Moroccan Journal of Algebra and Geometry with Applications
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco

Volume 2, Issue 2 (2023), pp 218-225

ISSN: 2820-7114

Title :

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Moroccan Journal of Algebra and Geometry with Applications
Vol. 2(2) (2023), 218–225
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco
ISSN: 2820-7114

Rings in which every regular ideal is finitely generated

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Communicated by Mohammed Tamekkante
(Received 15 February 2023, Revised 24 July 2023, Accepted 28 July 2023)

Abstract. In this paper, we introduce a weak version of Noetherianity that we call regular Noetherian property. A ring is called regular Noetherian, if every regular ideal is finitely generated. We investigate the stability of this property under localization and homomorphic image, and its transfer to various contexts of constructions such as trivial ring extensions, pullbacks and amalgamated duplication of a ring along an ideal. Our results generate examples which enrich the current literature with new and original families of rings that satisfy this property.
Key Words: UN-ring, strongly UN-ring, pullbacks, trivial ring extension, amalgamation.
2010 MSC: 13A15, 13A18, 13F05, 13G05, 13C20.

1 Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. Let R denote such a ring, we denote by $Reg(R)$ and $Z(R)$ the set of all regular elements of R and the set of all zero-divisors of R respectively. By a “local" ring we mean a (not necessarily Noetherian) ring with a unique maximal ideal.

Recall that a ring R is Noetherian if every ideal of R is finitely generated. In view of this we introduce a weak version of Noetherianity that we call regular Noetherian property. A ring is called regular Noetherian, if every regular ideal is finitely generated. A Noetherian ring is naturally a regular Noetherian ring, and in the domain context, these two forms coincide.

Some of our results use the $R \propto M$ construction. Let R be a ring and M be an R-module. Then $R \propto M$, the *trivial* (*ring*) *extension of* R *by* M, is the ring whose additive structure is that of the external direct sum $R \oplus M$ and whose multiplication is defined by $(r_1, m_1)(r_2, m_2) := (r_1r_2, r_1m_2 + r_2m_1)$ for all $r_1, r_2 \in R$ and all $m_1, m_2 \in M$. The basic properties of trivial ring extensions are summarized in the books [14, 15]. Mainly, trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [4, 5, 9, 10, 11, 14, 15, 16, 17, 18].

Let T be a ring and let M be an ideal of T. Denote by π the natural surjection $\pi : T \longrightarrow T/M$. Let D be a subring of T/M . Then, $R := \pi^{-1}(D)$ is a subring of T and M is a common ideal of R and T, such that $D = R/M$. The ring R is known as the pullback associated to the following pullback diagram:

$$\begin{array}{ccc} R := \pi^{-1}(D) & \xrightarrow{\pi|_R} & D = R/M \\ \downarrow i & & \downarrow j \\ T & \xrightarrow{\pi} & T/M \end{array}$$

where i and j are the natural injections.
A particular case of this pullback is the $D+M$-construction, when the ring T is of the form $K+M$, where K is a field and M is a maximal ideal of T, and R takes the form $D+M$. See for instance [14].

Let A be a ring and I an ideal of A. The following ring construction called the amalgamated duplication of A along I was introduced and investigated by D'Anna in [7] with the aim of applying it to curve singularities (over algebraic closed fields) where he proved that the amalgamated duplication of an algebroid curve along a regular canonical ideal yields a Gorenstein algebroid curve [7, Theorem 14 and Corollary 17]. It is the subring $A \bowtie I$ of $A \times A$ given by

$$A \bowtie I = \{(a, a+i)/a \in A \textit{ and } i \in I\}.$$

This extension has been studied, in the general case, and from the different point of view of pullbacks, by D'anna and Fontana [8]. One main difference of this construction, with respect to the idealization, is that the ring $A \bowtie I$ can be reduced (and it is always reduced if A is an integral domain). If J is an ideal of A, then $J \bowtie I := \{(j, j+i) \mid j \in J, i \in I\}$ is an ideal of $A \bowtie I$ with $\frac{A \bowtie I}{J \bowtie I} \cong \frac{A}{J}$. Under the natural injection $A \hookrightarrow A \bowtie I$ defined by $i(a) = (a, a)$, we identify A with its respective image in $A \bowtie I$; and the natural surjection $A \bowtie I \twoheadrightarrow A$ yields the isomorphism $\frac{A \bowtie I}{(0) \bowtie I} \cong A$. See for instance [6, 7, 8, 10, 12, 13, 19].

In this paper, we investigate the possible transfer of regular Noetherian property to the direct product of rings and various trivial extension constructions. Also, we examine the transfer of regular Noetherian property to a particular pullbacks and to the amalgamated duplication of ring along an ideal. Using these results, we construct several classes of examples of non-Noetherian regular Noetherian rings.

2 Main Results

A ring is called regular Noetherian and noted reg-Noetherian, if every regular ideal is finitely generated (that is every proper regular ideal is finitely generated). Now we give the following natural results.

Proposition 2.1. 1. *A reg-Noetherian ring provided a Noetherian ring.*

2. *Assume that R is an integral domain. Then R is reg-Noetherian if and only if R is Noetherian.*

3. *All total ring is reg-Noetherian.*

Proof. Straightforward. □

A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, valuation rings, and Prüfer/semihereditary rings. See for instance [1, 2, 14, 17] (see figure 1 below). Hence, we have :

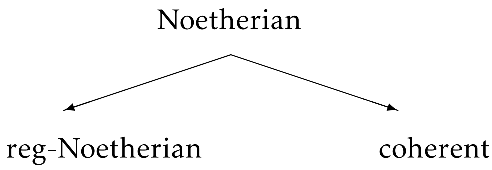

Figure 1:

Now, we give a new example of a non-Noetherian reg-Noetherian ring. Also, we show that the notions reg-Noetherian and coherent are not comparable.

Example 2.2. Let (A, M) be a local ring with a non finitely generated maximal ideal M (for instance, take $A = K[[X_1, ..., X_n, ...]]$ be a power series ring with infinite indeterminates over a field K) and set $R := A/M^2$. Then:

1. R is a local total ring with maximal ideal M/M^2. In particular, R is reg-Noetherian.
2. R is a non-Noetherian ring since M/M^2 is not finitely generated.

Example 2.3. Let R be a non-Noetherian coherent domain (for instance, take $R = K[[X_1, ..., X_n, ...]]$ be a power series ring with infinite indeterminates over a field K). Then:

1. R is coherent.
2. R is non-reg-Noetherian since it is a non-Noetherian domain.

Example 2.4. Let (A, M) be a local ring, $E := (A/M)^{\infty}$ be an infinite (A/M)-vector space, and $R := A \propto E$ be a trivial ring extension of A by E. Then:

1. R is a local total ring. In particular, R is reg-Noetherian.
2. R is non-coherent by [17, Theorem 2.6(2)] since E is an (A/M)-vector space with infinite rank.

Now, we study the transfer of reg-Noetherian notion to a direct product.

Proposition 2.5. *Let $R := \prod_{i=1}^{n} R_i$ the direct product of a rings R_i. Then R is a reg-Noetherian ring if and only if so is R_i, for every $i = 1, ..., n$.*

Proof. By induction, it suffices to show the proof for $n = 2$. Assume that R_1 and R_2 are reg-Noetherians and let J be a regular ideal of R. Then, it is easy to see that $J = I_1 \times I_2$, where I_i is a regular ideal of R_i for $i = 1, 2$. Hence, I_i is a finitely generated ideal of R_i and so $J := I_1 \times I_2$ is a finitely generated ideal of R, as desired.
Conversely, assume that R is reg-Noetherian and let I_1 be a regular ideal of R_1. Then, $I_1 \times R_2$ is a regular ideal of a reg-Notherian ring R, hence $I_1 \times R_2$ is a finitely generated ideal of R. Therefore, I_1 is a finitely generated ideal of R_1, as desired.
By the same argument, we show that R_2 is also a reg-Noetherian ring which completes the proof. □

We know that a localization of a Noetherian ring is Noetherian. Now, we give an example showing that the localization of a reg-Noetherian ring is not always a reg-Noetherian.

Example 2.6. Let $A = K[[X_1,...,X_n,...]] = K + M$ be a local power series ring with infinite indeterminates $(X_i)_{i\in\mathbb{N}}$ over a field K, where M is its maximal ideal generated by $(X_i)_{i\in\mathbb{N}}$ over a field K. Set $E := (A/M)^\infty (= K^\infty)$ be a K-vector space with infinite rank and set $R = A \propto E$ be the trivial ring extension of A by E. Let $S_0 := \{X_1^n / n \in \mathbb{N}\}$ be a multiplicative set of A and set $S := S_0 \propto 0$ a multiplicative set of R. Then:

1. R is reg-Noetherian since R is a total ring.
2. $S^{-1}R \cong S_0^{-1}A$ is a non-Noetherian integral domain. In particular, $S^{-1}R$ is a non-reg-Noetherian ring.

Proof. 1. Straightforward.

2. If we take $S_0 = \{X_1^n / n \in \mathbb{N}\}$ and $S = S_0 \propto 0$, we have $S^{-1}R \cong S_0^{-1}A = [S_0^{-1}(K[X_1])][X_2,...X_n,...]$ which is a non-Noetherian integral domain. Hence, $S^{-1}R$ is a non-reg-Noetherian ring, as desired.

□

Now, we study the transfer of reg-Noetherian property in trivial ring extension.

Theorem 2.7. Let A be a ring, E be an A-module and set $R := A \propto E$ be the trivial ring extension of A by E. Then:

1. Assume that A be an integral domain which is not a field, $K := qf(A)$, E be a K-vector space and $R := A \propto E$ be the trivial ring extension of A by E. Then:
   i) $R := A \propto E$ is a reg-Noetherian ring if and only if A is Noetherian.
   ii) R is non-coherent. In particular, R is non-Noetherian.
2. Assume that A is an integral domain, E is a finitely generated torsion free A-module, and set $R := A \propto E$ be the trivial ring extension of A by E. Then the following assertions are equivalents:
   i) R is reg-Noetherian.
   ii) A is Noetherian.
   iii) R is Noetherian.
3. Assume that (A,M) is a local ring, E is an (A/M)-vector space, and set $R := A \propto E$ the trivial ring extension of A by E. Then R is reg-Noetherian.

Proof. 1. i) Assume that A is Noetherian and let J be a proper regular ideal of R. Then there exists $(a,e) \in J$ such that $a \neq 0$ (since $(0 \propto E)(0,e) = 0$). Since $(a,e)R = aA \propto E$, hence $J := I \propto E$ for some proper ideal I of A. Therefore, $I := \sum_{i=1}^n Aa_i$ for some $a_i \in I \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0\}$ since A is Noetherian and so $J := \sum_{i=1}^n R(a_i, 0)$ is a finitely generated ideal of R, as desired.
Conversely, assume that R is a reg-Noetherian ring and let I be a proper ideal of A. Then, $J := I \bigotimes_A R = IR = I \propto E$ (since R is a flat A-module) is a regular ideal of R and so J is a finitely generated ideal of R since R is a reg-Noetherian ring. Assume that $J := \sum_{i=1}^n R(a_i, e_i)$ for some $(a_i, e_i) \in J := I \propto E$ and $n \in \mathbb{N} \setminus \{0\}$. Hence, $I := \sum_{i=1}^n Aa_i$ is finitely generated, as desired.
ii) R is non-coherent by [17, Theorem 2.8(1)]. In particular, R is non-Noetherian.

2. $i) \Rightarrow ii)$:
Assume that R is reg-Noetherian and let I be a proper ideal of A. Then, $J := I \propto IE$ is a proper

ideal of R which is regular. Indeed, let $(a,0) \in J$, where $a \in I - \{0\}$ and let $(b,f) \in R$ such that $(0,0) = (a,0)(b,f) = (ab,af)$. Hence, $ab = 0$ in A and $af = 0$ in E which imply that $b = 0$ (since A is an integral domain) and $f = 0$ (since E is a torsion free A-module). Hence, $(a,0)$ is a regular element in J and so J is a regular ideal of R. Therefore, $J := \sum_{i=1}^{n} R(a_i,e_i)$ since R is reg-Noetherian, where $(a_i,e_i) \in J$, and so $I = \sum_{i=1}^{n} Aa_i$, as desired.

$ii) \Leftrightarrow iii)$:
Clear since E is a finitely generated A-module.

$iii) \Rightarrow i)$:
Straightforward.

3. Straightforward since R is a total ring.

□

Corollary 2.8. *Let A be an integral domain, E be a finitely generated free A-module, and set $R := A \propto E$ be the trivial ring extension of A by E. Then:*

1. *R is reg-Noetherian.*
2. *A is Noetherian.*
3. *R is Noetherian.*

By Theorem 2.7(1), we obtain the following example:

Example 2.9. Let $R = \mathbb{Z} \propto \mathbb{Q}$. Then:

1. R is reg-Noetherian.
2. R is non-coherent. In particular, R is non-Noetherian.

We know that the homomorphic image of a Noetherian ring is Noetherian. The next example shows that the homomorphic image of a reg-Noetherian ring is not always a reg-Noetherian.

Example 2.10. Let (A,M) be a non-Noetherian local integral domain (For instance, take $A = \mathbb{Z}_2 + XQ[[X]]$ where X is an indeterminate over $\mathbb{Q}$), E be an (A/M)-vector space, and $R := A \propto E$ be the trivial ring extension of A by E. Then:

1. R is a reg-Noetherian ring since R is a total ring.
2. $R/(0 \propto E) \cong A$ is non-reg-Noetherian since it is an integral domain which is non-Noetherian.

Now, we study the transfer of reg-Noetherian property in a particular case of pullbacks.

Theorem 2.11. Let $T = K + M$ be a local ring, where K is a field and M is a maximal ideal of T such that for each $m \in M$, there exists $n \in M$ such that $mn = 0$ (take for instance $M^n = 0$ for some a positive integer n). Let $D \subsetneq K$ be a subring of K and set $R = D + M$. Then R is reg-Noetherian if and only if D is Noetherian.

Proof. Assume that R is a reg-Noetherian ring and let I be a proper ideal of D. Set $J = I + M$ be an ideal of R and we claim that J is a regular ideal of R. Indeed, let $d \in I - \{0\} \subseteq J$ and let $a + m \in R$ such that $d(a+m) = 0$, where $a \in D$ and $m \in M$. Then $0 = da + dm$ and so $da = 0$ in D and $dm = 0$ in M. Therefore, $a = 0$ since D is an integral domain and $d \in D - \{0\}$ and $m = 0$ since $0 = dm \in M$ and d is

invertible in K, hence d is a regular element in J. Hence, J is a finitely generated ideal of R since R is reg-Noetherian, that is $J = \sum_{i=1}^{n} R(a_i + m_i) = (\sum_{i=1}^{n} Da_i) + (\sum_{i=1}^{n} Rm_i + Ma_i) = \sum_{i=1}^{n} Da_i + M$, for some a positive integer n, $a_i \in I$ and $m_i \in M$, and so $I = \sum_{i=1}^{n} Da_i$ a finitely generated ideal of D. Hence, D is a Noetherian domain.
Conversely, assume that D is Noetherian and let J be a proper regular ideal of R. Then $J \subsetneq M$ since J is a regular ideal of R and so there exists $d + m \in J$, where $d \in D - \{0\}$ and $m \in M$. Hence, $J \supseteq (d + m)M = dM + mM = M$ (since $mM \subseteq M = dM$) and so $J = I + M$, where I is a proper ideal of D. Hence, $I = \sum_{i=1}^{n} Dd_i$ for some a positif integer n and $d_i \in D - \{0\}$ since D is Noetherian and so $J = I + M = \sum_{i=1}^{n} Dd_i + M = \sum_{i=1}^{n} Rd_i$ since $d_iM = M$ for each $d_i \in D - \{0\}$.
Therefore, J is a finitely generated ideal of R and so R is a reg-Noetherian ring which completes the proof of theorem 2.11. □

Example 2.12. Let $T = \frac{\mathbb{Q}[[\mathbb{X}]]}{<X^n>} = \mathbb{Q} + XT$, where X is an indeterminates over $\mathbb{Q}$, $\mathbb{Q}[[\mathbb{X}]]$ is the power series ring over $\mathbb{Q}$, and $< X^n >= X^n\mathbb{Q}[[X]]$ where n is a positive integers. Set $R = \mathbb{Z} + XT$. Then:

1. R is a reg-Noetherian ring by theorem 2.11.
2. R is non-Noetherian.

We end this work by studying the transfer of reg-Noetherian to the duplication.

Theorem 2.13. Let A be a ring, I be an ideal of A, and $R := A \bowtie I$ be the duplication of A by I. Then:

1. A is reg-Noetherian provided so is $A \bowtie I$.
2. Assume that A is a total ring and $I \subseteq J(A)$, where $J(A)$ is the Jacobson radical of A. Then $R := A \bowtie I$ is a reg-Noetherian ring.

Proof. 1) Let b be a regular element of a regular ideal I_0 of A. Set J be the ideal of R generated by $\{(a,a)/a \in I_0\}$. Hence, J is a regular ideal of R since (b,b) is a regular element in J since b is a regular element in I_0. Therefore, J is a finitely generated ideal of R since R is reg- Noetherian and so I_0 is a finitely generated ideal of A, as desired.
2) Assume that A is a total ring and $I \subseteq J(A)$. To show that $R := A \bowtie I$ is reg-Noetherian, it suffices to show that $R := A \bowtie I$ is a total ring.
Let $(x, x+i) \in A \bowtie I$. Two cases are then possible:
Case 1: $x \in Z(A)$:
In this case, $(x, x+i) \in Z(A \bowtie I)$ since $Z(A) \bowtie I \subseteq Z(A \bowtie I)$, as desired.
Case 2: x is invertible in A:
In this case, let $y := x^{-1}$ and set $j := -iy^2(1 + yi)^{-1}$. Since $I \subseteq J(A)$, then $j \in I$. Further, we have $(x, x+i)(y, y+j) = (1,1)$ and so $(x, x+i)$ is invertible in $R := A \bowtie I$, completing the proof of Theorem 2.13. □

Now, we construct a non-Noetherian reg-Noetherian ring by using the above Theorem 2.13.

Example 2.14. Let $A_0 = K[[X_1, ..., X_n, ...]] = K + M$ be a local power series ring with infinite indeterminate $\{X_i/i \in \mathbb{N}^*\}$ over a field K, where M is its maximal ideal generated by $\{X_i/i \in \mathbb{N}^*\}$. Set $A := \frac{A_0}{M^n} = K + \frac{M}{M^n}$ be a local ring with $I := \frac{M}{M^n}$ its maximal ideal, where $n \geq 3$ be a positive integers and set $R := A \bowtie I$. Then:

1. R is a reg-Noetherian ring by theorem 2.13 since A is a local total ring with maximal ideal $I := \frac{M}{M^n}$ and $I^n = 0$.

2. R is a non-Noetherian ring by [8, Corollary 3.3(b)] since A is a non-Noetherian ring (since I is non-finitely generated ideal of A).

Remark 2.15. The above example is neither a trivial ring extension (since $I^2 = \frac{M^2}{M^n} \neq 0$) nor a pullback of the type studied in this paper.

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