} \in B_\sigma$ satisfies $\beta_\sigma(y_{\nu,\sigma}) = \beta_\sigma(g_{k_\nu,\sigma}(x_{\nu,k_\nu})) = \beta_{k_\nu}(x_{\nu,k_\nu}) = x_\nu$. Next, observe that

$$\beta_\sigma(A_\sigma) = \beta_\sigma h_\sigma(A_\sigma) = h\alpha_\sigma(A_\sigma) = \alpha_\sigma(A_\sigma) \subseteq A,$$

whence, $A_\sigma \subseteq \beta_\sigma^{-1}(A)$. Consider the rings

$$\beta_\sigma^{-1}(A) = \beta_\sigma^{-1}(R_0) \subseteq \beta_\sigma^{-1}(R_1) \subseteq \beta_\sigma^{-1}(R_2) \subseteq \ldots \subseteq \beta_\sigma^{-1}(R_{n+1}) = \beta_\sigma^{-1}(B).$$

Once again, for the rest of this paragraph, fix $\nu \in \mathbb{N}_n$. We have $y_{\nu,\sigma} \in \beta_\sigma^{-1}(R_{\nu+1})$, since $\beta_\sigma(y_{\nu,\sigma}) = x_\nu \in R_{\nu+1}$. However, $y_{\nu,\sigma} \notin \beta_\sigma^{-1}(R_\nu)$ since $\beta_\sigma(y_{\nu,\sigma}) = x_\nu \notin R_\nu$.

Recall that $A_\sigma \subseteq \beta_\sigma^{-1}(A)$. Of course, $\beta_\sigma^{-1}(R_{n+1}) \subseteq B_\sigma$ (since the domain of β_σ is B_σ). But the existence of the elements $y_{0,\sigma}, y_{1,\sigma}, \ldots, y_{n,\sigma}$ ensures that

$$A_\sigma \subseteq \beta_\sigma^{-1}(R_0) \subset \beta_\sigma^{-1}(R_1) \subset \beta_\sigma^{-1}(R_2) \subset \ldots \subset \beta_\sigma^{-1}(R_{n+1}) \subseteq B_\sigma.$$

Omitting "$A_\sigma \subseteq$" and "$\subseteq B_\sigma$" from the last display produces a chain $\{\beta_\sigma^{-1}(R_\lambda) \mid 0 \leq \lambda \leq n+1\}$ in $[A_\sigma, B_\sigma]$ of length $n+1$, contradicting the hypothesis that $\ell([A_\sigma, B_\sigma]) \leq n$. The proof is complete. □

We next isolate the titular result. It is the case $n = 1$ of Theorem 2.1.

Corollary 2.2. *Let n be a nonnegative integer. Let $(I, \leq)$ be a directed set. For each $i \in I$, let $h_i : A_i \hookrightarrow B_i$ be a ring extension such that either $A_i = B_i$ or $A_i \subset B_i$ is a minimal ring extension. If $i \leq j$ in I, let the transition maps be ring homomorphisms denoted by $f_{ij} : A_i \to A_j$ and $g_{ij} : B_i \to B_j$, such that f_{ii} and g_{ii} are identity maps. If $i \leq j \leq k$ in I, suppose that $f_{jk}f_{ij} = f_{ik} : A_i \to A_k$ and $g_{jk}g_{ij} = g_{ik} : B_i \to B_k$. Consider the direct limits $A := \varinjlim_{i\in I} A_i$ and $B := \varinjlim_{i\in I} B_i$. For all $i \in I$, let $\alpha_i : A_i \to A$ and $\beta_i : B_i \to B$ denote the canonical ring homomorphisms (such that, for all $j \leq k$ in I, $\alpha_k f_{jk} = \alpha_j$ and $\beta_k g_{jk} = \beta_j$). View the induced injective ring homomorphism $h : A \to B$ (such that, for all $i \in I$, $h\alpha_i = \beta_i h_i : A_i \to B$) as an inclusion map. Then either $A = B$ or $A \subset B$ is a minimal ring extension.*

With the above two results having been painstakingly stated, we will be more relaxed in describing the direct limits that appear below. While the necessary terminology has not become absolutely standard, certain notions have appeared widely (albeit with a variety of names). For instance, one finds in [2, Exercise 18, page 33] the notion of a "homomorphism of directed systems" (albeit of module homomorphisms). In the same spirit but changing the ambient category, one could speak of a "homomorphism of directed systems of ring homomorphisms", or of its special case of a "homomorphism of directed systems of ring extensions." To describe this notion, we will find it convenient to use, instead, the term "directed system of ring extensions." Often, the directed systems in question, $\{A_j \mid j \in I\}$ and $\{B_j \mid j \in I\}$, will be directed unions (in the sense that the associated transition maps f_{ij} and g_{ij} are inclusion maps whose codomains are respective subrings of given rings, say $\mathcal{A}$ and $\mathcal{B}$, such that A_i is a subring of $\mathcal{A}$ and B_i is a subring of $\mathcal{B}$ for all $i \in I$). To describe this kind of special case, it will be appropriate to use the name "directed union of ring extensions", since its associated induced injective ring homomorphism $h : A \to B$ can be identified with the inclusion map $A = \varinjlim_{i \in I} A_i = \cup_{i \in I} A_i \hookrightarrow B = \varinjlim_{i \in I} B_i = \cup_{i \in I} B_i$ of subrings of the universe $\mathcal{B}$. In discussing a "directed system of (possibly minimal) ring extensions" or a "directed union of (possibly minimal) ring extensions", we will feel free to use, without further explanation, the symbols f_{ij}, A, f_i, g_{ij}, B, g_i, h_i and h (along with the associated compatibility conditions) from the statements of Theorem 2.1 and Corollary 2.2, while also always assuming that the index set I is a directed set.

There are some obvious situations where the inequalities asserted in the statements of Theorem 2.1 and Corollary 2.2 are trite, in the sense that $\ell([A,B]) = \ell([A_i,B_i]) = n$ for all $i \in I$. This happens, for instance, if $\ell([A_i,B_i]) = n$ for all $i \in I$ and I is finite (for there then exists $k \in I$ such that $i \leq k$ for all $i \in I$, whence $h : A \to B$ can be identified with $h_k : A_k \to B_k$). This also happens in the following (only slightly less trivial) situation where $\ell([A_i,B_i]) = n$ for all $i \in I$ and each transition map (that is, each f_{ij}, along with each g_{ij}) is an isomorphism (for once again, h can be identified with an h_k). It is much less trivial (and possibly surprising) that the conclusion $\ell([A,B]) \leq n$ in Theorem 2.1 can sometimes be sharpened to become a strict inequality even if one has $\ell([A_i,B_i]) = n$ for all $i \in I$. The next result uses some classic material from multiplicative ideal theory to show this in dramatic fashion.

Example 2.3. Let N be a positive integer. Then there exists a denumerable directed union of ring extensions $A_i \subseteq B_i$, $i \leq -1$ (with $A_i \subset A_{i+1}$ and $B_i \subset B_{i+1}$ for each negative integer i) such that

$$\ell([A_i,B_i]) = N \text{ for all negative integers } i, \text{ but}$$

$A := \varinjlim_{i \leq -1} A_i = \cup_{i \leq -1} A_i$ and $B := \varinjlim_{i \leq -1} B_i = \cup_{i \leq -1} B_i$ satisfy $\ell([A,B]) = 0$. It can further be arranged that there exists a (commutative) valuation domain V such that all the rings A_i and B_i are valuation domains and, in fact, overrings of V (that is, unital V-subalgebras of the quotient field of V). One way to construct such data is the following. Take V to be a valuation domain with maximal ideal M such that $\mathrm{Spec}(V)$ consists of the denumerably many prime ideals

$$P_\infty := 0 \subset \ldots \subset P_{n+1} \subset P_n \subset \ldots \subset P_1 \subset P_0 := M;$$

and, for each nonnegative integer n, take the entry in the sequence of inclusion maps $A_i \hookrightarrow B_i$ which corresponds to the index n to be

$$V_{P_n} \hookrightarrow V_{P_{N+n}}.$$

Proof. It is well known that if W is a valuation domain with quotient field L, then: each overring of W (that is, each ring T such that $W \subseteq T \subseteq L$) is a valuation domain, necessarily of the form $W_{\mathfrak{P}}$ for some $\mathfrak{P} \in \mathrm{Spec}(W)$ (cf. [18, Theorem 17.6 (a)], [25, Theorems 64 and 65]); and if $P, Q \in \mathrm{Spec}(W)$, then either $P \subseteq Q$ or $Q \subseteq P$, with $P \subseteq Q$ if and only if $W_Q \subseteq W_P$ (cf. [18, Exercise 12, page 59]). It follows that the set of overrings of a valuation domain W is, as a poset under inclusion, order-anti-isomorphic to $\mathrm{Spec}(W)$ (when viewed as a poset under inclusion). As a consequence, we will next prove that there exists a valuation domain V whose prime spectrum is as in the second display in the statement of this example; equivalently, that there exists a valuation domain V whose set of overrings is (when viewed as a poset under inclusion) order-anti-isomorphic to the poset in the second display in the statement of this example. Notice that this poset (and the poset obtained by reversing its order) each have the following three properties: being a linearly ordered set with a (unique) minimal element and a (unique) maximal element; being closed under suprema and infima of nonempty (necessarily linearly ordered) subsets; and whenever two of its elements satisfy $a < b$, then there exist elements c and d in the poset such that $a \leq c < d \leq b$ and no element e of the poset satisfies $c < e < d$. It is classically known (cf. also [28, Theorem 3.1, (a) $\Rightarrow$ (c)]) that posets that have these three properties are order-isomorphic to the prime spectrum of some valuation domain. Hence, the existence of a valuation domain V with the asserted properties has been shown. It remains only to establish the assertions concerning the ℓ operator.

It suffices to observe that

$$A := \cup_{n\geq 0} A_n = \cup_{n\geq 0} V_{P_n} = \cup_{n\geq 0} V_{P_{N+n}} = \cup_{n\geq 0} B_n =: B,$$

although for each nonnegative integer n, we have $\ell([A_n, B_n]) = N$ since the only maximal chain in $[A_n, B_n]$ is

$$A_n := V_{P_n} \subset V_{P_{1+n}} \subset V_{P_{2+n}} \subset \ldots \subset V_{P_{N+n}} =: B_n,$$

which is of length N. The proof is complete. □

Remark 2.4. (a) It seems natural to ask what sort of example can be built by reversing the (linear) partial order that was used in the proof of Example 2.3. In detail: by invoking [28, Theorem 3.1, (a) $\Rightarrow$ (c)] once again, we obtain a valuation domain W, say with maximal ideal N and quotient field L, such that $\mathrm{Spec}(W)$ consists of the denumerably many prime ideals

$$P_0 := 0 \subset P_1 \subset P_2 \subset \ldots \subset P_n \subset P_{n+1} \subset \ldots \subset P_\infty = N.$$

Hence, by the information recalled at the start of the proof of Example 2.3, the set of overrings of W consists of the following (valuation) domains:

$$V = V_N = V_{P_\infty} \subset \ldots \subset V_{P_{n+1}} \subset V_{P_n} \subset \ldots \subset V_{P_1} \subset V_{P_0} = L.$$

The assembled data leads to a (downward-)directed union of minimal ring extensions $A_n := V_{P_{n+1}} \hookrightarrow V_{P_n} =: B_n$ (as n runs through the set of nonnegative integers). Notice that the index (directed) set is obtained by reversing the natural order on the set of nonnegative integers. Also, this directed union of minimal ring extensions satisfies

$$\varinjlim_{n\geq 0} A_n = \cup_{n\geq 0} V_{P_{n+1}} = V_{P_1} \subset V_{P_0} = \cup_{n\geq 0} V_{P_n} = \cup_{n\geq 0} B_n = \varinjlim_{n\geq 0} B_n.$$

Thus, we have an explicit example of a denumerable directed union of minimal ring extensions producing a minimal ring extension. This shows, in conjunction with the case $N = 1$ of Example 2.3, that Corollary 2.2 is best possible. (An interested reader may seek additional examples to determine whether Theorem 2.1 is best possible, but we choose to focus on minimal ring extensions for the rest of this paper.) While it has been somewhat interesting to see where a “dualization" of the proof of Example 2.3 would lead, it is perhaps more interesting to note that the construction here in (a) of a concrete example of a direct limit of minimal ring extensions that produces a minimal ring extension is of a third trivial kind, a companion to two other kinds of trivial examples that were noted prior to Example 2.3. Indeed, for the construction in (a), the singleton set consisting of the index $n = 0$ is cofinal, so that the canonical inclusion map $A \subseteq B$ of direct limits is immediately identified with the minimal ring extension $A_0 \hookrightarrow B_0$, that is, with $V_{P_1} \hookrightarrow L$.

(b) In principle, any (infinite) sequence $R_1 \subset R_2 \subset \ldots \subset R_n \subset \ldots$ of minimal ring extensions (involving not necessarily commutative rings, such that some “large" ring R has each R_n as a subring) can be used to construct an example of a directed union of minimal ring extensions $A_n \hookrightarrow B_n$ such that the canonical map $h : \varinjlim_{n\geq 1} A_n \hookrightarrow \varinjlim_{n\geq 1} B_n$ is an identity map, as follows. For each positive integer n, let $A_n := R_n$ and let $B_n := R_{n+1}$. It is clear that $\{A_n \hookrightarrow B_n \mid n \geq 1\}$ (that is, $\{R_n \hookrightarrow R_{n+1} \mid n \geq 1\}$) satisfies the definition of a denumerable (upward-)directed union of minimal ring extensions. (In detail: the compatibility conditions are clearly satisfied since each transition map is an inclusion map.) Moreover, h is an identity map (that is, $A := \varinjlim_{n\geq 1} A_n = \varinjlim_{n\geq 1} B_n =: B$) since

$$\varinjlim_{n\geq 1} A_n = \cup_{n\geq 1} R_n = \cup_{n\geq 2} R_n = \cup_{n\geq 1} R_{n+1} = \cup_{n\geq 1} B_n = \varinjlim_{n\geq 1} B_n.$$

(c) In practice, naturally occurring sequences $R_1 \subset R_2 \subset \ldots \subset R_n \subset \ldots$ of rings of the kind posited in (b) are somewhat uncommon. Accordingly, we will devote some of the following results to some examples of (or built using) such sequences. In part to simplify matters, we will often consider certain such sequences where each ring R_n is a field.

In studies of minimal ring extensions, it has often been fruitful (especially when the relevant rings are commutative) to begin with a context where the base ring is a field. Recall the first classification result for minimal ring extensions, due to Ferrand and Olivier [15, Lemme 1.2]: if K is a field and S is a commutative ring such that $K \subseteq S$, then $K \subset S$ is a minimal ring extension if and only if (exactly) one of the following three conditions holds: S is K-algebra isomorphic to $K[X]/(X^2)$ where X is a commuting indeterminate over K (and where we view $K \subseteq K[X]/(X^2)$ via the unique K-algebra homomorphism $K \to K[X]/(X^2)$); S is K-algebra isomorphic to $K \times K$ (where we view $K \subseteq K \times K$ via the unique K-algebra homomorphism $K \to K \times K$); S is K-algebra isomorphic to a minimal field extension of K. Moreover, if R is a finite commutative (quasi-)local ring, then the class of commutative R-algebras represented by commutative minimal ring extensions of R is infinite if and only if R is a field [6, Corollary 2.6]. In addition, for any such R, the so-called “ramified" (resp., “decomposed") analogues of $K[X]/(X^2)$ (resp., $K \times K$) account for only finitely many such R-algebra isomorphism classes [6, Proposition 2.2]; and it follows that the collection of R-algebra isomorphism classes in question is infinite if and only if the so-called “inert" analogues of the minimal field extensions of K represent infinitely many of those R-algebra isomorphism classes. The above pieces of the historical record serve to motivate our upcoming focus on (denumerable) sequences of minimal field extensions.

(d) We next record analogues of Theorem 2.1 in some other concrete categories besides the category of rings. Recall (cf. [23, Definition 7.6, page 55]) that a *concrete category* is a category $\underline{C}$ with an associated “underlying object" functor from $\underline{C}$ to the category of sets. Thus, roughly speaking, a concrete category is a category whose objects are sets with some “enriched" structure and whose morphisms are functions that preserve that “enriched" structure in some sense (and whose composition is essentially that of the underlying functions). Many of the categories that are of fundamental interest in various areas of algebra (such as the categories of rings, left modules over a given ring, commutative rings, etc.) are concrete categories and it makes sense to ask if analogues of Theorem 2.1 hold for these categories. In many instances (including the category of left modules over a given ring and the category of commutative rings), the answer is in the affirmative; in fact, the proof of Theorem 2.1 carries over, *mutatis mutandis*, to those other contexts.

It has long been known that direct limits $\varinjlim_{i\in I} A_i$ can be constructed in essentially the same way in categories that are sufficiently like the category of abelian groups. In that regard, see [21, Proposition 1.8, pages 133-134], where it is assumed that the A_i are objects in an abelian category that has arbitrary coproducts. By imposing a somewhat stronger axiom, Grothendieck also showed in [21, Proposition 1.8, pages 133-134] that if $\{A_i \to B_i \mid i \in I\}$ is a directed family of monomorphisms in an abelian category that satisfies axiom AB5, then the induced morphism from $\varinjlim_i A_i$ to $\varinjlim_i B_i$ is a monomorphism (and, more generally, that the direct limit functor is an exact functor under these conditions). Since the concept of a “minimal morphism" makes sense in many concrete categories besides the category of rings, it seems natural to seek analogues of Corollary 2.2 even in some concrete categories that are not abelian categories. Note that if R is any nonzero commutative ring, then the category of commutative R-algebras is not even a pre-additive category: see [9, Proposition 2.2].

Readers interested in notions generalizing direct limits in some "'less concrete" categorical situations should search for work on "colimits." A valuable early contribution along those lines can be found in Freyd's discussion of "roots" (with "right root" generalizing a direct limit) in [16, pages 75-78]. This completes the remark.

For reference purposes, this paragraph will summarize some background about commutative minimal ring extensions, especially the ramified-decomposed-inert trichotomy that was alluded to in Remark 2.4 (c). This paragraph is the result of lightly editing two paragraphs from the introduction of one of my earlier papers. If $A \subset B$ is a minimal ring extension of commutative rings, it follows from [15, Théorème 2.2 (i) and Lemme 1.3] that there exists a maximal ideal M of A (called the *crucial maximal ideal* of $A \subset B$) such that the canonical injective ring homomorphism $A_M \to B_M$ ($:= B_{A \setminus M}$) can be viewed as a minimal ring extension while the canonical ring homomorphism $A_P \to B_P$ is an isomorphism for all prime ideals P of R except M. In a survey article in 2009, I gave an easy proof, via globalization and a case analysis, that conversely, a minimal ring extension of commutative rings can be characterized as a ring extension involving commutative rings for which there exists a crucial maximal ideal (in the above sense). For our purposes, since our base rings will often be fields in the rest of this paper, we will restrict attention here to the integral minimal ring extensions. Recall that Ferrand-Olivier [15, Lemme 1.2] proved that if K is a field, then a commutative ring extension $K \subset B$ is a minimal ring extension of K if and only if B is K-algebra isomorphic to (exactly one of) $K[X]/(X^2)$, $K \times K$ or a minimal field extension of K. Now, let $A \subset B$ be an integral ring extension of commutative rings, with the conductor $M := (A : B)$. By a standard homomorphism theorem, $A \subset B$ is a minimal ring extension if and only if $A/M \subset B/MB$ ($= B/M$) is a minimal ring extension. In fact (cf. also [15, Lemme 1.2 and Proposition 4.1], [12, Lemma II.3]), the above-mentioned classification result of Ferrand-Olivier leads to the following trichotomy: $A \subset B$ is a (an integral) minimal ring extension if and only if M is a maximal ideal of A and (exactly) one of the following three conditions holds: $A \subset B$ is said to be respectively *ramified*, *decomposed* or *inert* if B/MB ($= B/M$) is isomorphic, as an algebra over the field $K := A/M$, to $K[X]/(X^2)$, $K \times K$ or a minimal field extension of K.

We next present our second main result. It sharpens the conclusion of Corollary 2.2 for certain directed unions where the relevant rings are commutative and the given minimal ring extensions are integral. Indeed, Theorem 2.5 will establish that both the "ramified" and "decomposed" properties of minimal ring extensions are inherited by those directed unions.

Theorem 2.5. Let $(I, \leq)$ be a directed set. Let $\{A_i \hookrightarrow B_i \mid i \in I\}$ be an (upward-)directed union of integral minimal ring extensions $h_i : A_i \hookrightarrow B_i$, with M_i denoting the crucial maximal ideal of $A_i \subset B_i$ for each $i \in I$. Assume that if $i \leq j$ in I, then $f_{ij} : A_i \hookrightarrow A_j$ is an integral ring extension. Consider the direct limits $A := \varinjlim_{i \in I} A_i$ ($= \cup_{i \in I} A_i$) and $B := \varinjlim_{i \in I} B_i$ ($= \cup_{i \in I} B_i$); for all $i \in I$, let $\alpha_i : A_i \to A$ and $\beta_i : B_i \to B$ denote the canonically induced inclusion maps; and let $h : A \to B$ denote the canonically induced inclusion map. (More precisely, assume also that if $i \leq j$ in I, then the transition functions $f_{ij} : A_i \to A_j$ and $g_{ij} : B_i \to B_j$ are inclusion maps, while $f_{ii} : A_i \to A_i$ and $g_{ii} : B_i \to B_i$ are identity maps; assume also that if $i \leq j \leq k$ in I, then $f_{jk} f_{ij} = f_{ik} : A_i \to A_k$ and $g_{jk} g_{ij} = g_{ik} : B_i \to B_k$; assume also that for all $j \leq k$ in I, $\alpha_k f_{jk} = \alpha_j$ and $\beta_k g_{jk} = \beta_j$); and assume also that for all $i \in I$, $h\alpha_i = \beta_i h_i$.) Assume also that if $i \leq j$ in I, then $M_j \cap A_i = M_i$. (Note that the assumption in the preceding sentence holds automatically if A_i is quasi-local and, in particular, it holds if A_i is a field.) Then:

(a) If $i \leq j$ in I and the canonical map $B_i/M_i \to B_j/M_j$ is an injection (that is, if $i \leq j$ in I and $M_j \cap B_i = M_i$) and the minimal ring extension $A_i \subset B_i$ is ramified, then the minimal ring extension $A_j \subset B_j$ is ramified.

(b) If $M_j \cap B_i = M_i$ whenever $i \leq j$ in I and if the minimal ring extension $A_i \subset B_i$ is ramified for each $i \in I$, then $A/M \subseteq B/M$ is a ramified minimal ring extension and $A \subseteq B$ is a minimal ring extension.

(c) If $i \leq j$ in I with $M_j \cap B_i = M_i$ and if the minimal ring extension $A_i \subset B_i$ is decomposed, then the minimal ring extension $A_j \subset B_j$ is decomposed.

(d) If $M_j \cap B_i = M_i$ whenever $i \leq j$ in I and if the minimal ring extension $A_i \subset B_i$ is decomposed for each $i \in I$, then $A/M \subseteq B/M$ is a decomposed minimal ring extension and $A \subseteq B$ is a minimal ring extension.

(e) Suppose that $M_j \cap B_i = M_i$ whenever $i \leq j$ in I. Suppose also that there does not exist $i \in I$ such that the minimal ring extension $A_i \subset B_i$ is inert. Then $A/M \subseteq B/M$ is a minimal ring extension which is either ramified or decomposed, and $A \subset B$ is a minimal ring extension.

Proof. Let us begin by dispatching the parenthetical assertion. Suppose that $i \leq j$ in I with $(A_i, \mathcal{M})$ and $(A_j, \mathcal{N})$ each quasi-local. Then, by default, $\mathcal{M}$ is the crucial maximal ideal of the minimal ring extension $A_i \subset B_i$ and $\mathcal{N}$ is the crucial maximal ideal of the minimal ring extension $A_j \subset B_j$. As maximal ideals lie over maximal ideals in integral extensions involving commutative rings (cf. [2, Corollary 5.8], [18, Theorem 11.4], [25, Theorem 44]), one necessarily has $\mathcal{N} \cap A_i = \mathcal{M}$, that is, $M_j \cap A_i = M_i$.

Next (no longer assuming that each A_i is quasi-local), observe that for each $i \in I$, $M_i = (A_i : B_i)$ is a common ideal of A_i and B_i, whence by [12, Lemma II. 3], $A_i/M_i \subseteq B_i/M_i$ inherits the property of being an integral minimal ring extension from $A_i \subseteq B_i$. Consider $M := \varinjlim_{i \in I} M_i = \cup_{i \in I} M_i$. It is easy to see that M is a common ideal of $\cup_{i \in I} A_i$ $(= A)$ and $\cup_{i \in I} B_i$ $(= B)$. We will next review some categorical background material to establish that M is a maximal ideal of A and that (what is thus the field) A/M is canonically isomorphic to $\varinjlim_{i \in I} A_i/M_i$ (that is, to the directed union of the fields A_i/M_i).

One application of a comment in Remark 2.4 (d) is that direct limit is an exact functor when applied to directed systems of exact sequences of modules over any ring. For each i, consider the exact sequences

$$0 \to M_i \to A_i \to A_i/M_i \to 0 \text{ and } 0 \to M_i \to B_i \to B_i/M_i \to 0.$$

So, we have exact sequences $0 \to \varinjlim_i M_i \to \varinjlim_i A_i \to \varinjlim_i A_i/M_i \to 0$ and $0 \to \varinjlim_i M_i \to \varinjlim_i B_i \to \varinjlim_i B_i/M_i \to 0$ (of abelian groups), that is,

$$0 \to M \to A \to \varinjlim_i A_i/M_i \to 0 \text{ and } 0 \to M \to B \to \varinjlim_i B_i/M_i \to 0.$$

Moreover, in view of the monomorphisms $A \hookrightarrow B$ and $\varinjlim_i A_i/M_i \hookrightarrow \varinjlim_i B_i/M_i$, it is easy to see that the maps in the second of the just-displayed exact sequences induce the corresponding maps in the first of those exact sequences. Put differently, the isomorphism $B/M \to \varinjlim_i B_i/M_i$ (obtained via the First Isomorphism Theorem) restricts to the isomorphism $A/M \to \varinjlim_i A_i/M_i$ (which is also obtained via the First Isomorphism Theorem). It is easy to check that these isomorphisms of abelian groups each preserve multiplication and send 1 to 1; that is, these are ring isomorphisms. As $\varinjlim_i A_i/M_i = \cup_i A_i/M_i$ is a directed union of fields, it is itself a field, and so A/M is a field. Consequently, M is a maximal ideal of A. It is clear that if $i \in I$, then $M \cap A_i = M_i$ (in detail: observe that $M \cap A_i \supseteq M_i$, $M \cap A_i$ is an ideal of A_i, $1 \notin M \cap A_i$, and M_i is a maximal ideal of A_i), and so we have a natural algebraic field extension $A_i/M_i \hookrightarrow A/M$. (This could also have been obtained by composing the inclusion map $A_i/M_i \hookrightarrow \varinjlim_j A_j/M_j$ with the isomorphism $\varinjlim_j A_j/M_j \to A/M$.)

Some readers may have noticed that the facts established in the preceding paragraph are widely known and can be proven faster. Anyone interested in devising such alternative arguments is advised to consult folklore about the prime ideals of direct limits of rings (cf. [22, Proposition 6.1.2, page 128]) and to use the behavior of prime ideals in integral extensions.

Recall that for each $i \in I$, it follows from [12, Corollary II. 3] that $A_i/M_i \subseteq B_i/M_i$ inherits the property of being an integral minimal ring extension from $A_i \subseteq B_i$. Moreover, since M_i is the crucial maximal ideal of $A_i \subset B_i$, these two extensions ($A_i/M_i \subseteq B_i/M_i$ and $A_i \subset B_i$) are the same kind of integral minimal ring extension (that is, ramified, decomposed or inert). Similarly, the integral ring extension $A \subseteq B$ is a minimal ring extension if and only if (the integral ring extension) $A/M \subseteq B/M$ is a minimal ring extension; and, *if* these conditions hold with M being the crucial maximal ideal of $A \subset B$, then the extensions $A \subseteq B$ and $A/M \subseteq B/M$ are the same kind of integral minimal ring extension. It will be convenient to denote the field A/M by F and to denote its extension ring B/M by E. Just as we let h denote the canonical integral ring homomorphism $A \hookrightarrow B$, it will be convenient to let h^* denote the canonical integral (in fact, algebraic) ring homomorphism $F \hookrightarrow E$. As a final piece of notational convenience, let $F_k := A_k/M_k$ and $E_k := B_k/M_k$ for all $k \in I$.

(a) By hypothesis, $i \leq j$ in I and the minimal ring extension $F_i \subset E_i$ is ramified, and our task is to prove that the (integral) minimal ring extension $F_j \subset E_j$ is ramified. Since F_i and F_j are fields, it follows from Ferrand-Olivier's classification result [15, Lemme 1.2] that we can assume that E_i is F_i-algebra isomorphic to $F_i[X]/(X^2)$ (for some indeterminate X over F_i) and our task can be reformulated as needing to show that $F_j \subset E_j$ is neither decomposed nor inert (that is, that E_j is not F_j-algebra isomorphic to either $F_j \times F_j$ or a minimal field extension of F_j).

By hypothesis, E_i is the internal direct sum $F_i \oplus F_i e$ as an F_i-module, where $e \in E_i$ is a nonzero element such that $e^2 = 0$. Suppose that the assertion fails. The hypothesis that $M_j \cap B_i = M_i$ ensures that E_i can be viewed as a subring of E_j. Thus E_j cannot be (isomorphic to) a field, since any ring containing E_i as a subring must contain the nonzero nilpotent element e. Therefore, it cannot be the case that E_j is F_j-algebra isomorphic to a minimal field extension of F_j. So, necessarily, E_j is F_j-algebra isomorphic to $F_j \times F_j$. Hence, by considering the preimages of $(1,0)$ and $(0,1)$ in E_j for such an isomorphism, we see that there exist nonzero elements c and d in E_j such that $cd = 0$ and, moreover, that there exist $\xi, \eta \in F_j$ such that $e = \xi c + \eta d$. Since $e \neq 0$, either $\xi \neq 0$ or $\eta \neq 0$ (or both). However, since $cd = 0$, we have

$$0 = e^2 = (\xi c + \eta d)(\xi c + \eta d) = \xi^2 c^2 + 2\xi\eta cd + \eta^2 d^2 = \xi^2 c^2 + \eta^2 d^2.$$

Applying the above F_j-algebra isomorphism $E_j \to F_j \times F_j$ converts the (extreme members of the) just-displayed equation into

$$(0,0) = \xi^2 \cdot (1,0)^2 + \eta^2 \cdot (0,1)^2 = (\xi^2, \eta^2) \in F_j \times F_j,$$

whence $\xi^2 = 0 = \eta^2 \in F_j$, whence $\xi = 0 = \eta \in F_j$ (since a field cannot contain a nonzero nilpotent element), the desired contradiction.

(b) Without loss of generality, $I \neq \emptyset$. By hypothesis, the minimal ring extension $A_i \subset B_i$ is ramified for some $i \in I$, and so by (a), $A_j \subset B_j$ is ramified for each $j \in I$ such that $i \leq j$. As the set of such j is cofinal in (the directed set) I, we can assume, without loss of generality, that $A_j \subset B_j$ is ramified for each $j \in I$; that is, $F_j \subset E_j$ is ramified for each $j \in I$. In particular, for some $i \in I$, there exists a nonzero element $e_i \in E_i$ such that $e_i^2 = 0$. The assumption that $M_j \cap B_i = M_i$ whenever $i \leq j$ in I ensures that the canonical map $\beta_i^* : E_i \hookrightarrow E$ is an injection. Hence, we can use this map to view $e_i = \beta_i^*(e_i) \in E$, still such that $e_i^2 = 0 \neq e_i$. Consequently, E is not a field. However, since F *is* a field, we have $F \neq E$. Hence, by Corollary 2.2, $F \subset E$ is a minimal ring extension. Therefore, by the above comments, $A \subset B$ is a minimal ring extension. It remains to prove that the minimal ring extension $F \subset E$ is ramified.

As $F \subset E$ is an integral extension, it suffices to prove that the minimal ring extension $F \subset E$ is neither decomposed nor inert, that is, that E is not F-algebra isomorphic to either $F \times F$ or a minimal field extension of F. Neither of these options is tenable, since no ring that is isomorphic to either $F \times F$ or a field can contain a nonzero nilpotent element (such as the above e_i). Thus, by the process of elimination, $F \subset E$ is ramified, as asserted.

(c) By hypothesis, $i \leq j$ in I and the minimal ring extension $F_i \subset E_i$ is decomposed, and our task is to prove that the (integral) minimal ring extension $F_j \subset E_j$ is decomposed. Since F_i and F_j are fields, it follows from Ferrand-Olivier's classification result [15, Lemme 1.2] that we can assume that E_i is F_i-algebra isomorphic to $F_i \times F_i$ and our task can be reformulated as needing to show that $F_j \subset E_j$ is neither ramified nor inert (that is, that E_j is not F_j-algebra isomorphic to either $F_j[X]/(X^2)$ (for some indeterminate X over F_j) or a minimal field extension of F_j). By applying an F_i-algebra isomorphism between E_i and $F_i \times F_i$ and considering the counterparts of $(1,0)$ and $(0,1)$, we find nonzero elements c and d in E_i such that $cd = 0$ and $c^2 = c$. (We could also arrange that $c + d = 1$ and $d^2 = d$, but we will not need these additional properties.) Of course, no field can contain such elements. Therefore, since the hypothesis that $M_j \cap B_i = M_i$ ensures that E_i is a subring of E_j, it follows that E_j is not (isomorphic to) a field.

Suppose that the assertion fails. Then, necessarily, E_j is F_j-algebra isomorphic to $F_j[X]/(X^2)$ for some indeterminate X over F_j. Hence, E_j is the internal direct sum $F_j \oplus F_j e$ as an F_j-module, where $e \in E_j$ is a nonzero element such that $e^2 = 0$. As $c, d \in E_i \subseteq E_j$ (where the inclusion is a consequence of the hypothesis that $M_j \cap B_i = M_i$), there exist $\xi_1, \eta_1, \xi_2, \eta_2 \in F_j$ such that

$$c = \xi_1 + \eta_1 e \text{ and } d = \xi_2 + \eta_2 e.$$

Since $e^2 = 0$, we get

$$0 = cd = (\xi_1 + \eta_1 e)(\xi_2 + \eta_2 e) = \xi_1^2 + (\xi_1 \eta_2 + \xi_2 \eta_1)e.$$

As 1 and e are linearly independent over F_j, $\xi_1^2 = 0$ and $\xi_1 \eta_2 + \xi_2 \eta_1 = 0$. Since the field F_j cannot contain a nonzero nilpotent element, $\xi_1 = 0$. Thus $c = \eta_1 e$. Hence

$$0 \neq c = c^2 = (\eta_1 e)^2 = \eta_1^2 e^2 = \eta_1^2 \cdot 0 = 0,$$

the desired contradiction.

(d) Without loss of generality, $I \neq \emptyset$. By hypothesis, the minimal ring extension $F_i \subset E_i$ is decomposed for some $i \in I$ and so, as in the proof of (c), there exist nonzero elements $c_i, d_i \in E_i$ such that $c_i^2 = c_i$ and $c_i d_i = 0$. The hypothesis that $M_j \cap B_i = M_i$ whenever $i \leq j$ in I ensures that the canonical map $\beta_i^* : E_i \hookrightarrow E$ is an injection. Considering this map as an inclusion map allows us to view c_i and d_i as elements of E. By tweaking the approach in the first paragraph of the proof of (b), we use Corollary 2.2 to show that $F \subset E$ and $A \subset B$ are minimal ring extensions. It remains only to prove that the minimal ring extension $F \subset E$ is decomposed. This can be done by the process of elimination, as follows. By tweaking the second paragraph of the proof of (c), we get that the fact that $E_i \subset F_i$ is decomposed leads to the conclusion that the minimal ring extension $F \subset E$ is not ramified (that is, E is not F-algebra isomorphic to $F[X]/(X^2)$ for some indeterminate X over F). Finally, the presence of the nontrivial zero-divisors c_i and d_i in E shows that E is not a field, whence the minimal ring extension $F \subset E$ is not inert, thus completing the proof of (d).

(e) Without loss of generality, $I \neq \emptyset$. By hypothesis (and the ramified-decomposed-inert trichotomy for integral minimal ring extensions involving commutative rings), there exists $i \in I$ such that the minimal ring extension $A_i \subset B_i$ is either ramified or decomposed. Therefore, since I is directed, appeals to (a) and (c) show that the cofinal subset J of I consisting of the indices $j \in I$ such that $i \leq j$ has the following property: either $A_k \subset B_k$ is ramified for all $k \in J$ or $A_k \subset B_k$ is decomposed for all $k \in J$. Therefore, appeals to (b) and (d) show that the canonical map $\varinjlim_{k \in J} A_k/M_k \to \varinjlim_{k \in J} B_k/M_k$ is a minimal ring extension which is either ramified or decomposed. Since J is a cofinal subset of I, the just-mentioned map can be canonically identified with $h : A/M \hookrightarrow B/M$. Thus, $A/M \subset B/M$ is a minimal ring extension which is either ramified or decomposed. Moreover, by [12, Lemma II.3], $A \subset B$ is also a minimal ring extension. The proof is complete. □

If $i \leq j$ in I, one can say in general only that $M_j \cap B_i \supseteq M_i$. The possibility that "$\supseteq$" could be "$\supset$" in some example is genuine, and that possibility can lead to a situation where $A = B$. This is, in fact, what occurs with the kind of construction that was described in Remark 2.4 (b) (where $B_{n+1} = A_n$

for each $n \geq 1$) if each $A_n \subset A_{n+1}$ is ramified (resp., decomposed). A family of examples of directed unions of ramified (resp., decomposed) extensions satisfying the condition "$M_j \cap B_i = M_i$ for all $i \leq j$ in I" will be given in Corollary 2.8. That corollary will be our main application of Theorem 2.5.

One natural question about the data and conclusions in Theorem 2.5 asks for conditions under which M is the crucial maximal ideal of $A \subset B$. Although that question is open at this time, that fact will not deter us from building explicit applications of Theorem 2.5. Those will be given in Corollary 2.8 by using the methods developed in Proposition 2.6 and the background material collected in Lemma 2.7.

Proposition 2.6. *Let $(I, \leq)$ be a directed set. Let $\{A_i \mid i \in I\}$ be an (upward-)directed set of commutative rings, with transition map $f_{ij} : A_i \hookrightarrow A_j$ if $i \leq j$ in I (such that if $i \leq j \leq k$ in I, then $f_{jk}f_{ij} = f_{ik}$ and f_{ii} is an identity map). Assume that if $i \leq j$ in I, then A_j is integral over A_i. Consider the direct limit $A := \varinjlim_{i \in I} A_i\, (= \cup_{i\in I} A_i)$. For each $i \in I$, view the canonical injective ring homomorphism $\alpha_i : A_i \to A$ as an inclusion map (such that if $i \leq j$ in I, then $\alpha_j f_{ij} = \alpha_i$). Let M be a maximal ideal of A. For each $i \in I$, consider the maximal ideal $M_i := (\alpha_i)^{-1}(M) (= M \cap A_i)$ of A_i. (Note that if $i \leq j$ in I, then $M_j \cap A_i = M_i$ and $\varinjlim_{i\in I} M_i\, (= \cup_{i\in I} M_i) = M$.) Then there exist (upward-)directed unions of integral minimal ring extensions of commutative rings, $A_i \hookrightarrow B_i$ and $A_i \hookrightarrow C_i$, such that for each $i \in I$, $A_i \subset B_i$ is a ramified minimal ring extension with crucial maximal ideal M_i and $A_i \subset C_i$ is a decomposed minimal ring extension with crucial maximal ideal M_i, and also such that, for the direct limits $B := \varinjlim_{i\in I} B_i\, (= \cup_{i\in I} B_i)$ and $C := \varinjlim_{i\in I} C_i\, (= \cup_{i\in I} C_i)$, one has that $A \subset B$ is an (integral) ramified minimal ring extension with crucial maximal ideal M and $A \subset C$ is an (integral) decomposed minimal ring extension with crucial maximal ideal M.*

Proof. Despite its title, the main result in [5] is that if Λ is any *nonzero* commutative ring, then for any maximal ideal $\mathcal{M}$ of Λ (and at least one such $\mathcal{M}$ exists, thanks to Zorn's Lemma), $\Lambda \subset \Lambda(+)\Lambda/\mathcal{M}$ is an integral minimal ring extension; moreover, by a result of G. Picavet and M. Picavet-L'Hermitte [30, Lemma 2.1], this extension is subintegral, and so we can conclude that it is a ramified minimal ring extension, necessarily with crucial maximal ideal $(\Lambda : \Lambda(+)\Lambda/\mathcal{M}) = \mathcal{M}$. (To check this calculation of the conductor, note that for any module $\mathcal{E}$ over a commutative ring Γ, one views Γ as a subring of the idealization $\Gamma(+)\mathcal{E}$ via the canonical injective (unital) Γ-algebra homomorphism $\Gamma \to \Gamma(+)\mathcal{E}$, given by $\gamma \mapsto (\gamma, 0)$ for all $\gamma \in \Gamma$.) For each $i \in I$, let $B_i := A_i(+)A_i/M_i$. If $i \leq j$ in I, we have that $M_j \cap A_i = M_i$, and so the inclusion map $A_i \subseteq A_j$ and the canonical injective A_i-algebra homomorphism $A_i/M_i \to A_j/M_j$ can be used to get a canonical injective ring homomorphism (which is defined coordinatewise)

$$g_{ij} : B_i := A_i(+)A_i/M_i \to A_j(+)A_j/M_j =: B_j.$$

It is easy to see that if $i \leq j \leq k$ in I, then $g_{jk}g_{ij} = g_{ik}$ and g_{ii} is an identity map. The direct limit (that is, the directed union)

$$B := \varinjlim_{i\in I} B_i = \cup_{i\in I} B_i = \cup_{i\in I}(A_i(+)A_i/M_i) =$$

$$\cup_{i\in I} A_i(+) \cup_{i\in I} A_i/M_i = A(+)((\cup_{i\in I} A_i)/(\cup_{i\in I} M_i)) = A(+)A/M.$$

By the main result of [5] (or, alternatively, by Theorem 1.5), $A \subset B$ is an integral ramified minimal ring extension. Its crucial maximal ideal is $(A : B) = (A : A(+)A/M) = M$.

The proof of the "decomposed" assertion has the same tempo as the above proof of the "ramified" assertion. For each $i \in I$, we define $C_i := A_i \times A_i/M_i$ and view $A_i \subseteq C_i$ via the canonical injective ring homomorphism $A_i \to A_i \times A_i/M_i$, given by $a \mapsto (a, a + M_i)$ for all $a \in A_i$. This ring extension is clearly integral. By the second sentence of the proof of [13, Corollary 2.5], $A_i \subseteq C_i$ is a minimal ring extension; moreover, the crucial maximal ideal of this integral minimal ring extension is $(A_i : C_i) = (A_i : A_i \times A_i/M_i) = M_i$. If $i \leq j$ in I, the identity map $A_i \to A_i$ combines with the canonical ring homomorphism $A_i/M_i \to A_j/M_j$ to produce a canonical injective ring homomorphism $A_i \times A_i/M_i \to$

$A_j \times A_j/M_j$ (defined coordinatewise); we view this homomorphism as an inclusion map and as the transition map $g^*_{ij} : C_i \to C_j$. It is easy to see that the transition maps satisfy the usual desired compatibility conditions, and so one has the direct limit

$$C := \varinjlim_{i \in I} C_i = \cup_{i\in I} C_i = \cup_{i\in I}(A_i \times A_i/M_i) =$$

$$(\cup_{i\in I} A_i) \times (\cup_{i\in I} A_i/M_i) = A \times ((\cup_{i\in I} A_i)/(\cup_{i\in I} M_i)) = A \times A/M.$$

By another appeal to the proof of [13, Corollary 2.5], $A \subseteq C$ is a minimal ring extension; moreover, the crucial maximal ideal of this integral minimal ring extension is $(A : C) = (A : A \times A/M) = M$. The proof is complete. □

It seems appropriate to record the fact that the constructions in Proposition 2.6 satisfy all the hypotheses from Theorem 2.5. We will do so now for the "ramified" part of the construction, leaving the similar details for the "decomposed" part to the interested reader. For the "ramified" part, the only missing detail ("missing" because it was not needed in the proof of Proposition 2.6) is to verify that "if $i \le j$ in I, then $M_j \cap B_i = M_i$." For each $k \in I$, the relevant maximal ideal of $A_k(+)A_k/M_k$ is $M_k(+)A_k/M_k$. Thus, our task is to show that if $i \le j$ in I, then

$$(M_j(+)A_j/M_j) \cap (A_i(+)A_i/M_i) = M_i(+)A_i/M_i.$$

This, in turn, holds since $M_j \cap A_i = M_i$ and $A_i/M_i \hookrightarrow A_j/M_j$.

The segue prior to Proposition 2.6 promised to build examples satisfying the hypotheses of Theorem 2.5. To complete the fulfillment of that promise, it will now suffice to construct algebraic field extensions $K \subset L$ such that there is a denumerable maximal chain of fields going from K to L. Before constructing or considering specific such chains

$$K = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n \subset A_{n+1} \subset \ldots \subset L,$$

it will convenient to recall some material from [17] and [31]. That material will be collected in Lemma 2.7. Before *that*, we devote two paragraphs to a variety of background material which is presupposed in Lemma 2.7.

The next paragraph will merely define two concepts that were introduced in the (unpublished) doctoral thesis of Michael S. Gilbert [17]. While that thesis was largely concerned with extensions of commutative rings, we will confine the contexts for the definitions in this paragraph and for the results quoted from [17] in Lemma 2.7 to the special cases whose contexts involve field extensions. Although [17] develops extensive information about finite-dimensional λ-field extensions, Lemma 2.7 will emphasize the infinite-dimensional λ-field extensions (as those are the ones that will matter in our applications to nontrivial directed unions in Corollary 2.8).

Let $K \subseteq L$ be fields. Then $K \subseteq L$ is said to be a *λ-extension* (or a *λ-field extension*) if the poset $[K, L]$ is linearly ordered (by inclusion). We say that $K \subseteq L$ is a *μ-extension* (or a *μ-field extension*) if there exists an element $\alpha \in L \setminus K$ such that $\alpha \in F$ for every field F such that $K \subset F \subseteq L$; when this condition holds, some of the analogous ring-theoretic literature would say that K is "a subfield of L that is maximal without α." Note that for any field K, $K \subseteq K$ is a λ-extension but not a μ-extension.

This second paragraph of background material concerns fields of characteristic $p > 0$. In [17, Proposition 3.26], Gilbert reported the following information from [3]. If K a field of characteristic $p > 0$, then there exists a purely inseparable (algebraic) field extension $K \subseteq \Pi$ such that Π is a perfect field; such Π is unique up to K-algebra isomorphism and is called *a perfect closure of K*; given an algebraic closure $\overline{K}$ of K, one construction for such a field takes $\Pi := \{u \in \overline{K} \mid u$ is purely inseparable over $K\}$, and this Π is called *the perfect closure of K (in $\overline{K}$)*. The proofs of these facts and some related facts in [3] and [17] use the following definition. If $K \subseteq L$ are fields of characteristic $p > 0$ (if L is not specified, it is assumed to be an algebraic closure of K) and if r is any positive integer, then

$$K^{(p^{-r})} := \{u \in L \mid u^{(p^r)} \in K\}.$$

It is known (and easy to prove) that if L and r are as above, then $K^{(p^{-r})}$ is a field; $K^{(p^{-r})} \subseteq K^{(p^{-(r+1)})}$ for all $r > 0$; and L is purely inseparable over K (if and) only if $L = \cup_{r>0} K^{(p^{-r})}$.

Lemma 2.7. (*a*) (*Gilbert [17, Theorem 3.1 (1), (2)]*) *Let $K \subseteq L$ be a λ-field extension. Then L is algebraic over K and every element of $[K,L]$ is a field.*

(*b*) (*Gilbert [17, Theorem 3.1 (3) (ii)]*) *Let $K \subseteq L$ be a λ-field extension such that $\dim_K(L) = \infty$. Then* (*and only then*) *$[K,L]$ is a denumerable set, consisting of*

$$K = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_n \subset F_{n+1} \subset \ldots \subset L,$$

where $\dim_K(F_n) < \infty$ for each integer $n \geq 0$, and $L = \cup_{n\geq 0} F_n$.

(*c*) (*Gilbert [17, Proposition 3.15 (1)]*) *Let $K \subseteq L$ be a μ-field extension. Then L is algebraic over K.*

(*d*) (*Gilbert [17, Proposition 3.15 (2)]*) *Let $K \subset L$ be a λ-field extension* (*such that $K \neq L$*). *Then $K \subseteq L$ is a μ-extension.*

(*e*) (*Gilmer and Heinzer [20, page 96, lines 6-11], cited by Gilbert in [17, Remarks 3.16 (i)]*) *There exists a μ-field extension that is not a λ-field extension.*

(*f*) (*Gilbert [17, Proposition 3.17 (2)]*) *Let $K \subset L$ be a λ-field extension. Then either L is purely inseparable over K or K is purely inseparably closed in L* (*the latter option meaning, by definition, that no element of $L \setminus K$ is purely inseparable over K*).

(*g*) (*Gilbert [17, Proposition 3.17 (3)]*) *A field extension $K \subset L$ is a μ-extension if and only if there is a* (*necessarily unique*) *field K_0 such that $K \subset K_0 \subseteq L$ and $K_0 \subseteq F$ for each field F such that $K \subset F \subseteq L$; when these conditions hold, K_0 is the unique minimal proper field extension of K that is a subfield of L. If $K \subset L$ is a μ-field extension and $\alpha \in L$, then K is a subfield of L that is maximal without α if and only if $\alpha \in K_0 \setminus K$; when these conditions hold, $K_0 = K(\alpha) = K[\alpha]$. Moreover, if $K \subset L$ is a μ-field extension, then either L is purely inseparable over K or K is purely inseparably closed in L.*

(*h*) (*Gilbert [17, Theorem 3.25]*) *Let $K \subset L$ be a purely inseparable* (*algebraic*) *extension of* (*distinct*) *fields of characteristic $p > 0$. Then the following four conditions are equivalent:*

(1) *There exists $u \in K^{(p^{-1})}$ such that $K^{(p^{-1})} = K(u)$;*

(2) *$[K^{(p^{-1})} : K] = p$;*

(3) *$K \subset L$ is a μ-extension and $K^{(p^{-1})} = K_0$* (*where K_0 is as in* (*g*));

(4) *$K \subset L$ is a μ-extension.*

(*i*) (*Gilbert [17, Proposition 3.26]*) *Each field of characteristic $p > 0$ has a perfect closure.*

(*j*) (*Gilbert [17, Theorem 3.29]*) *Let K be a non-perfect field of characteristic $p > 0$ and let Π be a perfect closure of K. Then the following four conditions are equivalent:*

(1) *$K \subset \Pi$ is a λ-extension;*

(2) *$K \subset \Pi$ is a μ-extension;*

(3) *$[K : K^p] = p$;*

(4) *There exists $u \in K$ such that $K = K^p(u)$.*

(*k*) (*Gilbert [17, Corollary 3.30 and Proposition 3.24]*) *Let K be a non-perfect field of characteristic $p > 0$, let Π be a perfect closure of K, and suppose that $K \subset \Pi$ is a λ-extension. Then $[K,\Pi]$ is a denumerable set, consisting of*

$$K \subset K^{(p^{-1})} \subset K^{(p^{-2})} \subset \ldots \subset K^{(p^{-n})} \subset K^{(p^{-(n+1)})} \subset \ldots \subset \Pi,$$

where $\dim_K(K^{(p^{-n})}) = p^n$ for each integer $n > 0$, and $L = \cup_{n>0} K^{(p^{-n})}$.

(*l*) (*Gilbert [17, Examples 3.31 (b)]*) *Let K be a field of characteristic $p > 0$ such that $[K : K^p] = p$, and let Π be the perfect closure of K inside an algebraic closure $\overline{K}$ of K. Then Π is an infinite-dimensional purely* *inseparable (algebraic) λ-extension of K (and μ-extension of K) and, for each positive integer n, $K^{(p^{-n})}$ is the unique p^n-dimensional field extension of K inside $\overline{K}$.*

(m) (*Gilbert [17, Theorem 3.36]*) *Let $K \subset L$ be a proper finite-dimensional Galois field extension. Then $K \subset L$ is a λ-extension $\Leftrightarrow$ $K \subset L$ is a μ-extension $\Leftrightarrow$ the Galois group of L/K is cyclic of prime-power order.*

(n) (*Gilbert [17, Theorems 3.37 and 3.38]; cf. also Gilmer and Heinzer [20, Theorem 2.5]*) *Let $K \subset L$ be an infinite-dimensional algebraic Galois field extension. Then $K \subset L$ is a λ-extension $\Leftrightarrow$ $K \subset L$ is a μ-extension $\Leftrightarrow$ the Galois group of L/K is isomorphic (as a topological profinite group) to the additive group of the ring of p-adic integers, for some prime number p.*

(o) (*Gilbert [17, Theorem 3.25]*) *Let L be a purely inseparable algebraic field extension of a proper subfield K of characteristic $p > 0$. Then: $K \subseteq L$ is a μ-extension $\Leftrightarrow$ $K^{(p^{-1})} = K(u)$ for some element $u \in K^{(p^{-1})}$ $\Leftrightarrow$ $[K^{(p^{-1})} : K] = p$.*

(p) (*Gilbert [17, Theorem 3.29, Proposition 3.27, Corollary 3.30, Examples 3.31 (a)]*) *Let K be a non-perfect field of characteristic $p > 0$ and let Π be the perfect closure of K (in some algebraic closure of K). Then: $K \subseteq \Pi$ is a λ-extension $\Leftrightarrow$ $K \subseteq \Pi$ is a μ-extension $\Leftrightarrow$ $[K : K^p] = p$ $\Leftrightarrow$ $K = K^p(u)$ for some element $u \in K$. If these equivalent conditions hold, then $[\Pi : K] = \infty$. For each prime number p, an example of K (and Π) satisfying these equivalent conditions can be built as follows: take $K := F(X)$ where F is any perfect field of characteristic p and X is an indeterminate over F (and Π is a perfect closure of K).*

(q) (*Quigley [31, Theorems 1, 2 and 3]*) *Let $K \subset L$ be a proper field extension such that L is algebraically closed. Then $K \subseteq L$ is a μ-extension $\Rightarrow$ $K \subseteq L$ is a λ-extension. (Also, by [17, Proposition 3.15 (2)], $K \subset L$ is a λ-extension $\Rightarrow$ $K \subseteq L$ is a μ-extension.) Also (by [17, Proposition 3.15 (1)] and Artin-Schreier theory, as in [24, pages 269-278]): if $K \subset L$ is a μ-extension (with L still being assumed to be algebraically closed) and $[L : K] < \infty$, then K is a real closed field (hence of characteristic 0) and $L = K(u)$ for some $u \in L$ such that $u^2 = -1$.*

Proof. We will comment here only about any deviations that the above statements may manifest in comparison with their cited sources.

(b): The statement of [17, Theorem 3.1 (3) (ii)] did not include the parenthetical phrase "and only then", as it would already have been known to readers of [17], the point being that [17, Theorem 3.1 (3) (ii)] pointed out, *inter alia*, that if $\dim_K(L) < \infty$ (and $K \subseteq L$ is a λ-field extension), then $[K,L]$ is a finite set.

(c): Gilbert's proof of (c) simply observed that Quigley's *proof* of [31, Lemma 1] can be applied to the present situation because that proof did not use the standing hypothesis in [31] that the "top" field L is algebraically closed. This "proof" is complete. □

Perhaps it is of interest to record that in [17, Remark 3.39], Gilbert used what we have just called Lemma 2.7 (n) (and a known nontrivial fact about profinite groups) in a proof that if ℓ is either 0 or a prime number, there exist fields $K \subset L$ of characteristic ℓ such that $K \subset L$ is an algebraic infinite-dimensional Galois field extension which is also a λ-extension.

We are now ready to complete our promise to "build examples satisfying the hypotheses of Theorem 2.5." This can be done by combining Proposition 2.6 and Lemma 2.7. In view of the abundance of infinite-dimensional λ-extensions of fields noted above, one can obtain several interesting applications in this way. To save space, we will make only three of those applications explicit in Corollary 2.8, thereby further highlighting what I consider to be some of the most important algebraic contributions of Gilbert and Quigley.

Corollary 2.8. *Let the proper field extension $K \subset L$ satisfy (at least) one of the following three conditions: (i) $K \subset L$ is an infinite-dimensional algebraic Galois field extension whose Galois group is isomorphic (as a topological profinite group) to the additive group of the ring of p-adic integers, for some prime number p; (ii) K is a non-perfect field of characteristic $p > 0$, $[K : K^p] = p$ and L is a perfect closure of K (an example satisfying (ii) can be found by taking $K := F(X)$ where F is any perfect field of characteristic p and X is an*

indeterminate over F (and L is a perfect closure of K));
(iii) The field K is not real closed and the field L is algebraically closed (and $K \subset L$).
Then it is known that $K \subset L$ is an algebraic field extension and $[K,L]$ is a denumerable set consisting of fields, as follows:

$$K = A_1 \subset A_2 \subset \ldots \subset A_n \subset A_{n+1} \subset \ldots \subset L,$$

with $A := \varinjlim_{i \in I} A_i \,(= \cup_{i \in I} A_i) = L$. Moreover, there exist upward-directed unions of integral minimal ring extensions of commutative rings, $A_i \hookrightarrow B_i$ and $A_i \hookrightarrow C_i$, such that for each $i \in I$, $A_i \subset B_i$ is a ramified minimal ring extension and $A_i \subset C_i$ is a decomposed minimal ring extension, and also such that, for the direct limits $B := \varinjlim_{i \in I} B_i \,(= \cup_{i \in I} B_i)$ and $C := \varinjlim_{i \in I} C_i \,(= \cup_{i \in I} C_i)$, one has that $K \subset B$ is an (integral) ramified minimal ring extension and $K \subset C$ is an (integral) decomposed minimal ring extension.

Proof. The specific examples in (i), (ii) and (iii) of (proper) infinite-dimensional λ–field extensions $K \subset L$ were given, respectively, in [17, Theorem 3.38] (cf. also [20, Theorem 2.5]); [17, Proposition 3.26] (cf. also [3]) and [17, Theorem 3.29 and Examples 3.31 (b)]; and [31, Theorems 1, 2 and 3]. They can also be seen in parts (n); (j), (k), (l) and (p); and (q), respectively, of Lemma 2.7. (Perhaps I should have explicitly noted earlier that the other parts of Lemma 2.7 were included to make those six just-mentioned parts easier to understand and to provide supporting background about λ-field extensions and μ-field extensions.) Of course, once one knows that each of (i), (ii) and (iii) gives a denumerable maximal chain of fields going from K to L, an application of Proposition 2.6 completes the proof. □

The paper will close with three remarks. The first of these comments further on minimal ring extensions involving commutative rings.

Remark 2.9. (a) One should not be surprised that the ramified (resp., decomposed) minimal ring extension B (of A) that was found in Corollary 2.8 turned out to be $A(+)A/M$ (resp., $A \times A/M$). (This identification of B was obtained toward the end of the proof of Proposition 2.6.) For many rings A, this is, up to A-algebra isomorphism, the only possibility. In case A is a field (as it was in Corollary 2.8), that fact is a consequence of [15, Lemme 1.2]. Shapiro and the author showed that the same kind of classification as in [15, Lemme 1.2] holds if A is any (commutative) integral domain that is not a field. Subsequently, in [13, Theorem 2.4], Shapiro and the author showed that this overall kind of classification of the commutative minimal ring extensions could be extended to cover commutative base rings A that do not contain any maximal ideals that are also minimal prime ideals and do have von Neumann regular total quotient rings. Later, in [29], Lucas showed that sufficiently general commutative rings A could admit ramified or decomposed (integral) minimal ring extensions that are not A-algebra isomorphic to $A(+)A/M$ or $A \times A/M$ for some maximal ideal M of R.

(b) Apart from Example 2.3 and Remark 2.4 (a), we have not paid much attention here to commutative integrally closed minimal ring extensions. For a characterization of such extensions over quasi-local base rings with von Neumann regular total quotient rings which features Kaplansky transforms, see [13, Theorem 3.7]. More generally, for characterizations of commutative integrally closed minimal ring extensions of an arbitrary commutative ring, some of which feature generalized Kaplansky transforms, see [4].

(c) If $A \subset B$ is a minimal ring extension involving commutative rings, the integral closure of A in B is an element of $[A,B]$; so if such an extension is a minimal ring extension, it must be either integral or integrally closed. For this reason, the study of commutative minimal ring extensions has broken naturally into separate studies of the "integral" and the "integrally closed" contexts. No such facile bifurcation seems to be possible in analyzing noncommutative minimal ring extensions. For an overview of some noncommutative research on topics abutting that of integrality, one can see a brief recent survey by Jarboui and the author in the second and third paragraphs of [10, Remark 2.11]. This completes the remark.

Remark 2.10 comments further on noncommutative minimal ring extensions. Its part (a) will establish that Theorem 2.5 (e) admits an analogous conclusion in a noncommutative setting. First, we anticipate that the following basic background information may be of help to some readers. Recall that a nonzero (not necessarily one-sided Artinian) ring R is said to be a *simple ring* if 0 and R are the only two-sided ideals of R. It is clear that a commutative simple ring is the same as a field. Recall also that a ring R is said to be a *prime ring* if $aRb \neq 0$ whenever a and b are nonzero elements of R. It is clear that a commutative prime ring is the same as a (commutative) integral domain. It is not difficult to prove (and it is well known) that each simple ring is a prime ring. The converse is false, even in the commutative case: consider $\mathbb{Z}$.

Remark 2.10. (a) Let $\{A_n \hookrightarrow B_n\}$ be a directed union of minimal ring extensions, with index set I being the set of positive integers (with its natural ordering), such that $B := \cup_n B_n$ is not commutative, B is not a prime ring and, for each n, A_n is a field and B_n is not a prime ring. Then (with $A := \cup_n A_n$ as usual): $A \subset B$ is a minimal ring extension and B is isomorphic to a (noncommutative) idealization $A \ltimes M$ for some simple A-A bimodule M.

We begin a proof of the above assertion by establishing its first part. As A is the directed union of fields, A is a field and, hence, a commutative ring. Since B is assumed noncommutative, it follows that $A \neq B$. Therefore, by Corollary 2.2, $A \subset B$ is a minimal ring extension.

The rest of the proof will assume some familiarity with [14]. Since A is a simple ring and $A \subset B$ is a minimal ring extension, the classification of the minimal ring extensions of a simple ring in [14, Theorem 6.1] shows that, up to "A-isomorphism," B satisfies one of four properties which are labeled (P), (PI), (SI) and (N). In the just-cited result, Dorsey and Mesyan noted that if a minimal ring extension $A \subset C$ (with A a simple ring, as it is here) satisfies either (P) or (PI), then C is a prime ring. Since B is assumed to not be a prime ring, $A \subset B$ must satisfy either (SI) or (N). However, it follows at once from the definition of (SI) that $A \subset B$ cannot satisfy (SI) since B is not "A-isomorphic" to $A \times A$ (since $A \times A$ is commutative and B is assumed to be noncommutative). Therefore, by the definition of (N), the proof is complete.

(b) The proof in (a) used the notion of an "A-isomorphism" $\Lambda \to \Omega$ when Λ and Ω are rings which each contain A as a subring. Although the terminology of "A-isomorphism" is used extensively in [14], I have been unable to find a definition of this terminology anywhere in [14]. It seems clear from reading [14] that its authors intended an "A-isomorphism" $\Lambda \to \Omega$ to have as many of the familiar properties of an algebra isomorphism (over a commutative base ring) that would be reasonable in a not-necessarily-commutative setting. This feeling is only reinforced by reading the definitions in [14, pages 3465-3466], given a ring R, of an R-ring, an R-rng, and an R-homomorphism from one R-rng to another. In using [14] (this includes my use of it in the second paragraph of (a)), one must be sure that the various "A-isomorphic to" assertions being quoted are referring to *unital* maps. This specific lack of detail in [14] is regrettable, as I believe that any objective reading of that paper leaves open this question: must an "A-isomorphism" of A-rings $\Lambda \to \Omega$ send the multiplicative identity element 1 in Λ to the multiplicative identity element 1 in Ω?

As indicated above, I am reasonably certain that the authors of [14] intended their "A-isomorphisms" of A-rings to be unital. I will be devoting an article (that is in preparation) to studying the noncommutative minimal ring extensions of a field. Of course, some of that work will appeal to the above-mentioned classification result over simple rings [14, Theorem 6.1]. But in some situations where the published proof of a nontrivial result in [14] has been marred by what I consider to be a lack of attention to unital behavior, I will address special cases as needed via explicit *ad hoc* methods to prove some of the assertions in the article that is in preparation.

I believe that much remains to be discovered about the noncommutative minimal ring extensions of a field. For instance, while my recent paper [8] answered in the affirmative the question from [14] as to whether, for every prime number p, $\mathbb{F}_p$ is, up to isomorphism, the only field of characteristic p with no noncommutative minimal ring extensions, I believe that the following companion question from [14] remains open: is $\mathbb{Q}$, up to isomorphism, the only field of characteristic 0 with no noncommutative minimal ring extensions?

(c) While my article in preparation will focus on the (cardinal) number of "isomorphism" classes of noncommutative minimal ring extensions of a given field, I hope that some readers will pursue some of the lines of inquiry that were begun in (a). The result in (a) should be regarded as a non-commutative analogue of Theorem 2.5 (e), since a commutative minimal ring extension of a field K which is not a prime ring must, by [15, Lemme 1.2], be K-algebra isomorphic to either $K \times K$ (which is property (SI) of [14, Theorem 6.1]) or $K[X]/(X^2)$ (and this latter option is K-algebra isomorphic to the idealization $K \ltimes K$, which is often denoted in the literature by $K(+)K$). I would expect that the setting in (a) could be changed from directed unions to a more general class of direct limits. Also, it may be of interest to allow the rings A_i in (a) to be simple rings (rather than fields), in view of the observation of Dorsey and Mesyan [14, page 3481, lines 23-24] that any direct limit of simple rings is a simple ring. In that regard, let me add the following: a thoroughly non-commutative variant of some of Theorem 2.5 can be obtained by applying the "$n \times n$ matrix operator" $\mathrm{M}_n(\dots)$ to each of the rings in Theorem 2.5, thanks to the result of Al-Kuleab and Jarboui [1, Corollary 1.2] that a ring extension $\Lambda \subset \Omega$ is a minimal ring extension if and only if, for some (equivalently, for each) positive integer n, $\mathrm{M}_n(\Lambda) \subset \mathrm{M}_n(\Omega)$ is a minimal ring extension. Finally, I would hope that someone who is better versed than I in noncommutative ring theory will be able to determine conditions under which one could remove (or at least alter) the hypothesis in (a) that B is not a prime ring. The remark is complete.

We close by viewing a result from [15] as being an antecedent for Corollary 2.2.

Remark 2.11. Ferrand and Olivier left the proof of the following assertion [15, Lemme 1.3] to the reader. If $A \subset B$ is a minimal ring extension with B (and A) commutative and S is a multiplicatively closed subset of A then, by viewing the canonical injective A-algebra homomorphism $A_S \to B_S$ as an inclusion map, we have that either $A_S = B_S$ or $A_S \subset B_S$ is a minimal ring extension. As the conclusion in this assertion has somewhat the same flavor as the conclusion in Corollary 2.2, it seems appropriate, for the sake of completeness, to give a detailed proof of [15, Lemme 1.3]. Since $A \subset B$ is a minimal ring extension, it follows by a process of elimination that it will suffice to show that if T is a ring such that $A_S \subseteq T \subseteq B_S$, then there exists a ring C such that $A \subseteq C \subseteq B$ and $T = C_S$. (Showing *that* will not need the hypothesis that $A \subset B$ is a minimal ring extension.) Let $h : B \to B_S$ be the (unique) B-algebra homomorphism from B to B_S. Consider $C := h^{-1}(T) = \{b \in B \mid b/1 \in T\}$. It follows from general principles that C is a ring and, hence, a subring of B. Also, $A \subseteq C$, since $A \subseteq h^{-1}(A_S) \subseteq h^{-1}(T) = C$. Next, if $b \in C$ and $s \in S$, then $b/s = (1/s)(b/1) \in A_S T \subseteq T^2 = T$. This proves that $C_S \subseteq T$. Lastly, if $t \in T$, then there exist $b \in B$ and $s \in S$ such that $t = b/s$, whence $b/1 = (s/1)t \in A_S T \subseteq T^2 = T$ and so $b \in C$, whence $t \in C_S$. This shows that $T \subseteq C_S$, completing the proof of [15, Lemme 1.3].

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