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When Two Definitions of an Additive Functor of Commutative Algebras Agree

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Abstract. Let R be a commutative ring and $\underline{C}_R$ the category of commutative unital R-algebras. We show that $\underline{C}_R$ is a pre-additive category if and only if R is a zero ring. When these conditions hold, a functor F from $\underline{C}_R$ to a pre-additive category $\underline{D}$ with finite products is an additive functor (in the classical sense) if and only if F is additive in the sense due to Chase-Harrison-Rosenberg (the latter sense of "additive functor" meaning that F commutes with finite products), if and only if $F(R)$ is a terminal object of $\underline{D}$. More generally, if $\underline{C}$ and $\underline{D}$ are additive categories (that is, pre-additive categories with finite products) and $F : \underline{C} \to \underline{D}$ is a functor, then F is additive if and only if F commutes with finite products. For such categories $\underline{C}$ and $\underline{D}$, we also give four other new characterizations of the additive functors $F : \underline{C} \to \underline{D}$.

Key Words: Commutative ring, unital algebra, pre-additive category, additive functor, CHR-additive functor, zero ring, sheaf.
2010 MSC: Primary 13A99; Secondary 18E05.

In memory of Jon Beck and his stimulating teaching

1 Introduction

Let $\underline{C}$ be a category, with $|\underline{C}|$ its class of objects; and for all $C_1, C_2 \in |\underline{C}|$, let $\underline{C}(C_1, C_2)$ denote the "hom-set" of all morphisms in $\underline{C}$ having domain C_1 and codomain C_2. It will be convenient to let Ab denote the category of abelian groups and (abelian) group (homo)morphisms. In the early days of homological algebra, one often said [19, page 32] (cf. also [4, page 19]) that a category $\underline{C}$ is an *additive category* if, for all $C_1, C_2 \in |\underline{C}|$, there is an "addition function" $+ = +_{C_1,C_2} : \underline{C}(C_1, C_2) \times \underline{C}(C_1, C_2) \to \underline{C}(C_1, C_2)$, with the accompanying notation $(\varphi, \psi) \mapsto \varphi + \psi := +(\varphi, \psi)$, such that $\underline{C}(C_1, C_2)$ is thereby an additive abelian group (that is, an object of Ab). Subsequently, the definition of an "additive category" evolved and now also includes, at least, the requirement that the following two properties hold for all (not necessarily pairwise distinct) objects C_1, C_2 and C_3 of $\underline{C}$:

$$f(g+h) = fg + fh \text{ for all } g, h \in \underline{C}(C_1, C_2) \text{ and all } f \in \underline{C}(C_2, C_3); \text{and}$$

$$(f+g)h = fh + gh \text{ for all } h \in \underline{C}(C_1, C_2) \text{ and all } f, g \in \underline{C}(C_2, C_3).$$

One should not regard the just-displayed properties as indicating a logical gap in either [19] or[4]. Indeed, those early texts had a special interest, for any unital ring R, in the category ${}_R\text{Mod}$ consisting of unital left R-modules and R-module homomorphisms (and in its special case ${}_{\mathbb{Z}}\text{Mod} = \text{Ab}$); of course, the naturally occurring "addition functions" in ${}_R\text{Mod}$ are given by pointwise addition (more precisely, if φ, ψ are each left R-module homomorphisms $A \to B$, then $(\varphi + \psi)(a) = \varphi(a) + \psi(a)$ for all $a \in A$) and evidently satisfy the two just-displayed properties. So, ${}_R\text{Mod}$ satisfies all the above axioms/properties. However, as category theory (and, with it, homological algebra) has continued to evolve, it would be more appropriate to use the terminology "$\underline{C}$ is a *pre-additive category*" [20, page 6] or "$\underline{C}$ is an Ab-*category*" [16, page 28] to describe a category $\underline{C}$ that satisfies all the above properties. Pre-additive categories will suffice as a general context for the ring-theoretic work that is the main object of this note. (For the sake of completeness, let us record that it is nowadays commonly agreed to define an *additive category* as a pre-additive category $\underline{C}$ that has a null object (that is, an object which is both an initial object of $\underline{C}$ and a terminal object of $\underline{C}$) and is such that, for all objects C_1 and C_2 of $\underline{C}$, there exists a biproduct (in the sense of [16, Definition, page 194]) of C_1 and C_2 in $\underline{C}$; cf. also [13, page 126] and [9, page 60].) Fortunately, all relevant references agree (cf. [16, page 29], [20, page 7]) that if $\underline{D}$ and $\underline{E}$ are pre-additive categories and $F : \underline{D} \to \underline{E}$ is a (covariant) functor, then F is called an *additive functor* if, for all objects D_1 and D_2 of $\underline{D}$, the assignment $f \mapsto Ff$ $(:= F(f))$ determines an abelian group homomorphism $\underline{D}(D_1, D_2) \to \underline{E}(F(D_1), F(D_2))$ (that is, if, for all objects D_1 and D_2 of $\underline{D}$ and for all $\varphi, \psi \in \underline{D}(D_1, D_2)$, one has $F(\varphi + \psi) = F(\varphi) + F(\psi)$).

As mentioned above, our main interest here is in studying rings. Indeed, **from this point on, all rings are assumed commutative and unital; all algebras are assumed to be commutative and unital; and all algebra homomorphisms and all modules are also assumed to be unital**. For any (commutative unital) ring R, we let $\underline{C}_R$ denote the category of (commutative unital) R-algebras and (unital) R-algebra homomorphisms. In particular, $\underline{C}_{\mathbb{Z}}$ is the category of (commutative unital) rings and (unital) ring homomorphisms. For a ring R, we have already mentioned one connection with the above material, namely, the fact that ${}_R\text{Mod}$ is a pre-additive category. It is well known that ${}_R\text{Mod}(R, B)$ $(= \text{Hom}_R(R, B)) \cong B$ for all R-modules B. Consequently, ${}_R\text{Mod}(R, B)$ is a singleton set (namely, $\{0\}$) if and only if $B = 0$. Thus ${}_R\text{Mod}(R, 0) = 0$ for any ring R. However, ${}_R\text{Mod}(D, E) = 0$ for all (unital) R-modules D and $E \Leftrightarrow R = 0$ (for if the identity map ι on R is identically 0 and $r \in R$, then $r = \iota(r) = 0$; and if $R = 0$, any (unital) R-module is (isomorphic to) 0). In other words, every hom-set of ${}_R\text{Mod}$ is a singleton set if and only if R is a singleton set (if and only if $1 = 0$ in R; if and only if R is a *zero ring*). It seems natural to ask whether $\underline{C}_R$ exhibits the same kind of categorical behavior as ${}_R\text{Mod}$. It is true that a (commutative unital) ring R is a zero ring if and only if every hom-set of $\underline{C}_R$ is a singleton set. (To find a proof of this, consider what it would mean to be a polynomial ring over a zero ring – such musings will lead to the proof in Remark 2.3.) However, it is also true that, unlike ${}_R\text{Mod}$, $\underline{C}_R$ is rarely a pre-additive category. Indeed, as recorded in Corollary 2.4, $\underline{C}_R$ is a pre-additive category if and only if R is a zero ring. Also, as recorded in Corollary 2.7 (b) and Remark 2.11 (b), if R *is* a zero ring, then some general conclusions can be drawn concerning the additive functors from $\underline{C}_R$ to some pre-additive categories (such as $\underline{C}_R$ or Ab).

While developing Galois Theory for rings, Chase, Harrison and Rosenberg introduced a rather different meaning for the terminology "additive functor" in [5, page 30]. To avoid confusion, we will call that notion a "CHR-additive functor." If $\underline{C}$ and $\underline{D}$ are categories with finite products, then a functor $F : \underline{C} \to \underline{D}$ is called a *CHR-additive functor* if F preserves finite products; that is, if, for all finite lists $C_1, \ldots, C_n$ (possibly with repetition) of objects of $\underline{C}$, the canonical morphism

$$F(\prod_{i=1}^{n} C_i) \to \prod_{i=1}^{n} F(C_i),$$

induced by applying F to the projection maps $\prod_{i=1}^{n} C_i \to C_j$ (for $1 \le j \le n$), is an isomorphism in $\underline{D}$. It was noted without proof in [5] that for any ring R, the unit functor and the Picard group functor are each (in the above terminolgy) CHR-additive functors from $\underline{C}_R$ to Ab. The assertion concerning the unit functor is clear; for the sake of completeness, a proof of the assertion concerning the Picard group functor was given in [6, Theorem 1.27].

Frankly, [6] paid little, if any, attention to zero rings and empty products (and perhaps that can also be said of [5]). It is easy to see that the unit functor and the Picard group functor each send any zero ring to 0 (that is, to a/the abelian group with only one element). The rest of this paragraph and the next paragraph will go beyond those contexts and develop some facts that will be used in

Section 2. Observe that for any category $\underline{C}$, an empty product in $\underline{C}$ is the same as a terminal object of $\underline{C}$. Thus, the case $n = 0$ in the definition in the preceding paragraph implies that if $\underline{C}$ and $\underline{D}$ are categories with finite products, then any CHR-additive functor $F : \underline{C} \to \underline{D}$ sends some, hence each, terminal object of $\underline{C}$ to a terminal object of $\underline{D}$. With an eye to applications in situations where ($\underline{C}$ and $\underline{D}$ are each assumed to have finite products and) $\underline{C}$ is a full subcategory of $\underline{C}_{\mathbb{Z}}$ (note that in any such situation, any zero ring in $|\underline{C}|$ is an empty product and a terminal object of $\underline{C}$), we can conclude that any CHR-additive functor $F : \underline{C} \to \underline{D}$ sends each zero ring in $|\underline{C}|$ to a terminal object of $\underline{D}$. In particular, in any such situation where $\underline{D} = \text{Ab}$, we can conclude that any CHR-additive functor sends any zero ring in $|\underline{C}|$ to 0.

It is natural to ask whether additive functors exhibit behavior that is somewhat like the behavior that was just noted for certain CHR-additive functors. A familiar argument (cf. the proof of Lemma 2.1 (a) below) shows that if $\underline{K}$ and $\underline{L}$ are pre-additive categories and $F : \underline{K} \to \underline{L}$ is an additive functor, then F sends any zero morphism f to a zero morphism (that is, if $K_1, K_2 \in |\underline{K}|$ and f is the neutral element in the abelian group $\underline{K}(K_1, K_2)$, then Ff is the zero morphism in $\underline{L}(F(K_1), F(K_2))$). Under these assumptions, it follows that if K_3 is a terminal object of $\underline{K}$, then $F(K_3)$ is a null object of $\underline{L}$. (Indeed, the preceding sentence implies that the identity map on K_3 (which is the only endomorphism of K_3 and, hence, is the neutral element in $\underline{K}(K_3, K_3)$) is sent to the neutral element in $\underline{L}(F(K_3), F(K_3))$ and, because F is a functor, it is also sent to the identity map on $F(K_3)$. Then [16, Proposition 1, page 194] can be applied to conclude that $F(K_3)$ is a null object of $\underline{L}$.) We have just shown that any additive functor sends any terminal object to a null object. When this fact is compared with what was shown in the preceding paragraph (specifically, that for categories with finite products, any CHR-additive functor sends any terminal object to a terminal object), it is natural to ask whether the concepts of “additive functor" and “CHR-additive functor" are equivalent in categorical settings where the context for the definition of each of these concepts is satisfied. For domain categories $\underline{C}$ of the form $\underline{C}_R$, that question will be answered in the affirmative in Corollary 2.7 (b) (i). The next paragraph will recall an apparently different categorical context for which [5] gave an affirmative answer. The final paragraph of the Introduction briefly summarizes the connections between the above material and the main results in this paper.

In [5], Chase, Harrison and Rosenberg noted one context where the notions of an additive functor and (what we have called) a CHR-additive functor agree, namely, for a functor $F : \underline{C} \to \underline{D}$ where both $\underline{C}$ and $\underline{D}$ are abelian categories. (Cf. also [9, Theorem 3.11].) Perhaps the most familiar examples of abelian categories are ${}_R\text{Mod}$ (for any unital ring R) and the category of Ab-valued sheaves on X (for any topological space X). The term “abelian category" is due to Grothendieck [13, page 127], who defined an abelian category as an additive category in which every morphism has a kernel and a cokernel and which satisfies the so-called AB2 axiom (which can be paraphrased as requiring that the First Isomorphism Theorem holds in the category). An equivalent definition of abelian categories was introduced slightly earlier by Buchsbaum in the appendix of [4], where (what are now called) abelian categories were called “exact categories": see, especially, [4, pages 379-381]. We recommend [9] as an excellent introduction to abelian categories. For instance, [9] contains a lucid proof that the dual of any abelian category is an abelian category; this conclusion can also be found elsewhere, for example in [16, page 201].

One problem suggested by the title is to study the rings R and the functors $F : \underline{C}_R \to \text{Ab}$ such that F is both additive (in the classical sense reviewed in the first paragraph) and also CHR-additive (in the sense defined four paragraphs ago). Unfortunately, the observation of Chase, Harrison and Rosenberg that was mentioned at the beginning of the preceding paragraph would seem to be of little help in such studies, since $\underline{C}_R$ is not an abelian category if R is a nonzero ring. (Indeed, since any abelian category is an additive category, it must have a null object. However, if R is a nonzero ring, then $\underline{C}_R$ does not have a null object, since the initial object R of $\underline{C}_R$ is not isomorphic to the terminal object 0 of $\underline{C}_R$.) Recall that the classical definition of an additive functor F requires the domain of F

to be a pre-additive category. We show in Corollary 2.4 that $\underline{C}_R$ can be given the structure of a pre-additive category (relative to some binary operation of "addition" in the hom-sets of $\underline{C}_R$) if and only if R is a zero ring. Our first main result, Corollary 2.7, shows, among other things, that if R is a zero ring, then a functor $\underline{C}_R \to \text{Ab}$ is additive if and only if it is CHR-additive. Despite the comparatively simple nature of $\underline{C}_R$ when R is a zero ring, Example 2.10 constructs, for any such ring R, two non-additive functors $\underline{C}_R \to \text{Ab}$. Our second main result, Theorem 2.18, examines the functors $F : \underline{C} \to \underline{D}$ in the most general relevant context, namely, where both $\underline{C}$ and $\underline{D}$ are pre-additive categories with finite products (and, hence, also with finite coproducts), that is, where both $\underline{C}$ and $\underline{D}$ are additive categories. The main contribution of Theorem 2.18 is the explication of five new characterizations of the additive functors $F : \underline{C} \to \underline{D}$ for any given additive categories $\underline{C}$ and $\underline{D}$.

2 Results

For the sake of completeness, we begin with a lemma that establishes some basic facts about pre-additive categories. Some other useful facts about pre-additive categories will be given in Proposition 2.5 and Lemma 2.6.

Following [16, page 33], we will use the following notation concerning dual categories. Let $\underline{C}$ be a category. Then the *dual category* of $\underline{C}$ is the category $\underline{C}^{\text{op}}$ which is defined as follows: $|\underline{C}^{\text{op}}| = |\underline{C}|$; for any $C_1, C_2 \in |\underline{C}^{\text{op}}|$, there is a bijection of hom-sets $\underline{C}(C_2, C_1) \to \underline{C}^{\text{op}}(C_1, C_2)$, denoted by $h \mapsto h^{\text{op}}$; and for all $C_1, C_2, C_3 \in |\underline{C}^{\text{op}}|$ and all $g^{\text{op}} \in \underline{C}^{\text{op}}(C_1, C_2)$ and $f^{\text{op}} \in \underline{C}^{\text{op}}(C_2, C_3)$, one defines $f^{\text{op}} g^{\text{op}} := (gf)^{\text{op}}$. It is harmless (and customary) to identify $(\underline{C}^{\text{op}})^{\text{op}}$ with $\underline{C}$ (so that a morphism $(h^{\text{op}})^{\text{op}}$ is identified with h).

Lemma 2.1. *Let $\underline{C}$ be a pre-additive category. Then:*

(*a*) *Let $C_1, C_2, C_3 \in |\underline{C}|$, $f \in \underline{C}(C_1, C_2)$, and $g \in \underline{C}(C_2, C_3)$. Also let n_1, n_2 and n_3, respectively, denote the neutral elements in the abelian groups $\underline{C}(C_1, C_2)$, $\underline{C}(C_2, C_3)$ and $\underline{C}(C_1, C_3)$. Then*

$$gn_1 = n_3 = n_2 f.$$

(*b*) *$\underline{C}^{\text{op}}$ is a pre-additive category.*

Proof. (a) Since $n_1 + n_1 = n_1$ and $\underline{C}$ is a pre-additive category, we have

$$gn_1 = g(n_1 + n_1) = gn_1 + gn_1.$$

By adding $-(gn_1)$ to the extreme members of the last displayed equations and then using group axioms to simplify the resulting expressions, we get

$$n_3 = -(gn_1) + gn_1 = -(gn_1) + (gn_1 + gn_1) = (-(gn_1) + gn_1) + gn_1 =$$

$n_3 + gn_1 = gn_1$. Similarly, $n_2 f = (n_2 + n_2)f = n_2 f + n_2 f$ leads to

$$n_3 = -(n_2 f) + n_2 f = -(n_2 f) + (n_2 f + n_2 f) = (-(n_2 f) + n_2 f) + n_2 f =$$

$n_3 + n_2 f = n_2 f$.

(b) If $C, D \in |\underline{C}^{\text{op}}|$, the canonical bijection $\underline{C}(D, C) \to \underline{C}^{\text{op}}(C, D)$ allows the abelian group structure on $\underline{C}(D, C)$ (which exists because $\underline{C}$ is assumed to be a pre-additive category) to be transferred to an abelian group structure on $\underline{C}^{\text{op}}(C, D)$. In detail: if $\lambda^{\text{op}}, \mu^{\text{op}} \in \underline{C}^{\text{op}}(C, D)$, then

$$\lambda^{\text{op}} + \mu^{\text{op}} := (\lambda + \mu)^{\text{op}}.$$

It only remains to show that for all objects C_1, C_2 and C_3 of $\underline{C}$, the following two "distributivity laws" hold:

$$f^{\text{op}}(g^{\text{op}} + h^{\text{op}}) = f^{\text{op}}g^{\text{op}} + f^{\text{op}}h^{\text{op}}$$

$$\text{for all } g^{\text{op}}, h^{\text{op}} \in \underline{C}^{\text{op}}(C_1, C_2) \text{ and all } f^{\text{op}} \in \underline{C}^{\text{op}}(C_2, C_3); \text{and}$$

$$(f^{\text{op}} + g^{\text{op}})h^{\text{op}} = f^{\text{op}}h^{\text{op}} + g^{\text{op}}h^{\text{op}}$$

$$\text{for all } h^{\text{op}} \in \underline{C}^{\text{op}}(C_1, C_2) \text{ and all } f^{\text{op}}, g^{\text{op}} \in \underline{C}^{\text{op}}(C_2, C_3).$$

We will prove the first of these "laws" and leave to the reader the (similar) proof of the second "law". An interesting feature of the proof will be that the first (resp., second) of the distributivity laws in $\underline{C}^{\text{op}}$ will follow from the second (resp., first) of the distributivity laws in $\underline{C}$.

Given $g^{\text{op}}, h^{\text{op}} \in \underline{C}^{\text{op}}(C_1, C_2)$ and $f^{\text{op}} \in \underline{C}^{\text{op}}(C_2, C_3)$, we have

$$f^{\text{op}}(g^{\text{op}} + h^{\text{op}}) = f^{\text{op}}(g + h)^{\text{op}} = ((g + h)f)^{\text{op}}.$$

As the morphisms in $\underline{C}$ satisfy both distributivity laws (because $\underline{C}$ is a pre-additive category), the right-most member in the last display can be expressed as

$$(gf + hf)^{\text{op}} = (gf)^{\text{op}} + (hf)^{\text{op}} = f^{\text{op}}g^{\text{op}} + f^{\text{op}}h^{\text{op}}.$$

The proof is complete. □

In any pre-additive category $\underline{C}$, it is customary to let 0 denote the neutral element in any hom-set $\underline{C}(C_1, C_2)$. One can do so because each such hom-set is an abelian group under some "addition" operation +. Using the "0" notation in this way allows us to restate the conclusion of Lemma 2.1 (a) as $g0 = 0 = 0f$ (for $\underline{C}$, f and g as supposed above). However, this sort of use of the notation "0" can be ambiguous if it is not clear which objects of $\underline{C}$ are intended to be the domain and codomain, respectively, of "0." For that reason, the above statement of Lemma 2.1 (a) used the symbols n_1, n_2 and n_3 (instead of the generic symbol 0), out of an abundance of caution. In most situations, no harm is likely in using the notation "0" in a pre-additive category. The proof of Proposition 2.2 will use the fact that $0f = 0$, but it will not need to explicitly use the fact that $g0 = 0$.

The proof of Lemma 2.1 (a) was necessarily quite fussy. Half of that fussiness could have been avoided if we had proved part (b) of Lemma 2.1 before proving part (a). (We did not choose that reorganization because our usual pedagogic/expository preference is to begin with the easier proofs.) Let us give the details of how one could use Lemma 2.1 (b) and the "$g0 = 0$" conclusion from Lemma 2.1 (a) to prove the "$0f = 0$" conclusion in Lemma 2.1 (a). In detail,

$$0f = (0^{\text{op}})^{\text{op}}(f^{\text{op}})^{\text{op}} = (f^{\text{op}}0^{\text{op}})^{\text{op}} = (0^{\text{op}})^{\text{op}} = 0.$$

The preceding details give a nice example of using dual categories to avoid excessive fussiness. In particular, observe that the third equality in the preceding display used the fact that 0^{op} is the neutral element in the appropriate hom-set of $\underline{C}^{\text{op}}$ (and applied the first equation in Lemma 2.1 (a) to the pre-additive category $\underline{C}^{\text{op}}$ and its morphism $g := f^{\text{op}}$).

The proof of Proposition 2.2 will also use [16, Proposition 1, page 194], which is the first result in the section on pre-additive categories in [16]. The proof of that result in [16] is short, slick, very clever and, in our opinion, somewhat incomplete in two ways, the first of which is very minor and the second of which is more noteworthy. First, the proof of [16, Proposition 1, page 194] uses the fact that $0f = 0$. We agree that this fact is available at that point in [16], because our statement of Lemma 2.1 (a) can be gleaned from the first half of the (in our opinion, very terse) final sentence preceding the statement of [16, Proposition 1, page 194]. Indeed, that half of that sentence can be interpreted, using the above terminology, as stating the following:"Again, a composite with the neutral element of a hom-set in a pre-additive category is necessarily the neutral element of the appropriate hom-set in that category". One can find what was apparently intended to serve as a proof of the assertion in the first half of the final sentence preceding the statement of [16, Proposition 1, page 194] by reading the following second half of the final sentence preceding the statement of [16, Proposition 1, page 194]: ", since composition is distributive over addition." We would agree that the preceding quotation does give the second most important step in the proof that $0f = 0$, but it has omitted the most important step (which would begin the argument by observing that $0f = (0+0)f$) and it has also omitted the last step of the proof (which is to use the abelian group structure of the hom-sets by adding $-(0f)$ to both sides of the equation $0f = 0f + 0f$ and then simplifying by using the group axioms satisfied by that hom-set). I suggest that a clearer wording for the second half of the final sentence preceding the statement of [16, Proposition 1, page 194] would have been the following: ": mimic the usual proof that $r0 = 0 = 0r$ for any element r of a ring R". Thus, my first complaint about the proof of [16, Proposition 1, page 194] is only a critique of the terse manner in which it justified the step asserting that $0f = 0$. The second "somewhat incomplete" aspect of that proof is more serious. The statement of [16, Proposition 1, page 194] is that four conditions, (i)-(iv), in a pre-additive category A are equivalent and, hence, that "In particular, any initial (or terminal) object in A is a null object." This "In particular" assertion justifies the main step in the proof given below of Proposition 2.2. Also, one can see at once that this "In particular" assertion is clear *if* one has truly proven that (i)-(iv) are equivalent (for the ambient pre-additive category). However, an objective report on the complete published three-sentence proof of [16, Proposition 1, page 194] is the following: its first sentence implicitly uses $0f = 0$ to show that (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv), its second sentence explicitly uses $0f = 0$ to prove that (iii) $\Rightarrow$ (ii), and its third sentence states that "The rest follows by duality." I agree that what Mac Lane has called "the rest" *would* follow by duality *if* one knew that the dual of the ambient pre-additive category is itself a pre-additive category. Unfortunately, I cannot find anything in [16] prior to its page 194 that would suggest that one should (or does) know that the class of pre-additive categories is stable under the formation of dual categories. That deficiency in the exposition in [16, page 194] is why Lemma 2.1 (b) was given above. With both parts of Lemma 2.1 in hand, one can now use the preceding comments to give what I would consider to be a complete proof of [16, Proposition 1, page 194]. With that in hand, the proof of Proposition 2.2 that is given below will be seen as also being complete. However, I am frankly concerned that such a category-laden approach to proving Proposition 2.2 may deter some inexperienced readers. So, let me mention here that another proof of Proposition 2.2 will be given in Remark 2.3 and this alternate proof will use only the fact that $0f = 0$ from Lemma 2.1 (a) and the universal mapping property of a polynomial ring over a nonzero commutative ring. I would like to end this long paragraph with a three-part *apologia* of sorts for its existence. This paragraph has allowed me to rectify what I have long considered to be one of the exceedingly rare blemishes in the writings of Saunders Mac Lane (I know of only one other serious blemish in his writings) – Mac Lane was an outstanding creative mathematician and expositor; this paragraph has allowed me a forum to publish something that I discovered in April 1967 during my first week of doctoral research (my doctoral advisor, who was one of the authors of [5], advised me that the contents of what are here called Remark 2.3 and Corollary 2.4 should not appear in my eventual doctoral thesis [6], and I was frankly too intimidated to request an explanation from him or to otherwise pursue the matter further at that time); and this paragraph has given me the opportunity to alert any commutative ring-theorists who would prefer to read as little category theory as possible that one can proceed to the proofs in Remark 2.3 and Corollary 2.4 at once after reading the proof of Lemma 2.1 (a).

Recall that a *zero ring* is a ring with a unique element (equivalently, a singleton set R, with the unique function $R \times R \to R$ necessarily taken as both the "addition" operation on R and the "multiplication" operation on R, and with the unique element of R necessarily playing both the additive role of 0 in R and the multiplicative role of 1 in R). It is easy to see (and we will need to use this triviality later) that if R and S are rings and R is a zero ring, then S is a zero ring if and only if $R \cong S$ (as rings).

We can now give a necessary condition for $\underline{C}_R$ to be a pre-additive category.

Proposition 2.2. *Let R be a ring such that $\underline{C}_R$ is a pre-additive category. Then R is a zero ring.*

Proof. There is a unique way to view R as a (commutative unital) R-algebra (namely, via $s \cdot r := sr$ for all $s, r \in R$). Observe that R is then an initial object of $\underline{C}_R$. Choose T to be any zero ring, and let $t \in T$ denote the unique element of T. There is exactly one way to view T as a (commutative unital) R-algebra (namely, via $r \cdot t := t$ for each $r \in R$). Observe that T is then a terminal object of $\underline{C}_R$. Since $\underline{C}_R$ is assumed to be a pre-additive category, it follows from [16, Proposition 1, page 194] that every initial object of $\underline{C}_R$ is a terminal object of $\underline{C}_R$. Hence, both R and T are terminal objects of $\underline{C}_R$. But any two terminal objects of a category are isomorphic in that category. Consequently, R and T are isomorphic in $\underline{C}_R$. Thus, by equating cardinalities, we get $|R| = |T| = 1$. Therefore, R is a zero ring, as asserted. □

The assertion in the preceding proof that every initial object of $\underline{C}_R$ is a terminal object of $\underline{C}_R$ (assuming that $\underline{C}_R$ is a pre-additive category) follows from the implication (i) ⇒ (ii) in [16, Proposition 1, page 194]. That implication was proved in the first and second sentences of the proof of [16, Proposition 1, page 194]. Thus, all that the above proof of Proposition 2.2 needed from [16, Proposition 1, page 194] was the first and second sentences of the latter's proof. If the proof of Proposition 2.2 had, instead, used the fact that any two *initial* objects of a category are isomorphic, we would have needed to know that any terminal object of a pre-additive category is an initial object of that category; that, in turn, would have required the proof of Proposition 2.2 to use/explicate the third sentence of the proof of [16, Proposition 1, page 194] (namely, the above-mentioned sentence, "The rest follows by duality."); that, in turn, would have required us to develop Lemma 2.1 (b). By the way, the above proof of Proposition 2.2 also needed (because of its role in proving that (i) ⇒ (ii)) the part of the second sentence of the proof of [16, Proposition 1, page 194] which asserted that $0f = 0$, which is half of the content of Lemma 2.1. In summary, a justifiable appeal to [16, Proposition 1, page 194] in the proof of Proposition 2.2 required us to develop half of Lemma 2.1 (a), while contemplation of the proof that was published for [16, Proposition 1, page 194] led us to develop Lemma 2.1 (b) and the "other" half of Lemma 2.1 (a). We hope that much of the rest of this paper will convince the reader that there is merit in our emphasis here on pre-additive categories, as that emphasis will lead to new technical information about pre-additive categories (in Proposition 2.5 and Lemma 2.6) and, ultimately, to the solution of the paper's motivating question in Corollary 2.7 (b) (i), along with several categorical characterizations of the category of zero rings in Corollary 2.7 (a), as well as more substantial categorical generalizations in Theorem 2.18.

We next give an alternate, less categorical proof of Proposition 2.2. The results and arguments given in Remark 2.3 and Corollary 2.4 were found by the author in April 1967.

Remark 2.3. While readers who are comfortable with the basics of category theory may find the above proof of Proposition 2.2 to be terse (and perhaps self-contained and elegant), we believe that many readers will find the following alternate proof of Proposition 2.2 to be more direct and accessible than the proof which was given above. We will give an indirect argument (that is, a "proof by contradiction"). So, we assume that $\underline{C}_R$ is a pre-additive category and that the ring R is not a zero ring, and our task is to find a contradiction. Let X be an indeterminate over the ring R. By the universal mapping property of a polynomial ring over a nonzero commutative ring (cf. [14, Theorem 5.5, page 152]), the assignment $\varphi \mapsto \varphi(X)$ determines a bijection $\underline{C}_R(R[X], R) \to R$. The inverse of this bijection sends any $r \in R$ to the R-algebra homomorphism $\psi_r : R[X] \to R$ such that $\psi_r(X) = r$. Since $\underline{C}_R$ is a pre-additive category, the hom-set $\underline{C}_R(R[X], R)$ is an abelian group (under some "addition" operation which we need not specify). The neutral element in this abelian group (which we hesitate to denote by the overworked symbol "0") is, by the preceding observation, of the form ψ_n for some uniquely detemined element $n \in R$. (This choice of notation is motivated by the condition that ψ_n is a *n*eutral element.) By the second assertion in Lemma 2.1 (a), we get that if A is any (commutative unital) R-algebra and $\rho : A \to R[X]$ is any (unital) R-algebra homomorphism, then $\psi_n\rho$ is the neutral element of the abelian group $\underline{C}_R(A,R)$. In particular, if $A = R[X]$ and ρ is taken to be the R-algebra endomorphism of $R[X]$ determined by $X \mapsto X+1$, then $\psi_n\rho$ is the neutral element of the abelian group $\underline{C}_R(R[X],R)$. In other words, $\psi_n\rho = \psi_n$. Applying these equal functions to $X \in R[X]$, we get that $\psi_n(X)+1 = \psi_n(X)+\psi_n(1) =$

$$\psi_n(X+1) = \psi_n(\rho(X)) = (\psi_n\rho)(X) = \psi_n(X) = \psi_n(X)+0,$$

whence $1 = 0$ in R, whence each $r \in R$ satisfies $r = r \cdot 1 = r \cdot 0 = 0$, whence R is a zero ring, the desired contradiction. This completes the proof. This completes the remark.

It is natural to ask if the converse of Proposition 2.2 is valid. The next result answers this question.

Corollary 2.4. *Let R be a (commutative unital) ring. Then the following conditions are equivalent:*
(1) *$\underline{C}_R$ is a pre-additive category;*
(2) *R is a zero ring.*

Proof. (1) $\Rightarrow$ (2): Apply Proposition 2.2 (or Remark 2.3).

(2) $\Rightarrow$ (1): Assume that R is a zero ring. It follows easily that each (unital) R-algebra (that is, each object of $\underline{C}_R$) is also a zero ring. Hence, it also follows easily that for any objects S and T of $\underline{C}_R$, the unique function $f_{S,T} : S \to T$ is an R-algebra homomorphism, and so the hom-set $\underline{C}_R(S,T)$ is the singleton set $\{f_{S,T}\}$. Of course, this singleton set can be given the structure of an additive abelian group in a unique way (by defining $f_{S,T} + f_{S,T}$ to be $f_{S,T}$). It remains only to prove that, with addition having been explicated (in fact, forced) in all the hom-sets of $\underline{C}_R$, both distributivity laws hold in $\underline{C}_R$. We will prove the first of those laws, leaving to the reader the (similar) proof of the second distributivity law.

Suppose, then, that S, T and U are objects of $\underline{C}_R$, with $g,h \in \underline{C}_R(S,T)$ and $f \in \underline{C}_R(T,U)$. It remains only to prove that $f(g+h) = fg+fh$. This, in turn, is evident, since $f(g+h)$ and $fg+fh$ are each elements of the singleton set $\underline{C}_R(S,U)$. The proof is complete. □

We would caution the reader not to rework the above proof of Corollary 2.4 by using copious occurrences of the symbol "0". A more careful approach (for example, using notation such as the above "$f_{S,T}$") will yield clear benefits in Corollary 2.7 where, among other things, we will answer this paper's motivating question. We would also caution the reader, in case R is a zero ring, not to view $\underline{C}_R$ as having a unique object. While it is true (cf. [2, Chapter II, 1.2]) that a strong version of the Axiom of Choice (specifically, that the universe can be well-ordered) implies that every category is equivalent to a "skeletal" category (that is, to a category in which any two isomorphic objects are equal) and it has been known for more than 80 years (cf. [12]) that such a strong Axiom of Choice is consistent with ZFC, we recommend that one should not decide to make such an additional foundational assumption simply because of a desire to simplify some notation.

We next give two categorical results. Proposition 2.5 shows one way in which CHR-additive functors and additive functors behave similarly in relevant contexts, where each of these properties of functors is shown to be stable under natural equivalence. While the proof of Proposition 2.5 will seem routine for readers who are comfortable with category theory, we will provide full details for that proof, in order to enhance accessibility. That result can be seen as motivation for some of Example 2.10. The path to our first main result, Corollary 2.7, will be eased by Lemma 2.6, which collects/states some facts that were proved in the Introduction.

Proposition 2.5. *(a) Let $\underline{C}$ and $\underline{D}$ be pre-additive categories, let $F : \underline{C} \to \underline{D}$ be an additive functor, and let $G : \underline{C} \to \underline{D}$ be a functor such that F and G are naturally equivalent. Then G is an additive functor.*

(b) Let $\underline{C}$ and $\underline{D}$ each be categories with finite products, let $F : \underline{C} \to \underline{D}$ be a CHR-additive functor, and let $G : \underline{C} \to \underline{D}$ be a functor such that F and G are naturally equivalent. Then G is a CHR-additive functor.

Proof. (a) By hypothesis, we can pick a natural equivalence $\eta : F \to G$. Thus for each object C of $\underline{C}$, one has a "natural" isomorphism $\eta_C : F(C) \to G(C)$ in $\underline{D}$. Our task is to prove that if $f, g \in \underline{C}(C_1, C_2)$ (for some objects C_1 and C_2 of $\underline{C}$), then $G(f+g) = G(f) + G(g)$. Of course, $F(f+g) = F(f) + F(g)$, since F is assumed to be an additive functor. Moreover, when the "naturality" of the above-mentioned isomorphisms of the form η_C is applied to the morphisms f, g and $f+g$, we get

$$G(f)\eta_{C_1} = \eta_{C_2}F(f), G(g)\eta_{C_1} = \eta_{C_2}F(g) \text{ and } G(f+g)\eta_{C_1} = \eta_{C_2}F(f+g).$$

Therefore, since composition distributes over addition of morphisms in (the pre-additive category) $\underline{D}$, we get

$$G(f) + G(g) = \eta_{C_2}F(f)(\eta_{C_1})^{-1} + \eta_{C_2}F(g)(\eta_{C_1})^{-1} =$$

$\eta_{C_2}(F(f) + F(g))(\eta_{C_1})^{-1}$. This simplifies to $\eta_{C_2}F(f+g)(\eta_{C_1})^{-1} = G(f+g)$, as desired.

(b) Let us first deal with the case of empty products. In that regard, it will suffice to show that if T is a terminal object of $\underline{C}$ such that $F(T)$ is a terminal object of $\underline{D}$, then $G(T)$ is also a terminal object of $\underline{D}$. This can be shown by using basic category theory (without the hypothesis that F is CHR-additive and without the hypothesis that the object T is terminal in $\underline{C}$), as follows. Our task is to show that if $D \in |\underline{D}|$, then $\underline{D}(D, G(T))$ is a singleton set. By hypothesis, $\underline{D}(D, F(T))$ is a singleton set. Let φ denote its unique element. Pick a natural equivalence $\eta : F \to G$. Then $\psi := \eta_T\varphi \in \underline{D}(D, G(T))$, where as usual, η_T denotes the "natural" isomorphism $F(T) \to G(T)$ given by η. It remains only to prove that if $\psi^* \in \underline{D}(D, G(T))$, then $\psi^* = \psi$ (that is, $\psi^* = \eta_T\varphi$). This, in turn, holds since $(\eta_T)^{-1}\psi^* = \varphi$, the point being that $(\eta_T)^{-1}\psi^* \in \underline{D}(D, F(T)) = \{\varphi\}$.

It remains to consider nonempty finite products. Let $C_1, C_2, \ldots, C_n$ be a finite list (possibly with repetition) of elements in $|\underline{C}|$, for some integer $n \geq 2$. Fix a (direct) product $P = \prod_{i=1}^n C_i$ in $\underline{C}$; also fix products $\prod_i F(C_i)$ and $\prod_i G(C_i)$ in $|\underline{D}|$. The structures of these products include projection maps $p_j : P \to C_j$, $\pi_j : \prod_i F(C_i) \to F(C_j)$ and $\rho_j : \prod_i G(C_i) \to G(C_j)$, for $j = 1, \ldots, n$. The universal mapping property of products gives uniquely determined morphisms

$$\alpha : F(P) \to \prod_{i=1}^n F(C_i) \text{ and } \beta : G(P) \to \prod_{i=1}^n G(C_i)$$

such that $\pi_j\alpha = F(p_j)$ and $\rho_j\beta = G(p_j)$ for $j = 1, \ldots, n$. By hypothesis, α is an isomorphism. Our task is to show that β is an isomorphism.

Pick a natural equivalence $\eta : F \to G$. Recall that η_C is an isomorphism for each $C \in |\underline{C}|$. The universal mapping property of products gives uniquely determined morphisms

$$\gamma : \prod_{i=1}^n F(C_i) \to \prod_{i=1}^n G(C_i) \text{ and } \delta : \prod_{i=1}^n G(C_i) \to \prod_{i=1}^n F(C_i)$$

such that $\rho_j\gamma = \eta_{C_j}\pi_j$ and $\pi_j\delta = (\eta_{C_j})^{-1}\rho_j$ for $j = 1, \ldots, n$.

We next make the following two claims: the composite morphisms $\gamma\delta$ and $\delta\gamma$ are each identity maps. As the proofs of these claims are similar, we will prove the claim about $\gamma\delta$ and leave the claim about $\delta\gamma$ to the reader. By the "uniqueness" aspect of the universal mapping property of products, the claim will follow if we show that $\rho_j(\gamma\delta) = \rho_j$ for $j = 1, \ldots, n$. For each j, we have

$$\rho_j(\gamma\delta) = (\rho_j\gamma)\delta = (\eta_{C_j}\pi_j)\delta = \eta_{C_j}(\pi_j\delta) = \eta_{C_j}((\eta_{C_j})^{-1}\rho_j) = \rho_j,$$

thus proving the above claim(s).

It follows (now that we have proved the above claims) that γ is an isomorphism, with $\gamma^{-1} = \delta$. Hence, being a composite of isomorphisms, $\gamma\alpha(\eta_P)^{-1}$ is an isomorphism. Therefore, to complete the proof, it will suffice to show that $\beta = \gamma\alpha(\eta_P)^{-1}$. By the universal mapping property of $\prod_i G(C_i)$, an equivalent task is to show that

$$\rho_j\beta = \rho_j(\gamma\alpha(\eta_P)^{-1}) \text{ if } j = 1, \dots, n.$$

Fix j. Note via the naturality of η that $G(p_j)\eta_P = \eta_{C_j}F(p_j)$, and so $G(p_j) = (\eta_{C_j}F(p_j))(\eta_P)^{-1}$. Hence,

$$\rho_j\beta = G(p_j) = \eta_{C_j}F(p_j)(\eta_P)^{-1} = \eta_{C_j}(\pi_j\alpha)(\eta_P)^{-1} = (\eta_{C_j}\pi_j)\alpha(\eta_P)^{-1} =$$

$$(\rho_j\gamma)\alpha(\eta_P)^{-1} = \rho_j(\gamma\alpha(\eta_P)^{-1}),$$

as desired. The proof is complete. □

Lemma 2.6. *(a) Let $\underline{C}$ and $\underline{D}$ each be categories with finite products, let $F : \underline{C} \to \underline{D}$ be a CHR-additive functor, and let C be a terminal object of $\underline{C}$. Then $F(C)$ is a terminal object of $\underline{D}$.*

(b) Let $\underline{C}$ and $\underline{D}$ be pre-additive categories and let $F : \underline{C} \to \underline{D}$ be an additive functor. Then F sends any zero morphism to a zero morphism (that is, if $C_1, C_2 \in |\underline{C}|$ and f is the neutral element of the abelian group $\underline{C}(C_1, C_2)$, then Ff is the zero morphism in $\underline{D}(F(C_1), F(C_2))$). Moreover, it follows that, under these assumptions, F sends any terminal object of $\underline{C}$ to a null object of $\underline{D}$.

Proof. (a) This assertion was proved in the fourth paragraph of the Introduction.

(b) This assertion was proved in the fifth paragraph of the Introduction. □

It will be convenient to let $\underline{\mathcal{Z}}$ denote the category of zero rings, that is, the full subcategory of $\underline{C}_{\mathbb{Z}}$ whose class of objects is the collection of zero rings.

We next present our first main result.

Corollary 2.7. *Let R be a (commutative unital) ring. Let $\underline{\mathcal{Z}}$ denote the category of zero rings. Then:*

(a) The following conditions are equivalent:

(1) R is a zero ring;

(2) $\underline{C}_R = \underline{\mathcal{Z}}$;

(3) $\underline{C}_R$ is a pre-additive category;

(4) $\underline{C}_R$ is an additive category;

(5) $\underline{C}_R$ is an abelian category;

*(6) $\underline{C}_R = {}_R$*Mod.

(b) Assume, moreover, that R is a zero ring. Then the following assertions, (i)-(iii), are valid:

(i) Let $\underline{D}$ be a pre-additive category with finite products. (For instance, $\underline{D}$ could be $\underline{C}_R$ or Ab.*) Let $F : \underline{C}_R \to \underline{D}$ be a functor. Then F is an additive functor if and only if F is a CHR-additive functor.*

(ii) Every hom-set $\underline{C}_R(S, T)$ in $\underline{C}_R$ is a singleton set, and every morphism in $\underline{C}_R$ is an isomorphism.

(iii) Let $\underline{D}$ be a pre-additive category (resp., let $\underline{D}$ be a category with finite products). Let $F : \underline{C}_R \to \underline{D}$ be an additive functor (resp., a CHR-additive functor). Let $S, T \in |\underline{C}_R|$ and let $f_{S,T}$ denote the unique morphism $S \to T$ in $\underline{C}_R$. Then $F(S)$ and $F(T)$ are each null objects (resp., terminal objects) of $\underline{D}$. Moreover $F(S) \cong F(T)$ in $\underline{D}$, and $F(f_{S,T}) : F(S) \to F(T)$ is an isomorphism in $\underline{D}$, with inverse $F(f_{T,S})$.

Proof. (a) (1) $\Leftrightarrow$ (3): Apply Corollary 2.4.

(1) $\Rightarrow$ (2): Suppose that R is a zero ring. If $S \in |\underline{C}_R|$ and $s \in S$, then $s = 1 \cdot s = 0 \cdot s = 0$, so S is a zero ring, that is, $S \in |\underline{\mathcal{Z}}|$. It follows easily that if S and T are objects of $|\underline{C}_R|$, then the unique function $S \to T$ is an R-algebra homomorphism (and hence a ring homomorphism). Thus, every object (resp., morphism) of $\underline{C}_R$ is an object (resp., morphism) of $\underline{\mathcal{Z}}$. Now, consider any object V of $\underline{\mathcal{Z}}$. Let 0_V

(resp., 0_R) denote the (unique) element of V (resp., of R). There is exactly one way to endow the zero ring V with the structure of an R-algebra, namely, via $0_R \cdot 0_V := 0_V$. Moreover, if zero rings V and $W = \{0_W\}$ are thus endowed with R-algebra structures, it is easy to see that the unique function $h : V \to W$ (that is, the unique ring homomorphism $V \to W$) is an R-algebra homomorphism. (In detail: $0_R \cdot h(0_V) = 0_R \cdot 0_W = 0_W = h(0_V) = h(0_R \cdot 0_V)$.) Thus, every object (resp., morphism) of $\underline{\mathcal{Z}}$ is an object (resp., morphism) of $\underline{C}_R$. This concludes the proof of (2).

(2) ⇒ (1): Suppose that $\underline{C}_R = \underline{\mathcal{Z}}$. Since R is a (unital) R-algebra (and hence R is an object of $\underline{C}_R$), it follows that R is an object of $\underline{\mathcal{Z}}$ (and hence R is a zero ring).

(6) ⇒ (5): It suffices to use the fact that for all (unital but not necessarily commutative) rings Λ, the category ${}_\Lambda$Mod of (left) R-modules (and R-module homomorphisms) is an abelian category.

(5) ⇒ (4) ⇒ (3): These implications hold for arbitrary categories.

(1) ⇒ (6): Of course, any object (resp., morphism) in $\underline{C}_R$ is an object (resp., morphism) in ${}_R$Mod. Assume (1), with 0_R denoting the unique element of R. Let $M \in |{}_R\text{Mod}|$. Let 0_M denote the additive identity element of M. Since M is an (unital) R-module and 0_R is the multiplicative identity element of R, each $m \in M$ satisfies $m = 0_R \cdot m = 0_M \in M$, and so $M = \{0_M\}$. Moreover, by defining "multiplication on M" to be the unique binary operation on M, one checks easily that M is a zero ring. Thus, since (2) ⇒ (1), $M \in |\underline{C}_R|$. It remains only to show that if $M, N \in |_R \text{ Mod}|$ and $f_{M,N} : M = \{0_M\} \to N = \{0_N\}$ is an R-module homomorphism (that is, if $f_{M,N}$ is the unique function $M \to N$), then $f_{M,N}$ is an R-algebra homomorphism. This, in turn, follows since

$$f_{M,N}(0_M \cdot 0_M) = f_{M,N}(0_M) = 0_N = 0_N \cdot 0_N = f_{M,N}(0_M) \cdot f_{M,N}(0_M).$$

Although a proof of (a) is complete at this point, we next provide an alternate direct proof that (1) ⇒ (5), in order to have a self-contained proof of the equivalence of conditions (1)-(5) in (a) that would avoid any mention of condition (6).

(1) ⇒ (5): Assume that R is a zero ring. As (1) ⇔ (3), we already know that $\underline{C}_R$ is a pre-additive category. We will prove that $\underline{C}_R$ is an abelian category. This can be done by verifying that $\underline{C}_R$ satisfies the conditions in the characterization of abelian categories in [13] that was mentioned in the Introduction. We will leave the details of that kind of verification to the reader. Instead, we will next sketch how to verify that $\underline{C}_R$ satisfies the conditions in the (possibly more accessible) five-part characterization of abelian categories that can be found in [16, Definition, page 198].

• $\underline{C}_R$ is a pre-additive category: This was observed above (since R is a zero ring).
• $\underline{C}_R$ has a null object: This holds since R being a zero ring ensures that R is a null object of $\underline{C}_R$.
• $\underline{C}_R$ has binary biproducts: This follows from [16, Theorem 2, page 194], since $\underline{C}_R$ is a pre-additive category and $S \times S \cong S$ for each object S of $\underline{C}_R$ (the latter fact being an easy consequence of the fact that $\underline{C}_R(S,T)$ is a singleton set for all objects S and T of $\underline{C}_R$).
• Each morphism in $\underline{C}_R$ has a kernel and a cokernel: This holds by the following reasoning. Since $\underline{C}_R(S,T)$ is a singleton set for all objects S and T of $\underline{C}_R$, it follows from the discussion of equalizers (resp., coequalizers) on page 70 (resp., page 64) of [16] that for each/the morphism $f : S \to T$ in $\underline{C}_R$, the identity map on S (resp., the identity map on T) is an equalizer (resp., a coequalizer) of the pair consisting of f and f, and thus that identity map is a kernel (resp., a cokernel) of f.
• Each monomorphism in $\underline{C}_R$ is a kernel and each epimorphism in $\underline{C}_R$ is a cokernel: This can be shown to hold by using the facts (including the references) in the proof of the preceding bulleted item. Indeed, one can thus show that each morphism $f : S \to T$ in $\underline{C}_R$ is both a kernel and a cokernel of the unique morphism $T \to S$. This completes the proof of (a).

(b) (i) It suffices to combine (a) with both parts of Lemma 2.6. For the sake of completeness, we next provide the details.

Suppose first that F is a CHR-additive functor. Since each object of $\underline{C}_R$ is a terminal object, Lemma 2.6 (a) ensures that F sends each object of $\underline{C}_R$ to a terminal object of $\underline{D}$. Let $S, T \in |\underline{C}_R|$. Since $F(T)$ is a terminal object, $\underline{D}(F(S), F(T))$ is a singleton set. On the other hand, $\underline{C}_R(S,T)$ is also a singleton set, with unique element, say, f. To show that F is an additive functor, it suffices to prove that $F(f+f) = F(f)+F(f)$. Necessarily, $f+f = f$. Thus, $F(f+f) = F(f)$. Moreover, since $\underline{D}(F(S),F(T))$ is a singleton set, $F(f)+F(f) = F(f)$. Hence, $F(f+f) = F(f)+F(f)$, as desired.

Conversely, suppose that F is an additive functor. Since each object of $\underline{C}_R$ is a terminal object, Lemma 2.6 (b) ensures that F sends each object of $\underline{C}_R$ to a terminal object of $\underline{D}$. Moreover, if $C_1,\dots,C_n$ is a nonempty finite list (possibly with repetition) of objects of $\underline{C}_R$, then $\mathcal{T} := \prod_{i=1}^n F(C_i)$ is a product of finitely many terminal objects of $\underline{D}$, and so $\mathcal{T}$ is a terminal object of $\underline{D}$. As $F(\prod_{i=1}^n C_i)$ is also a terminal object of $\underline{D}$, there exists an isomorphism $h : F(\prod_{i=1}^n C_i) \to \mathcal{T}$ (in $\underline{D}$). Any such h must be the unique element of $\underline{D}(F(\prod_{i=1}^n C_i),\mathcal{T})$. Thus, the canonical morphism $F(\prod_{i=1}^n C_i) \to \mathcal{T}$ must be h and so is an isomorphism, whence F is a CHR-additive functor.

To accommodate readers who may have preferred the second approach to the proof of (b) (i) (which avoided using condition (6) in (a)), we will next give proofs of parts (ii) and (iii) of (b) that will avoid explicit mention of that condition (6). Readers who preferred the first approach to the proof of (b) (i) (which used condition (6) in (a)) are advised that some of the details in the following self-contained proofs of (ii) and (iii) necessarily repeat some observations from the above proof that (1) $\Rightarrow$ (6) in (a).

(ii) Let $S,T \in |\underline{C}_R|$. By the implication (1) $\Rightarrow$ (2) in (a), $\underline{C}_R = \underline{\mathcal{Z}}$, and so both S and T are zero rings. It will be convenient to let 0_S (resp., 0_T) denote the unique element of S (resp., T). It is easy to check that the unique function $f_{S,T} : S \to T$ (sending 0_S to 0_T) is an R-algebra homomorphism, and so $\underline{C}_R(S,T) = \{f_{S,T}\}$, which is a singleton set. Necessarily, the composite functions $f_{S,T}f_{T,S}$ and $f_{T,S}f_{S,T}$ are identity maps (on T and S, respectively), and so $f_{S,T}$ (the typical morphism in $\underline{C}_R$) is an isomorphism in $\underline{C}_R$ (with $f_{T,S}$ serving as its inverse).

(iii) The nature of $f_{S,T}$ was exposed in the proof of (ii). To prove the assertion that $F(S)$ and $F(T)$ are each null objects (resp., terminal objects) of $\underline{D}$, use the first assertion in (ii) to conclude that S and T are each terminal objects of $\underline{C}_R$ and then apply part (b) (resp, part (a)) of Lemma 2.6. The final assertions can be proven by combining the following facts: the proof of (ii) showed that $f_{S,T}$ is an isomorphism with inverse $f_{T,S}$, and functors preserve isomorphisms and their inverses. Note that an alternate proof that $F(S) \cong F(T)$ is available, since any two null (resp., terminal) objects of a category are isomorphic. □

We can now give a companion for Proposition 2.5.

Corollary 2.8. *Let R be a zero ring and let $\underline{D}$ a pre-additive category with finite products. Let F and G be additive functors (equivalently, CHR-additive functors) $\underline{C}_R \to \underline{D}$. Then F and G are naturally equivalent.*

Proof. The parenthetical equivalence follows from Corollary 2.7 (b) (i). By using either part of Lemma 2.6, we see that for all objects T of $\underline{C}_R$, the functors F and G each send T to (a possibly different) terminal object of $\underline{D}$, whence there is a (unique) isomorphism $\eta_T : F(T) \to G(T)$ in $\underline{D}$. Now, consider any objects T_1, T_2 of $\underline{C}_R$ and the (unique) morphism $f \in \underline{C}_R(T_1,T_2)$. Since $G(T_2)$ is a terminal object of $\underline{D}$, we have $\eta_{T_2}F(f) = G(f)\eta_{T_1}$. Hence, η is a natural transformation from F to G. As η_T is an isomorphism for each object T, it follows that η is a natural equivalence. The proof is complete. □

Note, as a consequence of Corollary 2.7 (a), that the category $\underline{\mathcal{Z}}$ of zero rings is an abelian category.

In view of the equivalence (1) $\Leftrightarrow$ (5) in Corollary 2.7 (a), one sees that, for the special case where $\underline{D}$ is an abelian category, the conclusion in Corollary 2.7 (b) (i) follows from an observation of Chase, Harrison and Rosenberg [5] that we mentioned in the penultimate paragraph of the Introduction. The conclusion obtained above in Corollary 2.7 (b) (i) (where $\underline{D}$ a pre-additive category with finite products) is a stronger result. Indeed, in the Foreword to [10] (a reprint of [9]), Freyd [10, page 21 of Foreword] has given an example of a pre-additive category with finite products which is not an abelian category. For the sake of completeness, the next result states the specifics of Freyd's example.

Example 2.9. (Freyd [10, page 21 of Foreword]) There exists a pre-additive category with finite products which is not an abelian category. One way to construct such a category $\underline{D}$ is the following. Let K be a field; let $X_1, X_2, \ldots$ be denumerably many (commuting) algebraically independent indeterminates over K; let A be the polynomial ring $K[\{X_i \mid i \geq 1\}]$; let I be the ideal of A generated by $\{X_i X_j \mid 1 \leq i \leq j\}$; let $R := A/I$; and let $\underline{D}$ be the full subcategory of ${}_R$Mod whose class of objects is the collection of finitely presented R-modules. Indeed, the (endo)morphism in $\underline{D}(R,R)$ which is given by multiplication by $X_1 + I$ does not have a kernel in $\underline{D}$.

Let R be a zero ring and let $\underline{D}$ be a pre-additive category with finite products. Perhaps because of the nature of what has been emphasized in the existing literature, much of the above material has focused on functors $\underline{C}_R \to \underline{D}$ that are additive (equivalently, CHR-additive). We next give two examples showing that for suitable such $\underline{D}$ (for instance, take $\underline{D}$ to be Ab), a functor $\underline{C}_R \to \underline{D}$ need not be additive (equivalently, need not be CHR-additive). Some readers may find the following fact to be interesting. The functor constructed in Example 2.10 (a) sends every object to an object G such that $G \times G$ is not isomorphic to G, but the functor constructed in Example 2.10 (b) sends every object to an object G such $G \times G \cong G$.

Example 2.10. (a) Let R be a zero ring. Let $\underline{D}$ be a pre-additive category with finite products such that there exists an object G of $\underline{D}$ with the property that $G \times G$ and G are not isomorphic in $\underline{D}$. (For instance, take $\underline{D} =$ Ab and take G to be any nontrivial finite abelian group.) One can obtain a functor $F_1 : \underline{C}_R \to \underline{D}$ via the following construction. For each $S \in |\underline{C}_R|$, put $F_1(S) := G$; and for each morphism f in $\underline{C}_R$, define $F_1(f)$ to be the identity map on G. Then F_1 is not an additive functor and F_1 is not a CHR-additive functor.

(b) Let R be a zero ring. Let G be an infinite abelian group such that $G \times G \cong G$ in Ab. (For instance, take G to be the direct product of $\aleph_0$ many copies of $\mathbb{Z}/2\mathbb{Z}$.) One can obtain a functor $F_2 : \underline{C}_R \to$ Ab via the following construction. For each $S \in |\underline{C}_R|$, put $F_2(S) := G$; and for each morphism f in $\underline{C}_R$, define $F_2(f)$ to be the identity map on G. Then F_2 is not an additive functor and F_2 is not a CHR-additive functor.

Proof. (a) We can use considerations of cardinality to verify the parenthetical assertion that if G is a nontrivial finite abelian group, then $G \times G$ is not isomorphic to G in Ab. Indeed, since $n := |G|$ satisfies $2 \leq n < \infty$ by hypothesis, we have $|G| = n < n^2 = |G|^2 = |G \times G|$.

Let us now return to the main assertion. It is straightforward to check that F_1 is a functor. By Corollary 2.7 (b) (i), F_1 is not an additive functor if and only if F_1 is not a CHR-additive functor. We will show directly that the functor F_1 is neither additive nor CHR-additive. Recall that in any category $\underline{K}$, if T is a terminal object of $\underline{K}$, then the product $T \times T$ exists in $\underline{K}$ and $T \times T \cong T$ in $\underline{K}$. Thus, since $G \times G$ is not isomorphic to G in $\underline{D}$ by hypothesis, we get that G is not a terminal object of $\underline{D}$. Therefore, it follows from the second assertion in part (b) (resp., from part (a)) of Lemma 2.6 that F_1 is not an additive functor (resp., that F_1 is not a CHR-additive functor). For an alternative proof in case $\underline{D} =$ Ab and G is a nontrivial finite abelian group, combine the fact that the identity map $G \to G$ is not a zero morphism in Ab with the first assertion in Lemma 2.6 (b) to conclude that F_1 is not an additive functor (and then use Corollary 2.7 (b) (i) to conclude that F_1 is not CHR-additive).

(b) Let us first address the parenthetical assertion. This can be done via considerations of cardinality. Indeed, if G is the direct product in Ab of $\aleph_0$ many copies of $\mathbb{Z}/2\mathbb{Z}$, then in Ab, $G \times G$ is the direct product of $(\aleph_0)^2$ many copies of $\mathbb{Z}/2\mathbb{Z}$. However, a standard fact about arithmetic with infinite cardinal numbers (assuming, as we do, the ZFC foundations) gives $(\aleph_0)^2 = \aleph_0$, whence $G \times G \cong G$ in Ab, as desired.

Let us now return to the main assertion. Observe that the terminal objects in Ab are the trivial (necessarily abelian) groups. Of course, G is not a trivial group (since G is infinite), and so G is not a terminal object of Ab. To complete the proof, it is now straightforward to tweak the final two sentences of the above proof of (a). □

Much of what we have said here (and much of what we will say below) has to do with the fact that if R is a zero ring, then the category $\underline{C}_R$ has the property that each of its morphisms is an isomorphism. This property can be restated, using standard terminology in category theory, as saying that under the stated conditions, $\underline{C}_R$ is a *groupoid*. (Some users restrict the "groupoid" terminology to small categories in which each morphism is an isomorphism, but I will ignore that "small" foundational issue in this comment.) Many mathematicians are somewhat familiar with such a concept, having studied the fundamental group(oid) of a (possibly path-connected) topological space as part of an introduction to homotopy in a course on algebraic topology or winding numbers in a course on complex analysis. Interested readers are invited to examine how far one go in extending our methods here so as to generalize our results on $\underline{C}_R$ when R is a zero ring (that is, our work here on the category $\underline{\mathcal{Z}}$ of all zero rings) to categorical facts about groupoids in which each hom-set is a singleton set. See Remark 2.11 for an indication of what may/should be possible along these lines.

Remark 2.11. (a) Following Corollary 2.4, we mentioned that certain strong foundational assumptions that are consistent with ZFC allow one to prove that any category is equivalent to a skeletal category. In the spirit of the above comments about groupoids, part (a) of this remark will examine how the use of the above-mentioned foundational assumptions would simplify and strengthen the work in Example 2.10.

Consider the category $\underline{C}_R$ for some zero ring R. (By Corollary 2.7 (a), this $\underline{C}_R$ is equal to the category of all zero rings.) By the above-mentioned strong foundational assumptions, $\underline{C}_R$ is equivalent to a skeletal category, say $\underline{K}$. Since each hom-set in $\underline{C}_R$ is a singleton set, the "skeletal" property (in conjunction with the fact that a categorical equivalence is a certain kind of fully faithful functor) shows that $\underline{K}$ is the simplest kind of nonempty category, namely, a category with a unique object and a unique morphism. It is straightforward to verify directly that $\underline{K}$ is a pre-additive category with finite products. In fact, $\underline{K}$ is an abelian category. Let us consider what happens when the role of $\underline{C}_R$ in Example 2.10 is played instead by the equivalent category $\underline{K}$. The earlier roles of a pre-additive category $\underline{D}$ with finite products and of Ab will not change. The reader may wish to pause reading at this point in order to consider whether the possible additive or CHR-additive nature of a functor is (un)affected by this change of functorial domains. That issue will, in effect, be handled by the discussion in (b).

Because of the simple nature of $\underline{K}$, it is clear that the functors F from $\underline{K}$ to $\underline{D}$ (resp., to Ab) are in one-to-one correspondence with the objects G of $\underline{D}$ (resp., of Ab), via the assignment sending each F to its value at the unique object of $\underline{K}$. For each object G of $\underline{D}$, let F_G denote the functor F associated to G; that is, G is the value of F_G at the unique object of $\underline{K}$. It is a straightforward (and not overly long) exercise in basic category theory to show, by using both parts of Lemma 2.6 and [16, Proposition 1, page 194] (cf. also the second paragraph of the proof of Corollary 2.7 (b) (i)), that the following holds for each functor F as above (that is, $F = F_G$ for some object G): F is an additive functor $\Leftrightarrow$ F is an CHR-additive functor $\Leftrightarrow$ G is a terminal (equivalently, a null) object (of $\underline{D}$ or of Ab, depending on the context). This result suggests a underlying reason which explains why each of F_1 and F_2 in Example 2.10 was neither additive nor CHR-additive, namely, neither of the abelian groups G in parts (a) and (b) of Example 2.10 was a null object of Ab (that is, neither of those groups G was a trivial group). The present analysis via an equivalent skeletal category suggests/reveals that the key tool that has been introduced here for such questions is Lemma 2.6, whereas the fact that exactly one of the two abelian groups G in Example 2.10 satisfies $G \times G \cong G$ is, however interesting it may have seemed, only of peripheral importance.

(b) It seems reasonable to expect that some readers would not wish to make the strong foundational assumption that led us in (a) to replace $\underline{C}_R$ (for a zero ring R) with an equivalent skeletal category. Now that (a) has indicated what may be expected, we will proceed (with the help of some of the above material, especially Corollary 2.7) to show that those expectations are realized even if we use only the usual ZFC foundations. Once again, let R be a zero ring and let $\underline{D}$ be a pre-additive category with finite products. We will characterize when a functor from $\underline{C}_R \to \underline{D}$ is additive (resp., CHR-additive). As above, if $S, T \in |\underline{C}_R|$ (that is, if S and T are zero rings), we let $f_{S,T}$ denote the unique function (equivalently, the unique R-algebra homomorphism; equivalently, the isomorphism in $\underline{C}_R$) $S \to T$.

"Having" a functor $F : \underline{C}_R \to \underline{D}$ (or "letting" F be such a functor) is equivalent to having the following four items:

(i) a nonempty class $\mathcal{D}$ of pairwise isomorphic objects of $\underline{D}$;

(ii) for each (ordered) pair of objects $D_1, D_2 \in \mathcal{D}$, a singleton set $\{h_{D_1,D_2}\} \subseteq \underline{D}(D_1, D_2)$, such that

(ii)$_1$: for all $D_3, D_4, D_5, D_6 \in \mathcal{D}$, we have $h_{D_4,D_5} h_{D_3,D_4} = h_{D_3,D_5}$ and h_{D_6,D_6} is the identity map on D_6;

(iii) an assignment $S \mapsto F(S)$ sending each object S of $\underline{C}_R$ to some element $F(S) \in \mathcal{D}$, such that

(iii)$_1$: $\mathcal{D}$ is the collection of all objects of $\underline{D}$ that are of the form $F(S)$ for some object S of $\underline{C}_R$; and

(iv) for each (ordered) pair of objects S, T of $\underline{C}_R$, a function $\underline{C}_R(S,T) \to \underline{D}(F(S), F(T))$ sending the morphism $f_{S,T}$ to $h_{F(S),F(T)}$.

Indeed, the "object assignment" aspect of any functor F of the kind being considered must satisfy (i) and (iii), since a functor must preserve isomorphisms; and (ii)$_1$ and (iv) are required so that F behaves functorially on morphisms (that is, so that F behaves "homomorphically" on composites of morphisms and sends identity maps to the appropriate identity maps). Note that while the particular functors F_1 and F_2 that were constructed in Example 2.10 necessarily satisfied (i)-(iv), their construction was as simple as possible, in the sense that the corresponding sets playing the role of $\mathcal{D}$ were chosen to be singleton sets. The above (more complicated) characterization of relevant functors F in terms of (i)-(iv) will be needed in the next paragraph where we will characterize when an arbitrary such functor is additive (resp., CHR-additive). As a pedagogic aside, the explicitness of (i)-(iv) also serves as a reminder that the construction of a functor requires more than simply stipulating the associated object assignment. (Along those lines, one may examine the route that I took in proving the first significant result in my doctoral research, [7, Chapter I, Theorem 3.10]; we will have reason to mention [7] again in Remark 2.20.)

We can now state the desired result. Let R be a zero ring, let $\underline{D}$ be a pre-additive category with finite products, and let $F : \underline{C}_R \to \underline{D}$ be a functor. Let $\mathcal{D}$ be the collection of all objects of $\underline{D}$ that are of the form $F(S)$ for some object S of $\underline{C}_R$. Then the following five conditions are equivalent:

(1) F is an additive functor;

(2) F is a CHR-additive additive functor;

(3) Some (equivalently, every) element of $\mathcal{D}$ is a terminal object of $\underline{D}$;

(4) Some (equivalently, every) element of $\mathcal{D}$ is an initial object of $\underline{D}$;

(5) Some (equivalently, every) element of $\mathcal{D}$ is a null object of $\underline{D}$.

Let us very briefly sketch a proof of the result. Since $\underline{D}$ is a pre-additive category, it follows from [16, Proposition 1, page 194] that an object D of $\underline{D}$ is a terminal object of $\underline{D}$ if and only if D is an initial object of $\underline{D}$, if and only if D is a null object of $\underline{D}$. It follows that (3), (4) and (5) are equivalent. In view of the above methods and accumulated information (especially Corollary 2.7), we can give almost the same hint for a proof of the equivalence of (1), (2), (3) that was given for the proof of the analogous equivalences in (a): use both parts of Lemma 2.6, [16, Proposition 1, page 194], and the second and third paragraphs of the proof of Corollary 2.7 (b) (i). This completes the remark.

The constructions in Example 2.10, which gave functors F_1 and F_2 that were neither additive nor CHR-additive, had the property that F_1 and F_2 each sent all objects to an object that was not a terminal object. Nevertheless, one could use Corollary 2.7 (b) (i) to recover the fact that if R is a zero ring, then the restrictions of the unit functor and the Picard group functor each give a CHR-additive functor $\underline{C}_R \to$ Ab, since one can see directly and easily that each of these restrictions is an additive functor. For such a direct analysis, the underlying fact is that the unit group of a zero ring and the Picard group of a zero ring are each trivial groups. We will say more about "zero-ish" matters in Remark 2.19. In closing, Remark 2.20 will recall some facts about some variants/applications of

CHR-additive functors from some of our early work and point the way to possible future work at the interface of commutative algebra and algebraic geometry.

Before proceeding to the two above-mentioned remarks, we will give our second main result, Theorem 2.18, which addresses the more general categorical question that is suggested by the results in Corollary 2.7 (b) (i) and Remark 2.11 (b). Theorem 2.18 not only generalizes those earlier results, but it also serves to validate the insight of Chase, Harrison and Rosenberg in [5] that [9, Theorem 3.11] is relevant to finding contexts for which the concepts of “additive functor" and (what we have called) “CHR-additive functor" are equivalent.

The next sentence pertains to some terminology used in Lemma 2.12 (and later). We will use 1_E to denote the identity map $E \to E$ on an object E of a given category; and a “zero morphism," usually denoted by 0, in a pre-additive category $\underline{E}$ will refer to the neutral element (with respect to addition) in some hom-set of $\underline{E}$. Also, in a pre-additive category $\underline{E}$ with a null object N, the zero morphism in a hom-set $\underline{E}(E_1, E_2)$ is the same as a morphism $E_1 \to E_2$ that factors through N. (A proof of this fact can be found easily by using Lemma 2.1 (a).)

Lemma 2.12. *Let $F : \underline{C} \to \underline{D}$ be a functor, where $\underline{C}$ and $\underline{D}$ are each a pre-additive category with a null object. Then the following conditions are equivalent:*

(1) *F sends each zero morphism of $\underline{C}$ to a zero morphism of $\underline{D}$;*

(2) *F sends some null object of $\underline{C}$ to a null object of $\underline{D}$;*

(3) *F sends each null object of $\underline{C}$ to a null object of $\underline{D}$.*

Proof. (3) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (3): This equivalence can be proven by applying the following three facts in order. Any two null objects of a given category are isomorphic; functors preserve isomorphisms; and null objects are preserved by isomorphisms.

(1) $\Rightarrow$ (3): Assume (1). Let N be a null object of $\underline{C}$. Then 1_N is a zero morphism, by [16, Proposition 1, page 194]. Thus, by (1), $1_{F(N)} = F(1_N)$ is a zero morphism. Hence, by [16, Proposition 1, page 194], $F(N)$ is a null object.

(3) $\Rightarrow$ (1): Assume (3). Let $f = 0 \in \underline{C}(C_1, C_2)$. Our task is to show that $g := F(f) \in \underline{D}(F(C_1), F(C_2))$ satisfies $g = 0$. By the above comments, $f = f_2 f_1$ where $f_1 = 0 \in \underline{C}(C_1, N)$ and $f_2 = 0 \in \underline{C}(N, C_2)$ for some null object N of $\underline{C}$. As (3) ensures that $F(N)$ is a null object of $\underline{D}$, it follows that $1_{F(N)} = 0$ by [16, Proposition 1, page 194], and so $F(f_2) = F(f_2)1_{F(N)} = F(f_2)0 = 0$ by Lemma 2.1 (a). Therefore, $g = F(f_2)F(f_1) = 0F(f_1) = 0$ by Lemma 2.1 (a). The proof is complete. □

Much in the results 2.15-2.18 will concern when the “canonical morphisms" $\alpha : F(A \times B) \to F(A) \times F(B)$ and $\beta : F(A) \oplus F(B) \to F(A \oplus B)$ are isomorphisms. Since (direct) product and (direct) sum are defined only up to isomorphism (even when they exist), one may well ask the apparently more basic question of whether α and β are actually well-defined morphisms. A specialist in category theory would perhaps reply, for good reason, "Yes, up to isomorphism." But such a reply may worry some readers who wish to avoid the conceptual problems associated with the 19$^{\text{th}}$ century's studies (before the foundations of the theory of Riemann surfaces were rigorously developed) of the supposed domain and range of a so-called “many-valued function." Propositions 2.13 and 2.14 will carefully identify and then resolve some of the underlying issues. Readers who are familiar with the definitions of Amitsur cohomology in a functor or of Cech cohomology in a presheaf (or perhaps only in a sheaf) have already dealt with similar issues. After all, the definitions of those cohomology groups use the definitions of some underlying cochain complexes, and the latter definitions assume that it is meaningful to apply a functor to certain tensor products or to apply a presheaf to certain fiber (co)products, even though those tensor products or fiber products are only defined up to isomorphism. Such readers who have already made their peace with that aspect of the literature (perhaps by emulating the spirit of [16, pages 195-196]) will likely not be surprised by the proofs of Propositions 2.13 and 2.14.

Proposition 2.13 will prove in detail that "α is an isomorphism" is a well-defined property (even though α is only "defined up to isomorphism"). The proof of Proposition 2.14, while being largely left to the reader, does give hints for ways to prove that "β is an isomorphism" is a well-defined property (even though β is only "defined up to isomorphism"). Our approach to proving Propositions 2.13 and 2.14 will not take the somewhat draconian step of replacing each relevant category with an equivalent skeletal category; nor will we pretend that it would suffice to merely point out that two objects of a certain comma category are isomorphic. Instead, in Proposition 2.13, we will assume only that we are given a functor $F : \underline{C} \to \underline{D}$, along with (possibly equal) objects A and B of $\underline{C}$, such that a product of A and B exists in $\underline{C}$ and a product of $F(A)$ and $F(B)$ exists in $\underline{D}$; we will show that, under these assumptions, various versions of α are certainly well defined and that, if one of those versions of α is an isomorphism, then all the other versions of α are also isomorphisms. In short, Proposition 2.13 gives a precise sense in which "α is an isomorphism" is a well defined property and shows that this property holds under what are arguably the most general conditions for which such a study should be pursued. One can say that Proposition 2.14 will essentially do for β (resp., sums) what Proposition 2.13 will have done for α (resp., products). Indeed, Proposition 2.14 gives the corresponding conclusions about the various analogous versions of β (assuming the existence of the appropriate sums, $A \oplus B$ in $\underline{C}$ and $F(A) \oplus F(B)$ in $\underline{D}$).

Proposition 2.13. *Let $\underline{C}$ be a category, and let A and B be (possibly isomorphic) objects of $\underline{C}$. Let P_1 be a product of A and B in $\underline{C}$, with projection morphisms $p_1 : P_1 \to A$ and $p_2 : P_1 \to B$. Let P_2 be a (possibly different) product of A and B in $\underline{C}$, with projection morphisms $p_1^* : P_2 \to A$ and $p_2^* : P_2 \to B$. Let $\theta : P_1 \to P_2$ be the unique morphism (actually, an isomorphism, given by the universal mapping property of the product P_2) such that $p_1^*\theta = p_1$ and $p_2^*\theta = p_2$. Let $\underline{D}$ be a category and let $F : \underline{C} \to \underline{D}$ be a functor. Let Q_1 be a product of $F(A)$ and $F(B)$ in $\underline{D}$, with projection morphisms $\pi_1 : Q_1 \to F(A)$ and $\pi_2 : Q_1 \to F(B)$. Let Q_2 be a (possibly different) product of $F(A)$ and $F(B)$ in $\underline{D}$, with projection morphisms $\pi_1^* : Q_2 \to F(A)$ and $\pi_2^* : Q_2 \to F(B)$. Let $\psi : Q_1 \to Q_2$ be the unique morphism (actually, an isomorphism, given by the universal mapping property of the product Q_2) such that $\pi_1^*\psi = \pi_1$ and $\pi_2^*\psi = \pi_2$. Let $\alpha_1 : F(P_1) \to Q_1$ be the (uniquely determined, since Q_1 is a product) canonical morphism such that $\pi_1\alpha_1 = F(p_1)$ and $\pi_2\alpha_1 = F(p_2)$. Let $\alpha_2 : F(P_2) \to Q_2$ be the (uniquely determined, since Q_2 is a product) canonical morphism such that $\pi_1^*\alpha_2 = F(p_1^*)$ and $\pi_2^*\alpha_2 = F(p_2^*)$. Then:*

(a) $\alpha_2 F(\theta) = \psi\alpha_1$.

(b) *α_1 is an isomorphism (in $\underline{D}$) if and only if α_2 is an isomorphism (in $\underline{D}$).*

Proof. Of course, θ and ψ are isomorphisms because in any category, any two objects with the same universal mapping property are isomorphic (cf. [14, proof of Theorem 7.3, page 54; also page 57]). As functors preserve isomorphisms, $F(\theta)$ is also an isomorphism. This fact will be used in the proof of (b). Also, a piece of the above information can be rewritten as $\pi_k^* = \pi_k\psi^{-1}$ for $k \in \{1,2\}$; this fact will be used in the proof of (a).

(a) Since ψ is an isomorphism, our task can be rephrased as the requirement to prove that $\psi^{-1}\alpha_2 F(\theta) = \alpha_1$. Thus, in view of the above characterization of α_1, our task can be rephrased as the requirement to prove that

$$\pi_1\psi^{-1}\alpha_2 F(\theta) = F(p_1) \text{ and } \pi_2\psi^{-1}\alpha_2 F(\theta) = F(p_2).$$

To accomplish this task, note that for $k \in \{1,2\}$, we have

$$F(p_k) = F(p_k^*\theta) = F(p_k^*)F(\theta) = \pi_k^*\alpha_2 F(\theta) = \pi_k\psi^{-1}\alpha_2 F(\theta).$$

(b) It will be enough to assume that α_1 is an isomorphism and then prove that α_2 is an isomorphism. Recall that $F(\theta)$ and ψ are isomorphisms. Thus, by (a), $\alpha_2 = \psi\alpha_1(F(\theta))^{-1}$, which is a composition of isomorphisms, and so α_2 is an isomorphism. □

Proposition 2.14. *Let $\underline{C}$ be a category, and let A and B be (possibly isomorphic) objects of $\underline{C}$. Let S_1 be a (direct) sum of A and B in $\underline{C}$, with injection morphisms $i_1 : A \to S_1$ and $i_2 : B \to S_1$. Let S_2 be a (possibly different) sum of A and B in $\underline{C}$, with injection morphisms $i_1^* : A \to S_2$ and $i_2^* : B \to S_2$. Let $\varphi : S_1 \to S_2$ be the unique morphism (actually, an isomorphism, given by the universal mapping property of the sum S_1) such that $\varphi i_1 = i_1^*$ and $\varphi i_2 = i_2^*$. Let $\underline{D}$ be a category and let $F : \underline{C} \to \underline{D}$ be a functor. Let T_1 be a sum of $F(A)$ and $F(B)$ in $\underline{D}$, with injection morphisms $j_1 : F(A) \to T_1$ and $j_2 : F(B) \to T_1$. Let T_2 be a (possibly different) sum of $F(A)$ and $F(B)$ in $\underline{D}$, with injection morphisms $j_1^* : F(A) \to T_2$ and $j_2^* : F(B) \to T_2$. Let $\Psi : T_1 \to T_2$ be the unique morphism (actually, an isomorphism, given by the universal mapping property of the sum T_1) such that $\Psi j_1 = j_1^*$ and $\Psi j_2 = j_2^*$. Let $\beta_1 : T_1 \to F(S_1)$ be the (uniquely determined, since T_1 is a sum) canonical morphism such that $\beta_1 j_1 = F(i_1)$ and $\beta_1 j_2 = F(i_2)$. Let $\beta_2 : T_2 \to F(S_2)$ be the (uniquely determined, since T_2 is a sum) canonical morphism such that $\beta_2 j_1^* = F(i_1^*)$ and $\beta_2 j_2^* = F(i_2^*)$. Then:*

(*a*) $\beta_2 \Psi = F(\varphi)\beta_1$.

(*b*) *β_1 is an isomorphism (in $\underline{D}$) if and only if β_2 is an isomorphism (in $\underline{D}$).*

Proof. We leave to the reader the details involved in intuitively "dualizing" the proof of Proposition 2.13. Readers seeking a more rigorous approach to such "dualizing" arguments are encouraged to skip ahead to Remark 2.15 (a), to familiarize themselves with the one-to-one correspondence of functors $F \leftrightarrow \mathcal{F}$ (with $F : \underline{C} \to \underline{D}$ and $\mathcal{F} : \underline{C}^{\mathrm{op}} \to \underline{D}^{\mathrm{op}}$) which is established there, and then to use that correspondence to fashion an alternate proof of Proposition 2.14. The details of that alternate proof are also left to the reader. □

Some of the considerations in Theorem 2.18 will require us to go a step further than what was done in the preceding two results. In view of the statements of Propositions 2.13 and 2.14, one can ask whether, when given objects $A_1, \dots, A_n$ (possibly listed with repetition) of a pre-additive category $\underline{C}$ with finite products and a functor $F : \underline{C} \to \underline{D}$, where $\underline{D}$ is also a pre-additive category with finite products, one can say, for the "canonical morphisms" $\alpha : F(\prod_{i=1}^n A_i) \to \prod_{i=1}^n F(A_i)$ and $\beta : \coprod_{i=1}^n F(A_i) \to F(\coprod_{i=1}^n A_i)$, that "$\alpha$ is an isomorphism" and "β is an isomorphism" are well defined properties. (Notice that this question about α is basically asking whether the concept of a CHR-additive functor is well defined.) The answer(s) is/are in the affirmative, and we expect that the interested reader will be able to obtain this/these answer(s) by building on the proofs of Propositions 2.13 and 2.14. If necessary, some readers may wish to review the general associative laws (up to natural isomorphisms) for finite (direct) products and finite coproducts in such categories: in this regard, see Remark 2.15 (a) and the third paragraph of Remark 2.16.

In the spirit of the characterization of additive functors F of abelian categories given in [9, Theorem 3.11], one of the characterizations given in Theorem 2.18 will be that F carries biproduct diagrams to biproduct diagrams. Remark 2.15 (a) will recall the definition of a biproduct diagram (for the more general context of Theorem 2.18) and some of its useful consequences. Parts (b) and (c) of Remark 2.15 will develop a useful categorical technique that involves dual categories and will then examine some of its applications, especially to pre-additive categories with finite products.

Remark 2.15. (a) Let $\underline{C}$ and $\underline{D}$ be (possibly isomorphic) pre-additive categories with finite products. Let A and B be (possibly isomorphic) objects of $\underline{C}$. Fix a product P of A and B (in $\underline{C}$). By [16, Theorem 2, page 194] (and its proof), the existence of the product P leads to a biproduct diagram, say $\mathcal{D}$, for A and B in $\underline{C}$. According to the definition of a biproduct diagram (see [16, page 194]), $\mathcal{D}$ consists of injection morphisms, $i_1 : A \to P$ and $i_2 : B \to P$, and projection morphisms, $p_1 : P \to A$ and $p_2 : P \to B$, such that $p_1 i_1 = 1_A$, $p_2 i_2 = 1_B$ and $i_1 p_1 + i_2 p_2 = 1_P$ (and, necessarily, $p_1 i_2 = 0$ and $p_2 i_1 = 0$, as in [16, page 195, lines 3-4]). It follows that the maps i_1 and i_2 determine P as a sum (that is, as a coproduct) of A and B (in $\underline{C}$) while the maps p_1 and p_2 determine P as a product of A and B (in $\underline{C}$).

It will be interesting and useful to ask whether applying F to the data in $\mathcal{D}$ produces a biproduct diagram in $\underline{D}$. By definition, an affirmative answer would mean that the object $\mathcal{P} := F(P)$ of $\underline{D}$ has injection morphisms, $j_1 := F(i_1) : F(A) \to \mathcal{P}$ and $j_2 := F(i_2) : F(B) \to \mathcal{P}$, and projection morphisms, $\pi_1 := F(p_1) : \mathcal{P} \to F(A)$ and $\pi_2 := F(p_2) : \mathcal{P} \to F(B)$, such that $\pi_1 j_1 = 1_{F(A)}$, $\pi_2 j_2 = 1_{F(B)}$ and $j_1\pi_1 + j_2\pi_2 = 1_{\mathcal{P}}$ (and, necessarily, $\pi_1 j_2 = 0$ and $\pi_2 j_1 = 0$). Then $\mathcal{P}$ would be a sum of $F(A)$ and $F(B)$ (in $\underline{D}$) determined by the injection morphisms j_1 and j_2, while $\mathcal{P}$ would be a product of $F(A)$ and $F(B)$ (in $\underline{D}$) determined by the projection morphisms π_1 and π_2.

A nontrivial consequence of the reasoning two paragraphs ago is that if $\underline{C}$, A and B are as above (not necessarily such that F carries $\mathcal{D}$ to a biproduct diagram), then any product of A and B in $\underline{C}$ is a sum of A and B in $\underline{C}$. Combining this observation with Lemma 2.1 (b), we get (for $\underline{C}$, A and B as above) that any sum of A and B in $\underline{C}$ is a product of A and B in $\underline{C}$. It is not difficult to conclude therefrom that a pre-additive category with finite products is the same as a pre-additive category with finite coproducts. These basic facts will be used in Theorem 2.18. While the just-mentioned facts in this paragraph and the facts in the preceding two paragraphs are true, we believe that the above exposition of them here has been incomplete, owing to what we consider to be some incomplete or vague passages in [16]. (For the same reason, one should also comment further on the use of the proof of [16, Theorem 2, page 195] which will be implicitly appealed to later in part (c) of the present remark in order to have certain biproduct diagrams.) Remark 2.16 will, in our opinion, provide enough details to rectify matters.

(b) This paragraph will develop the following result: for *any* categories $\underline{K}_1$ and $\underline{K}_2$, there is a natural one-to-one correspondence between the (covariant) functors $F : \underline{K}_1 \to \underline{K}_2$ and the (covariant) functors $\mathcal{F} : \underline{K}_1^{\text{op}} \to \underline{K}_2^{\text{op}}$. (Of course, the notation $\underline{K}_i^{\text{op}}$ means $(\underline{K}_i)^{\text{op}}$.) To see this, let us begin by observing that any F as above induces a (covariant) functor $\mathcal{F} : \underline{K}_1^{\text{op}} \to \underline{K}_2^{\text{op}}$, where the object assignment of $\mathcal{F}$ is the same as the object assignment of F, while $\mathcal{F}$ is defined on morphisms by $\mathcal{F}(f^{\text{op}}) := (F(f))^{\text{op}}$. Noting that any category $\underline{K}$ satisfies $(\underline{K}^{\text{op}})^{\text{op}} = \underline{K}$, one checks easily that if a functor F induces $\mathcal{F} : \underline{K}_1^{\text{op}} \to \underline{K}_2^{\text{op}}$ as above, then the functor that $\mathcal{F}$ induces from $\underline{K}_1$ to $\underline{K}_2$ is F itself. Similarly, if one uses a functor $\mathcal{G} : \underline{K}_1^{\text{op}} \to \underline{K}_2^{\text{op}}$ to induce a functor $G : \underline{K}_1 \to \underline{K}_2$, then one checks easily that the functor which G induces from $\underline{K}_1^{\text{op}}$ to $\underline{K}_2^{\text{op}}$ is $\mathcal{G}$ itself. This completes (a sketch of) the proof of the above-asserted one-to-one correspondence between the functors $\underline{K}_1 \to \underline{K}_2$ and the functors $\underline{K}_1^{\text{op}} \to \underline{K}_2^{\text{op}}$. (We have not used notation such as "$\mathcal{F} = F^{\text{op}}$", nor will we do so, for the following reason. A number of speakers and authors have seen fit to convert a (possibly naturally occurring) "contravariant functor" $H : \underline{K}_1 \to \underline{K}_2$ to a (covariant) functor, which they have denoted by H^{op}, either from $\underline{K}_1^{\text{op}}$ to $\underline{K}_2$ or from $\underline{K}_1$ to $\underline{K}_2^{\text{op}}$. That sort of construction should not be confused with the above assignment $F \mapsto \mathcal{F}$ that induced the above one-to-one correspondence, as our construction of $\mathcal{F}$ in terms of F involved dualizing *both* the domain of F *and* the codomain of F. Apart from this parenthetical aside, all the functors considered in this paper are assumed to be covariant.)

This paragraph collects some material that can be useful in applying the preceding paragraph to pre-additive categories. First, recall from the proof of Lemma 2.1 (b) that the definition of addition of morphisms in the dual of a pre-additive category, which was essentially given by

$$\lambda^{\text{op}} + \mu^{\text{op}} := (\lambda + \mu)^{\text{op}},$$

led to the fact that a category $\underline{K}^{\text{op}}$ is a pre-additive category if (and only if) $\underline{K}$ is a pre-additive category. The next two observations will be useful for the context of Theorem 2.18, that is, whenever when $\underline{C}$ and $\underline{D}$ are each pre-additive categories with finite products and $F : \underline{C} \to \underline{D}$ is a functor. Let $\mathcal{F} : \underline{C}^{\text{op}} \to \underline{D}^{\text{op}}$ be the functor induced by F via the construction in the preceding paragraph. Loosely stated, here is the next observation: if F satisfies condition (2) in the statement of Theorem 2.18, then so does $\mathcal{F}$. More precisely put: if F carries each biproduct diagram in $\underline{C}$ to a biproduct diagram in $\underline{D}$, then $\mathcal{F}$ carries each biproduct diagram in $\underline{C}^{\text{op}}$ to a biproduct diagram in $\underline{D}^{\text{op}}$. (For a proof, combine the following three items: the definition of a biproduct diagram in [16, Definition, page 194], as recalled in (a) above; the equivalence of (1) and (2) in Theorem 2.18 (taken directly from [16, Proposition 4, page 197]); and the fact (which is a consequence of the second sentence in this paragraph) that additive functors preserve each of the three identities appearing in the definition of a biproduct diagram.) Here is another useful observation: it is easy to see that F sends each zero morphism in $\underline{C}$ to a zero morphism if and only if $\mathcal{F}$ sends each zero morphism in $\underline{C}^{\text{op}}$ to a zero morphism. (In view of the definition of $\mathcal{F}$ on morphisms, the following elementary categorical observation provides the appropriate detail to prove the preceding "easy" comment. It follows from Lemma 2.1 (b) and the definition of addition of morphisms in the dual of a pre-additive category $\underline{E}$ that if μ is the neutral element in a hom-set (abelian group) $\underline{E}(E_1, E_2)$ (that is, if μ is the zero morphism from E_1 to E_2 in $\underline{E}$), then μ^{op} is the zero morphism from E_2 to E_1 in $\underline{E}^{\text{op}}$.)

(c) Let $\underline{C}$ and $\underline{D}$ be (possibly isomorphic) pre-additive categories with finite products, let $H : \underline{C} \to \underline{D}$ be a functor, and fix (possibly isomorphic) objects A and B of $\underline{C}$. It will often be convenient to replace the previously used notations α and β with α_H and β_H, respectively. Thus, for fixed A and B as above, α_H denotes the canonical morphism $H(A \times B) \to H(A) \times H(B)$ in $\underline{D}$ and β_H denotes the canonical morphism $H(A) \oplus H(B) \to H(A \oplus B)$ in $\underline{D}$. If $F \leftrightarrow \mathcal{F}$ is the one-to-one correspondence constructed in (b) and if $A, B \in |\underline{C}|$, then $\alpha_F = (\beta_{\mathcal{F}})^{\text{op}}$ and (so, by replacing F with $\mathcal{F}$, which is permissible in view of Lemma 2.1 (b), we also get that) $\beta_F = (\alpha_{\mathcal{F}})^{\text{op}}$.

For a proof, we begin by fixing a product P of A and B in $\underline{C}$. This leads to a biproduct diagram, say $\mathcal{D}$, for A and B in $\underline{C}$. By definition, $\mathcal{D}$ consists of injection morphisms, $i_1 : A \to P$ and $i_2 : B \to P$, and projection morphisms, $p_1 : P \to A$ and $p_2 : P \to B$, such that $p_1 i_1 = 1_A$, $p_2 i_2 = 1_B$ and $i_1 p_1 + i_2 p_2 = 1_P$ (and, necessarily, $p_1 i_2 = 0$ and $p_2 i_1 = 0$). It follows that i_1 and i_2 determine P as a coproduct of A and B (in $\underline{C}$) while p_1 and p_2 determine P as a product of A and B (in $\underline{C}$). Next, fix a product $\mathcal{P}$ of $F(A)$ and $F(B)$ in $\underline{D}$. As above, an ensuing biproduct diagram (this time, in $\underline{D}$) features injection morphisms, $j_1 : F(A) \to \mathcal{P}$ and $j_2 : F(B) \to \mathcal{P}$, and projection morphisms, $\pi_1 : \mathcal{P} \to F(A)$ and $\pi_2 : \mathcal{P} \to F(B)$, such that $\pi_1 j_1 = 1_{F(A)}$, $\pi_2 j_2 = 1_{F(B)}$ and $j_1 \pi_1 + j_2 \pi_2 = 1_{\mathcal{P}}$ (and, necessarily, $\pi_1 j_2 = 0$ and $\pi_2 j_1 = 0$). Then $\mathcal{P}$ is a coproduct of $F(A)$ and $F(B)$ (in $\underline{D}$) determined by the injection morphisms j_1 and j_2, while $\mathcal{P}$ is a product of $F(A)$ and $F(B)$ (in $\underline{D}$) determined by the projection morphisms π_1 and π_2.

Let us repeat the above reasoning, this time focusing on the functor $\mathcal{F}$ (rather than on F) and on A and B as objects of $\underline{C}^{\text{op}}$ (rather than $\underline{C}$). The upshot is that the injection morphisms, $(i_1)^* := (p_1)^{\text{op}} : A \to P$ and $(i_2)^* := (p_2)^{\text{op}} : B \to P$ in $\underline{C}^{\text{op}}$, determine P as a *coproduct* of A and B in $\underline{C}^{\text{op}}$, while the injection morphisms, $(j_1)^* := (\pi_1)^{\text{op}} : \mathcal{F}(A) = F(A) \to \mathcal{P}$ and $(j_2)^* := (\pi_2)^{\text{op}} : \mathcal{F}(B) = F(B) \to \mathcal{P}$ in $\underline{D}^{\text{op}}$, determine $\mathcal{P}$ as a *coproduct* of $\mathcal{F}(A)$ and $\mathcal{F}(B)$ in $\underline{D}^{\text{op}}$.

It remains to prove that $\alpha_F = (\beta_{\mathcal{F}})^{\text{op}}$. It follows from the universal mapping properties of product and coproduct that α_F is (uniquely) determined by the conditions $\pi_1 \alpha_F = F(p_1)$ and $\pi_2 \alpha_F = F(p_2)$; and that $\beta_{\mathcal{F}}$ is determined by the conditions $\beta_{\mathcal{F}}(j_1)^* = \mathcal{F}((i_1)^*)$ and $\beta_{\mathcal{F}}(j_2)^* = \mathcal{F}((i_2)^*)$. It suffices to prove that $(\beta_{\mathcal{F}})^{\text{op}}$ has the just-mentioned properties which determine α_F. In other words, it suffices to prove that

$$\pi_1(\beta_{\mathcal{F}})^{\text{op}} = F(p_1) \text{ and } \pi_2(\beta_{\mathcal{F}})^{\text{op}} = F(p_2).$$

As the proofs of the two just-displayed equations are similar, we will give the first of those proofs next, while leaving the proof of the second equation to the reader. By applying the $^{\text{op}}$ operator, we see that our task is equivalent to proving that

$$(\pi_1(\beta_{\mathcal{F}})^{\text{op}})^{\text{op}} = (F(p_1))^{\text{op}}.$$

Accordingly, the proof concludes via the following calculation:

$$(\pi_1(\beta_{\mathcal{F}})^{\text{op}})^{\text{op}} = \beta_F(\pi_1)^{\text{op}} = \beta_F(j_1)^* = \mathcal{F}((i_1)^*) =$$

$\mathcal{F}((p_1)^{\text{op}}) := (F(p_1))^{\text{op}}$.

We are optimistic that additional uses of the one-to-one correspondence $F \leftrightarrow \mathcal{F}$ will be noticed and become popular in presentations of a variety of topics in category theory. This completes the remark.

The next remark fulfills the expository purposes that were mentioned in the third paragraph of Remark 2.15 (a).

Remark 2.16. Let A and B be objects of a pre-additive category $\underline{E}$. At first glance, the proof of [16, Theorem 2, pages 194-195] would seem to prove the following two things: the first sentence of the statement of that result, namely, that a (direct) product $A \prod B$ (also denoted by $A \times B$) exists in $\underline{E}$ if and only if A and B have a biproduct diagram (in the sense defined in [16, Theorem 2, page 194] and recalled in Remark 2.15 (a)) in $\underline{E}$; and the first part of the second sentence in the statement of [16, Theorem 2, page 194], which essentially explains how to extract a product of A and B from a biproduct diagram associated to A and B. The statement of [16, Theorem 2, page 194] asserts more (and the proof of Theorem 2.18 and some argumentation leading to that proof will need it), namely, for A, B and $\underline{E}$ as above, the following two things: the second part of the second sentence in the statement of [16, Theorem 2, page 194], which essentially explains how to extract a coproduct of A and B from a biproduct diagram associated to A and B; and the assertion that $A \prod B$ exists in $\underline{E}$ if and only if a (direct sum, that is, a) coproduct $A \coprod B$ (also denoted by $A \oplus B$) exists in $\underline{E}$. (Note that the "only if" part of the preceding equivalence was given in [9, Exercise A1, page 60], although Freyd's usage ruled out the easy case where $\underline{E}$ has at most one object. In the just-mentioned exercise, Freyd went on conclude that $A \coprod B$ is isomorphic to $A \prod B$ if the latter exists (cf. also [9, Theorem 2.35], which is a result on abelian categories). That same conclusion can be drawn from the statement of [16, Theorem 2, page 194].) To be fair, these extra assertions in the statement of [16, Theorem 2, page 194] can be viewed as proven, as the statement of that result includes the word "dually" and we can see, thanks to Lemma 2.1 (b), that such usage is appropriate. (For instance, a product of A and B in the pre-additive category $\underline{E}^{\text{op}}$ would be the same as a coproduct of A and B in $\underline{E}$; and an instructive calculation using the operator $^{\text{op}}$ reveals how a biproduct diagram associated to A and B in $\underline{E}^{\text{op}}$ leads to a biproduct diagram associated to A and B in $\underline{E}$.)

We next follow up on the above comment about "Freyd's usage". As category theory was a quickly developing field in the early 1960s, it is perhaps not surprising that with the passage of time, some of the terminology that had been used in [9] has changed its meaning. We will next give two instances of such changes. (These are pertinent to the statement of some results in [9] that are related to [9, Theorem 3.11]). First, what was called an "additive category" in [9, page 60] is what would nowadays be called a "pre-additive category with finite products and a null object". (Equivalently, by [16, Proposition 1, page 194], a result whose validity we explicated above with the aid of Lemma 2.1 (b), this kind of category would nowadays be called a "pre-additive category with finite products".) Second, what was called a "pre-additive category" in [9, page 60] would now be called a "pre-additive category with a null object and greater than one object."

Lastly, we address something that was mentioned in the Introduction and is implicit in the statement of conditions (5) and (7) of Theorem 2.18, namely, the fact that a pre-additive category $\underline{K}$ with finite products necessarily also has finite coproducts. The issue of the existence of an empty coproduct (that is, an initial object) of $\underline{K}$ is settled by [16, Proposition 1, page 194], which guarantees that any empty product in $\underline{K}$ (that is, any terminal object of $\underline{K}$) is an initial object of $\underline{K}$. The issue of the existence of binary coproducts $A \oplus B$ in $\underline{K}$ was handled two paragraphs ago. Finally, for integers $n \geq 3$, the existence of coproducts $\coprod_{i=1}^{n} A_i$ in $\underline{K}$ can be discerned from the proof that (6) $\Rightarrow$ (7) in Theorem 2.18 below. This completes the remark.

The next result contains the final technical information that will be needed in the proof of Theorem 2.18. Readers seeking a shorter path to Theorem 2.18 may be interested to know that the proof of Proposition 2.17 (a) was the last proof that I completed while doing this research, as it enabled me to complete the proof that (4) $\Rightarrow$ (3) in Theorem 2.18.

To avoid possible confusion, Proposition 2.17 and Theorem 2.18 will occasionally use the following enhanced notation for zero morphisms. If E_1 and E_2 are (possibly equal) objects of a pre-additive category $\underline{E}$, then the neutral element of the abelian group $\underline{E}(E_1, E_2)$ will be denoted by $0_{E_1,E_2}$.

Proposition 2.17. *Let $\underline{C}$ and $\underline{D}$ be (possibly isomorphic) pre-additive categories with finite products, and let $F : \underline{C} \to \underline{D}$ be a functor. Let $\mathcal{F} : \underline{C}^{\mathrm{op}} \to \underline{D}^{\mathrm{op}}$ be the functor that is induced by F by using the construction in Remark 2.15 (b). Let A and B be (possibly equal) objects of $\underline{C}$. Let P be a product of A and B in $\underline{C}$ (and hence also a sum of A and B in $\underline{C}$), with projection morphisms $p_1 : P \to A$ and $p_2 : P \to B$, and also with injection morphisms $i_1 : A \to P$ and $i_2 : B \to P$, such that the set of data $\{A, B, P, p_1, p_2, i_1, i_2\}$ gives a (uniquely determined) biproduct diagram $\mathcal{D}$ in $\underline{C}$ (that is, such that $p_1 i_1 = 1_A$, $p_2 i_2 = 1_B$ and $i_1 p_1 + i_2 p_2 = 1_P$, and, necessarily, $p_1 i_2 = 0_{B,A}$ and $p_2 i_1 = 0_{A,B}$). Put $\mathcal{P} := F(P)$. Let Q be a product of $F(A)$ and $F(B)$ in $\underline{D}$ (and hence also a sum of $F(A)$ and $F(B)$ in $\underline{D}$), with projection morphisms $\pi_1 : Q \to F(A)$ and $\pi_2 : Q \to F(B)$, and also with injection morphisms $j_1 : F(A) \to Q$ and $j_2 : F(B) \to Q$, such that the set of data $\{F(A), F(B), Q, \pi_1, \pi_2, j_1, j_2\}$ gives a (uniquely determined) biproduct diagram $\mathcal{E}$ in $\underline{D}$ (that is, such that $\pi_1 j_1 = 1_{F(A)}$, $\pi_2 j_2 = 1_{F(B)}$ and $j_1\pi_1 + j_2\pi_2 = 1_Q$, and, necessarily, $\pi_1 j_2 = 0_{F(B),F(A)}$ and $\pi_2 j_1 = 0_{F(A),F(B)}$). Let α be the canonical morphism $F(A \times B) \to F(A) \times F(B)$, viewed as the morphism $\alpha : \mathcal{P} \to Q$ in $\underline{D}$ that is uniquely determined by $\pi_1\alpha = F(p_1)$ and $\pi_2\alpha = F(p_2)$. Also, consider the canonical morphism $\beta : F(A) \oplus F(B) \to F(A \oplus B)$, viewed as the morphism $\beta : Q \to \mathcal{P}$ in $\underline{D}$ that is uniquely determined by $\beta j_1 = F(i_1)$ and $\beta j_2 = F(i_2)$. Then:*

(a) Suppose that B is a null object of $\underline{C}$ and that α is an isomorphism. Then $F(0_{A,B}) = 0_{F(A),F(B)}$.

(b) Suppose that A is a null object of $\underline{C}$ and that β is an isomorphism. Then $F(0_{A,B}) = 0_{F(A),F(B)}$.

(c) Suppose that α is an isomorphism, $F(0_{A,B}) = 0_{F(A),F(B)}$ and $F(0_{B,A}) = 0_{F(B),F(A)}$. Then β is an isomorphism.

(d) Suppose that β is an isomorphism, $F(0_{A,B}) = 0_{F(A),F(B)}$ and $F(0_{B,A}) = 0_{F(B),F(A)}$. Then α is an isomorphism.

(e) Suppose that F carries $\mathcal{D}$ to a biproduct diagram in $\underline{D}$. Then both α and β are isomorphisms.

(f) Suppose that α is an isomorphism, $F(0_{A,B}) = 0_{F(A),F(B)}$ and $F(0_{B,A}) = 0_{F(B),F(A)}$. Then F carries $\mathcal{D}$ to a biproduct diagram in $\underline{D}$.

(g) Suppose that β is an isomorphism, $F(0_{A,B}) = 0_{F(A),F(B)}$ and $F(0_{B,A}) = 0_{F(B),F(A)}$. Then F carries $\mathcal{D}$ to a biproduct diagram in $\underline{D}$.

Proof. Note that the existence of the product P (resp., Q) implies the existence of the biproduct diagram $\mathcal{D}$ (resp., $\mathcal{E}$) with the stated properties by virtue of the proof of [16, Theorem 2, pages 194-195] (as supplemented by Remark 2.16). Note also that the various assumptions or conclusions that α (resp., β) is an isomophism are unambiguous, by Proposition 2.13 (b) (resp., Proposition 2.14 (b)).

(a) Since B is a terminal object of $|\underline{C}|$, there is a unique morphism, say u, from A to B in $|\underline{C}|$. Since $\underline{C}$ is a pre-additive category, u is a zero morphism; that is, $u = 0_{A,B}$. Let $C \in |\underline{C}|$. Let v denote the unique morphism from C to B in $|\underline{C}|$. By the uniqueness of v, it is clear that $u\theta = v$ for each $\theta \in \underline{C}(C, A)$. It follows that for each $\lambda \in \underline{C}(C, A)$, there exists a unique $\psi \in \underline{C}(C, A)$ such that $1_A\psi = \lambda$ and $u\psi = v$, namely, $\psi = \lambda$. Therefore, A is a product of A and B in $\underline{C}$ when considered together with the projection maps $p_1^* = 1_A : A \to A$ and $p_2^* = u : A \to B$. This view of A as a product with respect to these projection maps leads to an associated biproduct diagram in $\underline{C}$, by the proof of [16, Theorem 2, pages 194-195] (cf. also Remark 2.15 (a) and the first paragraph of Remark 2.16). In view of Proposition 2.13 (b), it is clear that, in regard to the task of proving (a), there is no harm in taking $A = P$, $p_1^* = p_1$ and $p_2^* = p_2$, with the just-mentioned biproduct diagram being $\mathcal{D}$, together with the items i_1 and i_2 as in the statement of (a). Then we also have $\mathcal{P} := F(P) = F(A)$.

Let $\tilde{u}$ denote the morphism $F(u) : \mathcal{P} \to F(B)$. We claim that $\mathcal{P}$ is a product of $F(A)$ and $F(B)$ in $\underline{D}$ when considered together with the projection maps $1_{F(A)} : \mathcal{P} \to F(A)$ and $\tilde{u}$. Recall that $\alpha : \mathcal{P} = F(A \times B) \to F(A) \times F(B) = Q$ is assumed to be an isomorphism; and that Q is a product of $F(A)$ and $F(B)$ with associated projection morphisms $\pi_1 : Q \to F(A)$ and $\pi_2 : Q \to F(B)$. Consequently, $\mathcal{P}$ is a product of $F(A)$ and $F(B)$ in $\underline{D}$ when considered together with the projection maps $\pi_1\alpha : \mathcal{P} \to F(A) = \mathcal{P}$ and $\pi_2\alpha : \mathcal{P} \to F(B)$. Recall from Proposition 2.13 that the definition of α entails that $\pi_1\alpha = F(p_1)$ and

$\pi_2\alpha = F(p_2)$. Observe that $F(p_1) = F(p_1^*) = F(1_A) = 1_{F(A)}$; and $F(p_2) = F(p_2^*) = F(u) = \tilde{u}$. This completes the proof of the above claim.

In view of the above established claim, it is clear that, in regard to the task of proving (a), there is no harm in viewing $F(A) \times F(B)$ as $Q = \mathcal{P}$, together with the projection maps $\pi_1 = 1_{F(A)}$ and $\pi_2 = \tilde{u}$. Now, consider any object D of $\underline{D}$, along with any morphisms $\rho_1 \in \mathcal{D}(D, F(A))$ and $\rho_2 \in \mathcal{D}(D, F(B))$. Since $\mathcal{P}$ has the universal mapping property of a product of $F(A)$ and $F(B)$ in $\underline{D}$, there exists a unique morphism $\varphi : D \to \mathcal{P}$ in $\underline{D}$ such that $1_{F(A)}\varphi = \rho_1$ and $\tilde{u}\varphi = \rho_2$. Necessarily, $\varphi = 1_{F(A)}\varphi = \rho_1$, and so $\tilde{u}\rho_1 = \rho_2$.

Now, consider the special case of the result in the preceding paragraph when we take $D := F(A)$ and $\rho_1 := 1_{\mathcal{P}}$. With ρ now playing the role of ρ_2 from the preceding paragraph, we get the following conclusion: for each morphism $\rho : F(A) \to F(B)$ in $\mathcal{D}$, $\varphi = 1_{\mathcal{P}}$ is the unique morphism $\mathcal{P} \to \mathcal{P}$ in $\underline{D}$ such that $\tilde{u}\varphi = \rho$, and so $\rho = \tilde{u}\varphi = \tilde{u}1_{\mathcal{P}} = \tilde{u}$. Hence, $\tilde{u}$ is the only element of the set $\underline{D}(F(A), F(B))$. Since $\underline{D}$ is a pre-additive category, $\tilde{u}$ is a zero morphism; that is, $\tilde{u} = 0_{F(A),F(B)}$. Thus,

$$F(0_{A,B}) = F(u) = \tilde{u} = 0_{F(A),F(B)},$$

as desired. The proof of (a) is complete.

(b) We have an object P that is both a product of A and B in $\underline{C}$ and a sum of A and B in $\underline{C}$; and we also have an object Q that is both a product of $F(A)$ and $F(B)$ in $\underline{D}$ and a sum of $F(A)$ and $F(B)$ in $\underline{D}$. Note that P is both a sum of A and B in $\underline{C}^{\text{op}}$ and a product of A and B in $\underline{C}^{\text{op}}$; and Q is both a sum of $F(A)$ and $F(B)$ in $\underline{D}^{\text{op}}$ and a product of $F(A)$ and $F(B)$ in $\underline{D}^{\text{op}}$. Also, recall that $\mathcal{P} := F(P)$.

For reasons that will become clear, let α_F (rather than simply α) denote the canonical morphism $F(A \times B) \to F(A) \times F(B)$, viewed as $\alpha_F : \mathcal{P} \to Q$, in $\underline{D}$; and similarly, let β_F (rather than simply β) denote the canonical morphism $F(A) \oplus F(B) \to F(A \oplus B)$, viewed as $\beta_F : Q \to \mathcal{P}$, in $\underline{D}$. Let $\mathcal{F} : \underline{C}^{\text{op}} \to \underline{D}^{\text{op}}$ be the functor induced by F via the construction in Remark 2.15 (b). In the spirit of two sentences ago, we introduce the following notation: let $\alpha_{\mathcal{F}}$ denote the canonical morphism $\mathcal{F}(A \times B) \to \mathcal{F}(A) \times \mathcal{F}(B)$, viewed as $\alpha_{\mathcal{F}} : \mathcal{P} \to Q$, in $\underline{D}^{\text{op}}$; and let $\beta_{\mathcal{F}}$ denote the canonical morphism $\mathcal{F}(A) \oplus \mathcal{F}(B) \to \mathcal{F}(A \oplus B)$, viewed as $\beta_{\mathcal{F}} : Q \to \mathcal{P}$, in $\underline{D}^{\text{op}}$. Recall from Remark 2.15 (c) that $\alpha_F = (\beta_{\mathcal{F}})^{\text{op}}$ and $\beta_F = (\alpha_{\mathcal{F}})^{\text{op}}$.

There are two hypotheses in (b). The first of these is the assumption that A is a null object of $\underline{C}$, and this assumption is equivalent to A being a null object of $\underline{C}^{\text{op}}$. The second hypothesis in (b) is that β_F is an isomorphism in $\underline{D}$; equivalently, that $(\alpha_{\mathcal{F}})^{\text{op}}$ is an isomorphism in $\underline{D}$. We claim that this second hypothesis implies that $\alpha_{\mathcal{F}}$ is an isomorphism in $\underline{D}^{\text{op}}$. (Actually, one can strengthen that "implies" to "is equivalent to" here, but we will not need that stronger fact.) To prove the above claim, it suffices to show that if $h : E_1 \to E_2$ is an isomorphism in a category $\underline{E}$, then $h^{\text{op}} : E_2 \to E_1$ is an isomorphism in $\underline{E}^{\text{op}}$. To accomplish this task will require two elementary categorical observations. For those, see the next paragraph.

First, it follows easily from the definition of composition of morphisms in a dual category that if E is an object of a category $\underline{E}$ and $f := 1_E : E \to E$ is the identity morphism on E in $\underline{E}$, then $f^{\text{op}} : E \to E$ is the identity morphism on E in $\underline{E}^{\text{op}}$. Second, if $h : E_1 \to E_2$ is an isomorphism in a category $\underline{E}$ with inverse $h^{-1} : E_2 \to E_1$ (in $\underline{E}$), then it follows easily from the preceding sentence that $h^{\text{op}} : E_2 \to E_1$ is an isomorphism in $\underline{E}^{\text{op}}$ with inverse $(h^{-1})^{\text{op}} : E_1 \to E_2$ (in $\underline{E}^{\text{op}}$). It is clear that the above claim follows from the two observations in this paragraph.

We can use the above (established) claim that $\alpha_{\mathcal{F}}$ is an isomorphism in $\underline{D}^{\text{op}}$ by applying (a) to $\mathcal{F}$. (Note that (a) is applicable to $\mathcal{F}$ by Lemma 2.1 (b) since $\underline{C}^{\text{op}}$ and $\underline{D}^{\text{op}}$ inherit from $\underline{C}$ and $\underline{D}$, respectively, the property of being a pre-additive category with finite products. That application shows that $\mathcal{F}$ sends the zero morphism $u : B \to A$ in $\underline{C}^{\text{op}}$ to the zero morphism $v : \mathcal{F}(B) = F(B) \to \mathcal{F}(A) = F(A)$ in $\underline{D}^{\text{op}}$.

By the final comment in Remark 2.15 (b), it follows that if $\underline{E}$ is a pre-additive category and if μ is the neutral element in a hom-set (abelian group) $\underline{E}(E_1, E_2)$ (that is, μ is the zero morphism from E_1 to E_2 in $\underline{E}$), then μ^{op} is the zero morphism from E_2 to E_1 in $\underline{E}^{\text{op}}$. In particular, if z denotes the zero morphism from A to B in $\underline{C}$, then z^{op} is the zero morphism from B to A in $\underline{C}^{\text{op}}$ (that is, $z^{\text{op}} = u$); and if ζ denotes the zero morphism from $F(A)$ to $F(B)$ in $\underline{D}$, then ζ^{op} is the zero morphism from $F(B)$ to $F(A)$ in $\underline{D}^{\text{op}}$ (that is, $\zeta^{\text{op}} = v$, the zero morphism from $\mathcal{F}(B)$ to $\mathcal{F}(A)$ in $\underline{D}^{\text{op}}$). Therefore, by applying (a) to $\mathcal{F}$ and also using the definition of $\mathcal{F}$ on morphisms from Remark 2.15 (b) (together with several applications of the fact that any morphism g satisfies $(g^{\text{op}})^{\text{op}} = g$), we get

$$F(0_{A,B}) = F(z) = F(u^{\text{op}}) = (\mathcal{F}(u))^{\text{op}} = v^{\text{op}} = \zeta = 0_{F(A),F(B)},$$

as desired. The proof of (b) is complete.

(c) Since α is an isomorphism, $\mathcal{P}$ is a product (and hence also a sum) of $F(A)$ and $F(B)$ in $\underline{D}$. Therefore, it follows from Proposition 2.13 (b) and Proposition 2.14 (b) that, in proving (c), we can assume, without loss of generality, that $Q = \mathcal{P}$. Hence, α is the uniquely determined endomorphism of $\mathcal{P}$ in $\underline{D}$ such that $\pi_1\alpha = F(p_1)$ and $\pi_2\alpha = F(p_2)$; and β is the uniquely determined endomorphism of $\mathcal{P}$ in $\underline{D}$ such that $\beta j_1 = F(i_1)$ and $\beta j_2 = F(i_2)$.

We claim that it suffices to prove (under the assumptions that α is an isomorphism and F sends both $0_{A,B}$ and $0_{B,A}$ to zero morphisms) that $\alpha\beta\alpha = \alpha$. Indeed, if $\alpha\beta\alpha = \alpha$, then (since α is an isomorphism),

$$\beta = \alpha^{-1}(\alpha\beta) = \alpha^{-1}(\alpha\alpha^{-1}) = \alpha^{-1}1_{\mathcal{P}} = \alpha^{-1},$$

which is an isomorphism. This proves the above claim.

We will next prove that $\alpha\beta\alpha = \alpha$. This is equivalent to showing that $\pi_1\alpha\beta\alpha = F(p_1)$ and $\pi_2\alpha\beta\alpha = F(p_2)$. It is interesting to observe that the proof of the first (resp., second) of these equations will use the fact that F sends $0_{B,A}$ (resp., $0_{A,B}$) to a zero morphism. We will provide a detailed proof of the first of these equations, leaving the similar proof of the second equation to the reader.

Note that $F(p_1)F(i_1) = F(p_1i_1) = F(1_A) = 1_{F(A)}$ and, similarly, $F(p_2)F(i_2) = 1_{F(B)}$; and $F(p_1)F(i_2) = F(p_1i_2) = F(0_{B,A}) = 0_{F(B),F(A)}$ (by hypothesis) and, similarly, $F(p_2)F(i_1) = F(0_{A,B}) = 0_{F(A),F(B)}$ (by hypothesis). Therefore, using at a crucial point that composition of morphisms distributes over addition in a pre-additive category, we get that

$$\pi_1\alpha\beta\alpha = F(p_1)\beta(1_{\mathcal{P}})\alpha = F(p_1)\beta(j_1\pi_1 + j_2\pi_2)\alpha =$$

$$F(p_1)\beta j_1\pi_1\alpha + F(p_1)\beta j_2\pi_2\alpha.$$

Since $F(p_1)\beta j_1\pi_1\alpha = F(p_1)F(i_1)\pi_1\alpha = F(p_1i_1)\pi_1\alpha = F(1_A)\pi_1\alpha = 1_{F(A)}\pi_1\alpha = \pi_1\alpha = F(p_1)$, we need only show that $F(p_1)\beta j_2\pi_2\alpha = 0_{\mathcal{P},F(A)}$, the neutral element in the additive abelian group $\underline{D}(\mathcal{P}, F(A))$. That, in turn, holds (thanks, in part, to Lemma 2.1 (a)), since

$$F(p_1)\beta j_2\pi_2\alpha = F(p_1)F(i_2)\pi_2\alpha = F(p_1i_2)\pi_2\alpha = F(0_{B,A})\pi_2\alpha =$$

$0_{F(B),F(A)}\pi_2\alpha = 0_{\mathcal{P},F(A)}$. The proof of (c) is complete.

(d) We will explain how (d) follows from (c) in the same spirit of the above proof which explained how (b) follows from (a). As above, let α_F denote the canonical morphism $F(A \times B) \to F(A) \times F(B)$, viewed as $\alpha_F : \mathcal{P} \to Q$, in $\underline{D}$; and similarly, let β_F denote the canonical morphism $F(A) \oplus F(B) \to F(A \oplus B)$, viewed as $\beta_F : Q \to \mathcal{P}$, in $\underline{D}$. Let $\mathcal{F} : \underline{C}^{\text{op}} \to \underline{D}^{\text{op}}$ be the functor induced by F via the construction in Remark 2.15 (b). Let $\alpha_{\mathcal{F}}$ denote the canonical morphism $\mathcal{F}(A \times B) \to \mathcal{F}(A) \times \mathcal{F}(B)$, viewed as $\alpha_{\mathcal{F}} : \mathcal{P} \to Q$, in $\underline{D}^{\text{op}}$; and let $\beta_{\mathcal{F}}$ denote the canonical morphism $\mathcal{F}(A) \oplus \mathcal{F}(B) \to \mathcal{F}(A \oplus B)$, viewed as $\beta_{\mathcal{F}} : Q \to \mathcal{P}$, in $\underline{D}^{\text{op}}$. Recall from Remark 2.15 (c) that $\alpha_F = (\beta_{\mathcal{F}})^{\text{op}}$ and $\beta_F = (\alpha_{\mathcal{F}})^{\text{op}}$.

As we have assumed in (d) that $F(0_{A,B}) = 0_{F(A),F(B)}$ and $F(0_{B,A}) = 0_{F(B),F(A)}$, it follows from the reasoning in the final paragraph of Remark 2.15 (b) that $\mathcal{F}$ sends both the zero morphism $B \to A$ in $\underline{C}^{\text{op}}$ and the zero morphism $A \to B$ in $\underline{C}^{\text{op}}$ to zero morphisms (in $\underline{D}^{\text{op}}$). Moreover, by hypothesis, β_F is an isomorphism; that is, $(\alpha_{\mathcal{F}})^{\text{op}}$ is an isomorphism. Therefore, by reasoning as in the third and fourth paragraphs of the proof of (b), we get that $\alpha_{\mathcal{F}}$ is an isomorphism. Recall also that both $\underline{C}^{\text{op}}$

and $\underline{D}^{op}$ are pre-additive categories with finite products. Consequently, by applying (c) to $\mathcal{F}$, we get that $\beta_{\mathcal{F}}$ is an isomorphism; that is, $(\alpha_F)^{op}$ is an isomorphism. Hence, by another appeal to the just-mentioned part of the proof of (b), α_F is an isomorphism. The proof of (d) is complete.

(e) Recall that the biproduct diagram $\mathcal{D}$ in $\underline{C}$ is given by the data set $\{A,B,P,p_1,p_2,i_1,i_2\}$, such that $p_1i_1 = 1_A$, $p_2i_2 = 1_B$ and $i_1p_1 + i_2p_2 = 1_P$, and, necessarily, $p_1i_2 = 0_{B,A}$ and $p_2i_1 = 0_{A,B}$. Assume that F carries $\mathcal{D}$ to a biproduct diagram in $\underline{D}$; that is, that $F(p_1)F(i_1) = 1_{F(A)}$, $F(p_2)F(i_2) = 1_{F(B)}$ and $F(i_1)F(p_1)+F(i_2)F(p_2) = 1_{F(P)}$ $(=1_{\mathcal{P}})$, and, necessarily, $F(p_1)F(i_2) = 0_{F(B),F(A)}$ and $F(p_2)F(i_1) = 0_{F(A),F(B)}$. Our task is to show that both α and β are isomorphisms.

Recall from the proof of [16, Theorem 2, pages 194-195] that the fact that $\mathcal{D}$ is a biproduct diagram implies that P is a product of A and B (in $\underline{C}$) with projection morphisms $p_1 : P \to A$ and $p_2 : P \to B$. For the sake of completeness, we next show how to modify the just-cited argument in [16] to prove that the fact that $\mathcal{D}$ is a biproduct diagram implies that P is also a sum of A and B (in $\underline{C}$) with injection morphisms $i_1 : A \to P$ and $i_2 : B \to P$.

Consider any morphisms $f_1 : A \to C$ and $f_2 : B \to C$ for some object C of $\underline{C}$. To prove the above "sum" assertion, it suffices to show that there exists a unique morphism $h : P \to C$ such that $hi_1 = f_1$ and $hi_2 = f_2$. As for existence, it suffices to show that $h := f_1p_1 + f_2p_2$ satisfies $hi_1 = f_1$ and $hi_2 = f_2$. We will prove the first of these equations, leaving the similar proof of the second equation to the reader. We have

$$hi_1 = (f_1p_1 + f_2p_2)i_1 = f_1(p_1i_1) + f_2(p_2i_1) = f_1 1_A + f_2 0_{A,B} =$$

$f_1 + 0_{A,C} = f_1$. As for uniqueness, suppose that a morphism $h^* : P \to C$ satisfies $h^*i_1 = f_1$ and $h^*i_2 = f_2$. Then

$$h^* = h^*1_P = h^*(i_1p_1 + i_2p_2) = (h^*i_1)p_1 + (h^*i_2)p_2 = f_1p_1 + f_2p_2 = h.$$

This completes the proof of the above "sum" assertion.

Recall that F is assumed to carry $\mathcal{D}$ to a biproduct diagram, say Δ, in $\underline{D}$. Thus, by the reasoning in the preceding two paragraphs, $F(P)$ $(= \mathcal{P})$ is a product of $F(A)$ and $F(B)$ in regard to the projection morphisms $F(p_1) : \mathcal{P} \to F(A)$ and $F(p_2) : \mathcal{P} \to F(B)$, and $\mathcal{P}$ is also a sum of $F(A)$ and $F(B)$ in regard to the injection morphisms $F(i_1) : F(A) \to \mathcal{P}$ and $F(i_2) : F(B) \to \mathcal{P}$. We claim that, in view of Proposition 2.13 (b) and Proposition 2.14 (b), we can assume, without loss of generality, that $Q = \mathcal{P}$. To prove this claim, one must show that if one takes $Q = \mathcal{P}$, along with $\pi_1 = F(p_1)$, $\pi_2 = F(p_2)$, $j_1 = F(i_1)$ and $j_2 = F(i_2)$, then these changes of variables still give a biproduct diagram in $\underline{D}$. In other words, one must show that

$$F(p_1)F(i_1) = 1_{F(A)}, F(p_2)F(i_2) = 1_{F(B)}, F(i_1)F(p_1) + F(i_2)F(p_2) = 1_{\mathcal{P}},$$

(and, necessarily, $F(p_1)F(i_2) = 0_{F(B),F(A)}$ and $F(p_2)F(i_1) = 0_{F(A),F(B)}$). The three just-displayed desired equations are *precisely* what it means to say that Δ is a biproduct diagram in $\underline{D}$. This proves the above claim (that we can take $Q = \mathcal{P}$ along with the above identifications of the associated structural morphisms).

It remains to prove that $\alpha : \mathcal{P} \to \mathcal{P}$ and $\beta : \mathcal{P} \to \mathcal{P}$ are isomorphisms. Recall that α is uniquely determined by the conditions $\pi_1\alpha = F(p_1)$ and $\pi_2\alpha = F(p_2)$; and that β is uniquely determined by the conditions $\beta j_1 = F(i_1)$ and $\beta j_2 = F(i_2)$. Hence, in view of the above changes of variables, α is uniquely determined by $F(p_1)\alpha = F(p_1)$ and $F(p_2)\alpha = F(p_2)$; and β is uniquely determined by $\beta F(i_1) = F(i_1)$ and $\beta F(i_2) = F(i_2)$. The uniqueness of those determinations ensures that $\alpha = 1_{\mathcal{P}}$ and $\beta = 1_{\mathcal{P}}$. In particular, both α and β are isomorphisms. The proof of (e) is complete.

(f), (g): Suppose that α is an isomorphism (resp., β is an isomorphism), $F(0_{A,B}) = 0_{F(A),F(B)}$ and $F(0_{B,A}) = 0_{F(B),F(A)}$. Then by (c) (resp, by (d)), β is an isomorphism (resp., α is an isomorphism). Recall that the set of data $\{A,B,P,p_1,p_2,i_1,i_2\}$ gives a uniquely determined biproduct diagram $\mathcal{D}$ in $\underline{C}$ (that is, such that $p_1i_1 = 1_A$, $p_2i_2 = 1_B$ and $i_1p_1 + i_2p_2 = 1_P$, and, necessarily, $p_1i_2 = 0_{B,A}$ and $p_2i_1 = 0_{A,B}$). Our task is to show that F carries $\mathcal{D}$ to a biproduct diagram in $\underline{D}$; that is, that $F(p_1)F(i_1) = 1_{F(A)}$,

$F(p_2)F(i_2) = 1_{F(B)}$ and $F(i_1)F(p_1) + F(i_2)F(p_2) = 1_{F(P)}$ $(= 1_{\mathcal{P}})$. The first and second of these equations follow easily from the corresponding equations given above since F is a functor. Thus, it remains only to prove that

$$F(i_1)F(p_1) + F(i_2)F(p_2) = 1_{\mathcal{P}}.$$

To that end, let us use the determining conditions $\pi_1\alpha = F(p_1)$, $\pi_2\alpha = F(p_2)$, $\beta j_1 = F(i_1)$ and $\beta j_2 = F(i_2)$. These lead to

$$F(i_1)F(p_1) + F(i_2)F(p_2) = \beta j_1\pi_1\alpha + \beta j_2\pi_2\alpha = \beta(j_1\pi_1 + j_2\pi_2)\alpha =$$

$\beta 1_{\mathcal{P}}\alpha = \beta\alpha$. Hence, we need only prove that $\beta\alpha = 1_{\mathcal{P}}$. Since α is an isomorphism, we need only prove that $\alpha\beta\alpha = \alpha$. That, in turn, can be shown by repeating the final two paragraphs of the proof of (c). The proof is complete. □

Recall (cf. [16, page 196]) that an *additive category* is a pre-additive category with a null object and binary products. (Equivalently, one could define an additive category as a pre-additive category with finite products.) One could summarize Theorem 2.18, which is our second main result, as giving, for additive categories $\underline{C}$ and $\underline{D}$, five new characterizations of the additive functors $\underline{C} \to \underline{D}$.

Theorem 2.18. Let $\underline{C}$ and $\underline{D}$ be (possibly isomorphic) pre-additive categories with finite products, and let $F : \underline{C} \to \underline{D}$ be a functor. Then the following seven conditions are equivalent:

(1) F is an additive functor;

(2) F carries each biproduct diagram (in the sense defined in [16, Definition, page 194]) in $\underline{C}$ to a biproduct diagram in $\underline{D}$;

(3) F is a CHR-additive functor;

(4) If $A, B \in |\underline{C}|$, the canonical morphism $F(A \times B) \to F(A) \times F(B)$ is an isomorphism in $\underline{D}$;

(5) If $A_1, \ldots, A_n \in |\underline{C}|$ for some integer $n \geq 2$, then the canonical morphism $F(\prod_{i=1}^{n} A_i) \to \prod_{i=1}^{n} F(A_i)$ is an isomorphism in $\underline{D}$;

(6) If $A, B \in |\underline{C}|$, then the canonical morphism $F(A) \oplus F(B) \to F(A \oplus B)$ is an isomorphism in $\underline{D}$;

(7) If $A_1, \ldots, A_n \in |\underline{C}|$ for some integer $n \geq 2$, then the canonical morphism $\coprod_{i=1}^{n} F(A_i) \to F(\coprod_{i=1}^{n} A_i)$ is an isomorphism in $\underline{D}$.

Proof. We begin by proving the following useful facts: if (4) holds (resp., if (6) holds) and if A and B are (possibly equal) objects of $\underline{C}$, then $F(0_{A,B}) = 0_{F(A),F(B)}$. To see this, note first that $u := 0_{A,B}$ factors through some null object N of $\underline{C}$; that is, there exists a null object N of $\underline{C}$ such that $u = vw$ for some morphisms $w \in \underline{C}(A,N)$ and $v \in \underline{C}(N,B)$. Since $\underline{C}$ is a pre-additive category and N is a null object, we have $w = 0_{A,N}$ and $v = 0_{N,B}$. Moreover, since (4) holds (resp., since (6) holds) and N is a null object, it follows from part (a) (resp., part (b)) of Proposition 2.17 that $F(w) = 0_{F(A),F(N)}$ (resp., that $F(v) = 0_{F(N),F(B)}$). Therefore, by Lemma 2.1 (a), $F(0_{A,B}) = F(u) = F(vw) = F(v)F(w)$ equals $F(v)0_{F(A),F(N)} = 0_{F(A),F(B)}$ (resp., equals $0_{F(N),F(B)}F(w) = 0_{F(A),F(B)}$), as asserted.

(4) ⇔ (6) : By the preceding paragraph, part (c) (resp., part (d)) of Proposition 2.17 gives (4) ⇒ (6) (resp., gives (6) ⇒ (4)).

(4) ⇔ (5) : It is trivial that (5) ⇒ (4). Conversely, the implication (4) ⇒ (5) follows from the associativity, up to natural isomorphism, of nonempty (direct) products. (*That*, in turn, follows from the proof, not the statement, of [16, Proposition 1, page 73].)

(6) ⇔ (7) : It is trivial that (7) ⇒ (6). Conversely, the implication (6) ⇒ (7) can be proved by adapting the above proof that (4) ⇒ (5). That adaptation, which is being left to the reader, proceeds via a straightforward dualization that focuses on coproducts rather than products.

(1) ⇔ (2) : This equivalence was proved in [16, Proposition 4, page 197]. While the statement of [16, Proposition 4, page 197] includes fewer explicit assumptions than the statement of the present Theorem 2.18, an examination of the proof of [16, Proposition 4, page 197] reveals that it uses all the assumptions of our Theorem 2.18.

(2) ⇒ (3): Assume (2). Since (2) ⇒ (1), it follows from Lemma 2.6 (b) that F sends any terminal object N of (that is, any empty product in) $\underline{C}$ to a null (hence, terminal) object of (hence, an empty product in) $\underline{D}$. Hence, the canonical morphism in $\underline{D}$ from $F(N)$ to an empty product, being the unique morphism between two terminal (actually, null) objects in a pre-additive category, is necessarily an isomorphism. Thus, by the definition of a CHR-additive functor, our task of proving (3) has been reduced to proving (5). As we proved above that (4) ⇒ (5), the task of proving (3) can be reduced to proving (4). That, in turn, follows since Proposition 2.17 (e) ensures that (2) ⇒ (4) (and, incidentally, also that (2) ⇒ (6)).

(3) ⇒ (4): This implication follows at once from the definition of a CHR-additive functor.

(4) ⇒ (2): It suffices to combine the first paragraph of this proof with Proposition 2.17 (f). The proof is complete. □

The equivalence of conditions (1) and (3) in Theorem 2.18 makes precise a statement of the result that was promised in the penultimate sentence of the Abstract. I trust that it would not be considered immodest or inaccurate for me to add that, because of Example 2.9, one can conclude that the equivalence (1) ⇔ (3) in Theorem 2.18 gives a strict generalization of the above-mentioned observation of Chase, Harrison and Rosenberg [5] concerning abelian categories.

In regard to the results that were promised in the final sentence of the Abstract: the equivalence of (1), (4), (5), (6) and (7) in Theorem 2.18 provides four additional new characterizations of the additive functors $F : \underline{C} \to \underline{D}$ whenever $\underline{C}$ and $\underline{D}$ are pre-additive categories with finite products. Thus, for such categories, Theorem 2.18 has provided five new characterizations of the associated additive functors. Of course, the equivalence of conditions (1) and (2) in Theorem 2.18 also serves to characterize those additive functors, but as noted in the proof of Theorem 2.18, its equivalence (1) ⇔ (2) can be found in [16].

One consequence of the equivalence (1) ⇔ (4) in Theorem 2.18 is that, for the data in Example 2.10 (b), the canonical morphism $\alpha : F_2(A\times B) \to F_2(A)\times F_2(B)$ is not an isomorphism for some zero rings A and B. (This follows since it was shown in Example 2.10 (b) that the functor F_2 is not additive.) This consequence may seem surprising, as those data satisfy $G \times G \cong G$ in Ab and F_2 sends every object of $\underline{C}_R$ to G. So, it may be of interest to have the following short proof that if $A = B = N$ is an arbitrary (necessarily null) object of $\underline{C}_R$ (for the ambient zero ring R), then α is *not* an isomorphism. Let $p_1 : N \times N \to N$ and $p_2 : N \times N \to N$ be the projection morphisms that are pertinent to the product $N \times N$ in $\underline{C}_R$. Then $p_1 = p_2$ since N is a terminal object of $\underline{C}_R$. Then $F_2(p_1) = F_2(p_2)$. Recall that $F_2(N) = G$ in |Ab|. As G is a nontrivial (in fact, infinite) group, one can pick distinct elements $a, b \in G$. Suppose, contrary to the above assertion, that α is an isomorphism (in Ab). As α is then surjective, there exists a (in fact, unique) element $\xi \in F_2(N\times N)$ (= G) such that $\alpha(\xi) = (a, b)$. Let π_1 and π_2 be the projection morphisms $F_2(N) \times F_2(N) \to F_2(N)$ (that is, $G \times G \to G$) that are pertinent to the product $F_2(N) \times F_2(N)$ in Ab. Hence, $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$. By the universal mapping property of this (direct) product, α is determined by the two conditions $\pi_1\alpha = F_2(p_1)$ and $\pi_2\alpha = F_2(p_2)$. Therefore,

$$a = \pi_1(a, b) = \pi_1(\alpha(\xi)) = F_2(p_1)(\xi) = F_2(p_2)(\xi) = \pi_2(\alpha(\xi)) =$$

$\pi_2(a, b) = b$, the desired contradiction, thus completing the promised “short proof."

The result in the previous paragraph gives a sense in which we cannot weaken condition (4) in the statement of Theorem 2.18. In particular, the preceding paragraph shows that if $\underline{C}$ and $\underline{D}$ are each pre-additive categories with finite products and $F : \underline{C} \to \underline{D}$ is a functor such that $F(A\times B) \cong F(A)\times F(B)$ for all objects A and B of $\underline{C}$, then it need not be the case that F is an additive functor. Thus, the functor F_2 from Example 2.10 (b) illustrates the importance of requiring that the isomorphisms stipulated in condition (4) of Theorem 2.18 be “canonical" or “natural". The reader is invited to use the $F \leftrightarrow \mathcal{F}$ correspondence to construct an example that makes the analogous point about the isomorphisms stipulated in condition (6) of Theorem 2.18.

This paragraph and the next two paragraphs will discuss some theoretical and/or pedagogical matters that we believe to be of some interest. The multitude of conditions in the statements of Proposition 2.17 and Theorem 2.18 leads naturally to several different reasonable ways to organize the proofs of those results. Exploring those ways can lead to some material for use in a graduate course on category theory. First, in regard to the proof of Proposition 2.17, one could ask whether a proof of its part (g) would be possible (as was the case for the above proofs of its parts (b) and (d)) by using the $F \leftrightarrow \mathcal{F}$ correspondence from Remark 2.15 (b). The answer is in the affirmative, but for reasons of space, we will only sketch the relevant details in the next paragraph.

Let $F : \underline{C} \to \underline{D}$ be as in the setting for Proposition 2.17. Let $\mathcal{F} : \underline{C}^{\text{op}} \to \underline{D}^{\text{op}}$ be the functor induced by F using the construction in Remark 2.15 (b). Let $\mathcal{D}$ be a biproduct diagram in $\underline{C}$, featuring projection morphisms p_1 and p_2 and injection morphisms i_1 and i_2. We have seen in Proposition 2.17 and Theorem 2.18 that it can be fruitful to study whether F carries $\mathcal{D}$ to a biproduct diagram in $\underline{D}$. One can ask if it would be fruitful to ask the analogous question about $\mathcal{F}$. The answer is in the affirmative, but before stating it, we need to make precise what is meant by the opposite of a biproduct diagram. In short, by definition, $\mathcal{D}^{\text{op}}$ features projection morphisms i_1^{op} and i_2^{op} and injection morphisms p_1^{op} and p_2^{op}. One can show that F carries $\mathcal{D}$ to a biproduct diagram in $\underline{D}$ if and only if $\mathcal{F}$ carries $\mathcal{D}^{\text{op}}$ to a biproduct diagram in $\underline{D}^{\text{op}}$. (Since the correspondence $F \leftrightarrow \mathcal{F}$ is a one-to-one correspondence, it suffices to prove the "only if" part of the preceding statement.) This result can be used to show that part (f) of Proposition 2.17 (when applied to $\mathcal{F}$ instead of F) implies part (g) of Proposition 2.17 (and *vice versa*), thus answering a question that was raised in the preceding paragraph. The calculations involving in proving this result, as one may expect from having worked out the details in the above proofs of parts (b) and (d) of Proposition 2.17, give some worthwhile and instructive experience in dealing with the definitions of composition and addition of morphisms in a dual category, as well as the definition of the action of $\mathcal{F}$ on morphisms. For instance, a proof of the result that we just stated includes verifying that

$$\mathcal{F}(p_1^{\text{op}})\mathcal{F}(i_1^{\text{op}}) + \mathcal{F}(p_2^{\text{op}})\mathcal{F}(i_2^{\text{op}}) = (1_{F(P)})^{\text{op}}$$

and a complete verification of the just-displayed equation involves (at least) five steps.

Next, any search for alternate ways to present a proof of Theorem 2.18 comes down to asking for different ways of stating and organizing the various parts of Proposition 2.17. In that regard, our preliminary work did find direct proofs (for given objects A and B) that led to the conclusion that condition (2) in Theorem 2.18 implies each of conditions (3)-(7) in Theorem 2.18. For a lecture or homework, an instructor could ask a class to find some of those direct proofs and to see how such arguments could be used to create different presentations of Proposition 2.17 and Theorem 2.18. A somewhat harder assignment would be to ask for direct proofs that each of (4)-(7) implies (2). The latter task would be somewhat easier if the assignment allowed students to also assume that F sends both $0_{A,B}$ and $0_{B,A}$ to zero morphisms.

Remark 2.19. Although we have had reason to discuss several "zero-ish" concepts here, I hesitated to title this paper, "Much ado about zero", for two reasons. First, our work here would likely suffer in comparison with a similarly titled play by Shakespeare. Second (and more seriously), one must admit that there are some mathematical situations where consideration of a zero element or a zero ring would be inconvenient and, ultimately, irrelevant for the study at hand. (We will discuss a family of such situations in the next paragraph.) However, one must also admit that in some other mathematical situations, one cannot ignore zero rings, as they can provide answers to some natural questions (as in this paper's Corollaries 2.4 and 2.7.) Moreover, in the final paragraph of this remark, we will discuss an anecdote illustrating how objects such as zero rings can be part of some mathematicians' fundamental views about the basis and nature of mathematics. That anecdote will also serve to explain this paper's dedication.

Since the turn of the century, there has been considerable interest in, and research on, a variety of graphs that are defined in terms of the structure of a given nonzero (commutative unital) ring R. To a large extent, the history of such research began with a remarkable paper [3] on “colorings" by István Beck that was published in 1986. To some readers, the graphs that resulted from the methodology in [3] were unnecessarily complicated while studying the zero-divisors of R because that methodology required the element $0 \in R$ to be treated in the same way as each nonzero (possibly zero-divisor) element of R. A much more attractive approach to such questions was begun by D. F. Anderson and P. S. Livingston in a paper [1] that was published in 1999. The methodology that was introduced in [1] to study the set of zero-divisors of R produced more tractable graphs than those which would have resulted from [3] because Anderson and Livingston took the set of vertices of the appropriate graph to be the set of *nonzero* zero-divisors of R. By thus not allowing the element $0 \in R$ to be a vertex of the graph, [1] produced a more intelligible graph (with, for a nonzero ring, fewer vertices and typically many fewer edges). Once it had been decided to disallow consideration of the *element* $0 \in R$, it was clearly pointless (pun intended) to consider any zero ring as a possible R. (Indeed, if one deletes that element from a zero ring, one gets the empty set, and no one would seriously suggest that graph theory could/should be used to deeper our understanding of $\emptyset$.) Moreover, I believe that most ring theorists would find no merit in considering the set of zero-divisors of a *zero ring*. In that regard, perhaps Kaplansky said it best [15, Note, page 34]: “It is perhaps treacherous to try to talk about zero-divisors on the zero module".

During my final year in graduate school (1968-69), I was a student in a course on category theory that was taught by Jon Beck (not to be confused with István Beck). The highlight of the course was Professor Beck's presentation of his famous “tripleability theorem" (cf. [16, pages 151-159]). Much earlier in the course, one of the students interrupted a lecture by asking the following question: “Why do we need to consider zero to be a ring?". (Of course, everyone understood that, by “zero", the student meant “$\{0\}$".) Perhaps Professor Beck knew that the student who had asked the question was doing doctoral research in algebraic geometry (and so was I, with more of an emphasis on “algebraic" and less on “geometry" than the other student). Professor Beck immediately replied, “Because [the category of] Schemes needs an initial object." My initial reaction to that reply was that it must have been intended as a joke, as it could be translated, at least for affine schemes, via duality, as saying that the category of commutative unital rings needs a terminal object. Within moments, I understood more deeply that Professor Beck's reply had not been meant as a joke. After all, everyone agrees that $\emptyset$ is an initial object in the category of sets, and it is only a small step from there to agree that the empty scheme is an initial object in the category of (not necessarily affine) schemes. Of course, the empty scheme can be realized as $\mathrm{Spec}(R)$ where R is any zero ring. Elucidating the (admittedly easy/trivial) sheaf-theoretic details of the structure of that (empty) scheme as a local ringed space took me only a few more moments, and then I was able to resume listening to the lecture. The way in which Professor Beck handed the question has, over the years, given me much food for thought as I considered how to teach advanced graduate courses, because of the following aspects, each of which slowly dawned upon me as time passed. In saying just six words, Professor Beck had managed to do all of the following: he welcomed the question; he answered it in a way that was consistent with the course's point of view (and, as I learned later, with his personal point of view of mathematics); he treated the audience with respect, seeing them as young professionals by giving an answer that would be clear to some of the students but would possibly require other students to think and study before being able to understand his reply; and he exuded authenticity by using his view that category theory is central to mathematics in order to inform his teaching practices. Many readers will be familiar with the following saying of a 19^{th} century historian and journalist, Henry Adams: “A teacher affects eternity; he can never tell where his influence stops." Whenever I hear or read that saying, I think of three of my teachers. In chronological order, the first of these, who was female, was my History teacher in high school; the second of these directed my masters thesis; and the third of these memorable teachers was Jon Beck. This completes the remark.

We close with some background and a recommendation for one possible direction of future research.

Remark 2.20. For any field k, let $\underline{D}_k$ be the full subcategory of $\underline{C}_k$ whose objects are (isomorphic to) finite products of finite-dimensional separable field extensions of k, and let Ad be the category whose objects are the CHR-additive functors from $\underline{D}_k$ to Ab. See [7, Chapter I, especially Theorem 3.13, pages 29-30, also pages 56-57] for a result showing how $\underline{C}_k$ and $\underline{D}_k$ can be used (along with Cech cohomology in the étale topology for (Spec)(k)) to determine the cohomological dimension of k (that is, the cohomological dimension of the Galois group of the separable closure of k, in the sense of Serre and Tate). For base rings R (that need not be fields), [7, Chapter II, especially pages 86-87] developed tools for use on "R-based topologies" T (which are certain affine-inspired variants of Grothendieck topologies) and the associated notions of a T-additive functor (which is a certain kind of CHR-additive functor), a T-sheaf, and Cech cohomology in T. In Chapter III (resp., Chapter IV) of [7], a specific R-based topology T was introduced that reduced to the classical étale setting from Chapter I if R is a field but also, in case R is a certain kind of one-dimensional valuation domain, produced (by use of the above-mentioned tools, especially Cech cohomology in T-additive functors) a T-cohomological dimension of R that coincides with the classical cohomological dimension of the quotient field (resp., with the classical cohomological dimension of the residue field) of R.

Having thus demonstrated the applicability of the notion of a CHR-additive functor in various settings, we returned, in [8], to the context of a base field k. Let Ad denote the category of CHR-additive functors from $\underline{D}_k$ to Ab. This category is amenable to the classical methods of homological algebra, as it was shown in [8, Corollary 2.3] that Ad is a Grothendieck category with a generator. Moreover, precise categorical descriptions of Ad were obtained via an exact left adjoint functor in [8, Theorem 2.2] and a categorical equivalence in [8, Remark 2.5]. In [8, Section 3], these descriptions of Ad led to examples of behavior of Amitsur cohomology of certain finite-dimensional non-Galois field extensions (for certain associated CHR-additive functors) which were qualitatively different from the behavior of group cohomology, Grothendieck sheaf cohomology, or Cech cohomology of CHR-additive functors in the above-mentioned étale setting. The interpretation in terms of the étale topology of Spec(k) is that, while the direct limit of certain cohomology groups, when indexed by a geometrically interesting set of covers, may exhibit classical behavior, very different behavior can be exhibited by those cohomology groups when one focuses on only one cover which is a singleton set. An affine analogue of that conclusion is the following fact, which has surely been observed by many commutative algebraists. While an integral domain D may be severely restricted when one requires *every* overring of D (inside the quotient field of D) to have a certain property $\mathcal{P}$, there may exist more general integral domains D with some overrings that satisfy $\mathcal{P}$ and other overrings (of D) that do not satisfy $\mathcal{P}$. For instance, if every proper overring of a Noetherian domain D is Noetherian, then D and each of its overrings have Krull dimension at most 1 (cf. [15, Exercise 20, page 64], [11, page 363]), but any Noetherian domain of finite Krull dimension $n \geq 3$ has a proper Noetherian overring of Krull dimension 2: cf. [15, Theorem 85].

Recent decades have witnessed a variety of transfusions connecting commutative algebra with algebraic geometry and cohomology theories. In our opinion, much remains to be learned along these lines, that is, by suitably translating various results from multiplicative ideal theory into the modern language of algebraic geometry, and *vice versa*, and that some of that prospective research should involve CHR-additive functors. This opinion may receive some support from algebraic geometers, since many interesting sheaves are CHR-additive functors, as one can see from the method of proof of a result [6, Proposition 5.2, page 51] in the classical étale setting (cf. also [18, page 707]). In particular, given the intuitive geometric meaning of a "cover", it would seem reasonable in many situations to expect an Ab-valued T-sheaf of some commutative algebras (resp., an Ab-valued sheaf of some affine schemes) to send the 0 algebra (resp., the empty scheme) to the abelian group 0. In that regard and in view of the attention that was paid to null objects and zero morphisms leading up to and during the proof of Theorem 2.18, one should note our long-held interest in ensuring that certain cohomologically relevant functors of commutative algebras send the 0 algebra to the abelian group 0: cf. "$M^*(0) = 0$" in [7, Definition 3.8, page 24]. Note, however, as a closing counterpoint, that a sheaf in an R-based topology need not be a T-additive functor: see [7, page 176, line 4] (where the reference there to "page 33" was intended to be to page 33 of Chapter II, that is, to page 101 of that volume).

References

[1] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra 217 (2) (1999), 434–447.

[2] H. Bass, Lectures on topics in algebraic K-theory, Tata Inst. Fund. Research, Bombay, 1967.

[3] I. Beck, Colorings of commutative rings, J. Algebra 116 (1) (1988), 208-226.

[4] H. Cartan and S. Eilenberg, An introduction to homological algebra, Princeton University Press, Princeton, 1956.

[5] S. U. Chase, D. K. Harrison and A. Rosenberg, Galois theory and Galois cohomology of commutative rings, Mem. Amer. Math. Soc., Volume 52 (1965), 15–33.

[6] D. E. Dobbs, Cech cohomology and a dimension theory for commutative rings, Ph. D. thesis, Cornell University, Ithaca (NY), 1969.

[7] D. E. Dobbs, Cech cohomological dimensions for commutative rings, Lecture Notes in Math., Volume 147, Springer-Verlag, New York, 1970.

[8] D. E. Dobbs, Amitsur cohomology in additive functors, Canad. Math. Bull. 16 (3) (1973), 417–426.

[9] P. Freyd, Abelian categories: an introduction to the theory of functors, Harper & Row, New York, 1964.

[10] P. J. Freyd, Abelian categories: an introduction to the theory of functors, Repr. Theory Appl. Categ. 3, 2003.

[11] R. Gilmer, Multiplicative ideal theory, Dekker, New York, 1972.

[12] K. Gödel, The consistency of the axiom of choice and of the generalized continuum-hypothesis with the axioms of set theory, Princeton Univ. Press, Princeton, 1940.

[13] A. Grothendieck, Sur quelques points d'algèbre homologique, Tohoku Math. J. 9 (2) (1957), 119–221.

[14] T. W. Hungerford, Algebra, Graduate Texts in Math. **73**, Springer-Verlag, New York-Berlin, 1980.

[15] I. Kaplansky, Commutative rings, revised edition, Univ. of Chicago, Chicago-London, 1974.

[16] S. Mac Lane, Categories for the working mathematician, second edition, Graduate Texts in Math., Volume 5, Springer-Verlag, New York, 1998.

[17] R. A. Morris, The inflation-restriction theorem for Amitsur cohomology, Pacific J. Math. 41 (1972), 791–797.

[18] R. A. Morris, Reciprocity isomorphisms for separable field extensions, Rocky Mountain J. Math. 9 (4) (1979), 707–719.

[19] D. G. Northcott, An introduction to homological algebra, Cambridge University Press, London, 1966.

[20] J. J. Rotman, An introduction to homological algebra, Academic Press, New York, 1979.