Moroccan Journal of Algebra and Geometry with Applications
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco
Volume 2, Issue 1 (2023), pp 162-173
ISSN: 2820-7114

Title :

On *1*-absorbing prime submodules

Author(s):

Emel Aslankarayigit Ugurlu

Moroccan Journal of Algebra and Geometry with Applications
Vol. 2(1) (2023), 162–173

Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco **ISSN: 2820-7114**

On 1-absorbing prime submodules

Emel Aslankarayigit Ugurlu
Department of Mathematics, Marmara University, 34722, Istanbul, Turkey.
e-mail: *emel.aslankarayigit@marmara.edu.tr*

Communicated by Najib Mahdou
(Received 06 February 2023, Revised 12 April 2023, Accepted 17 April 2023)

Abstract. Our aim for this paper is to present the notion of *1-absorbing prime submodules* of A-module M. We display that the new notion is a generalization of prime submodules coinciding it is a sort of specialized 2-absorbing submodule. Along with some properties of them, we characterize the quasi local rings by the help of the new concept. Also, we investigate their behaviors under homomorphisms, in the localization of modules, and in a cartesian product of modules. After introducing the minimal 1-absorbing prime submodules, the radical$_1$ of ideals and submodules, we obtain some famous results for them. Furthermore, we obtain two characterizations of the concept in a multiplication module. Finally, we obtain a result for 1-absorbing prime submodules similar to the Prime Avoidance Theorem.

Key Words: prime ideal, prime submodule, 1-absorbing prime, prime avoidence theorem.
2010 MSC: Primary 13A15; Secondary 16P40, 16D60

1 Introduction

In abstract algebra, there are a great many publications addressing the structure of rings and modules, see [2, 4, 14, 15, 17, 18]. In this article, we focus on only commutative rings with a non-zero identity and non-zero unital modules. Let A always denote such a ring and let M denote such a A-module. The concept of prime ideals and its generalizations have a significant position in commutative algebra since they are used in understanding the structure of rings. Note that throughout this paper L (resp., Q) denotes a proper submodule (resp., ideal) of M (resp., A). Recall that Q is called a *prime ideal* if $ab \in Q$ yields $a \in Q$ or $b \in Q$, [3]. Some authors expanded the concept of prime ideals to modules, [8, 11, 13]. Also, L is called *prime* if whenever $ax \in L$ for any $a \in A, x \in M$, then $x \in L$ or $aM \subseteq L$, [13].

$Id(A)$ and $S(M)$ denote the lattice of all ideals of A and the lattice of all submodules in M, respectively. The *radical of* Q, denoted by $\sqrt{Q}$, is defined as the intersection of all prime ideals contain Q. Note that we have the equality $\sqrt{Q} = \{r \in A \mid r^k \in Q \text{ for some } k \in \mathbb{N}\}$, see [3]. For any $a \in A$, the principal ideal generated by a is denoted by (a). All unit elements of A is denoted by $U(A)$. For any element $m \in M$, the set $< m >= Am = \{rm : \forall r \in A\}$ is the cyclic submodule of M generated by m. If $M =< X >$, we say that M is a finitely generated A-module for any finite subset X of M. Note that the ideal $\{a \in A : aM \subseteq L\}$ is said to be the *residue* of L by M and we denote as $(L :_A M)$. If A is clear, it is written by only $(L : M)$. Especially, $Ann(M) := (0 :_A M)$ is said to be the *annihilator in* M. Whenever $Ann(M) = 0_A$, it is said to be a *faithful module*. Then for an element $m \in M$, the *annihilator of* m is defined as $Ann(m) := \{r \in A : rm = 0_M\}$ and it is an ideal of A. Moreover, we denote the *radical of* L as $rad(L)$.

In 2007, Badawi has introduced the concept of 2-absorbing ideals as a generalization of prime ideals: Q is called a *2-absorbing ideal* if whenever $r, s, t \in A$ and $rst \in Q$, then $rs \in Q$ or $rt \in Q$ or $st \in Q$, see [5]. Then in 2011, A. Y. Darani and F. Soheilnia defined the concept of 2-absorbing submodules as following: L is called *2-absorbing* if whenever $r, s \in A$; $x \in M$ with $rsx \in L$, either $rx \in L$ or $sx \in L$ or $rs \in (L : M)$, see [7]. Actually, the concept of 2-absorbing submodules is a generalization of prime submodules.

In 2021, Yassine, Nikmehr and Nikandish introduced a recent class of ideals that it is a class of ideals between 2-absorbing ideals and prime ideals: Q is called *1-absorbing prime* whenever for all non-units $r,s,t \in A$ with $rst \in Q$, $rs \in Q$ or $t \in Q$, see [19]. Afterwards, the authors introduced the notion of weakly 1-absorbing prime: whenever for all non-units $r,s,t \in A$ with $0 \neq rst \in Q$, $rs \in Q$ or $t \in Q$, see [10]. Note that every prime ideal is a 1-absorbing prime and every 1-absorbing prime ideal is a 2-absorbing ideal. Thus, we have a chain: prime ideals ⇒ 1-absorbing prime ideals ⇒ 2 absorbing ideals. On the other hand, we have a second chain: prime submodules ⇒ 2 absorbing submodules. Thus we realize that there is a missing part in the second chain, which it is between prime submodules and 2-absorbing submodules. Then we define the missing part of the chain as *1-absorbing prime submodules.*

In Section 2, after introducing the notion of *1-absorbing prime submodules*, we examine the main properties of the new class. For all non-unit elements $r,s \in A$ and $x \in M$, if $rsx \in L$, either $rs \in (L:M)$ or $x \in L$, then L is said to be *1-absorbing prime*, see Definition 2.1. Firstly, we investigate in Proposition 2.3, the relation between 1-absorbing prime submodules and other special submodules, for example prime submodules, 2-absorbing submodules. We prove that every prime submodule is a 1-absorbing prime submodule, but the converse is not true: To see this, consider the cyclic submodule of $\mathbb{Z}_4$-module $\mathbb{Z}_4[X]$ generated by X, that is, $< X >$. Indeed, it is a 1-absorbing prime submodule, but is not a prime submodule of $\mathbb{Z}_4$-module $\mathbb{Z}_4[X]$, see Example 2.4. Also, we show that every 1-absorbing prime submodule is a 2-absorbing submodule. However, it is not true that every 2-absorbing submodule is 1-absorbing prime. Consider the cyclic submodule of $\mathbb{Z}$-module $\mathbb{Z}_{30}$ generated by $\overline{6}$, it is 2-absorbing but not 1-absorbing prime, see Example 2.5. Actually, by the help of Proposition 2.3, the second chain is completed. For the completed picture of these algebraic structures, see Figure 1. Among other results in this section, we give a characterization of 1-absorbing prime submodules, see Theorem 2.8. In Theorem 2.9, we characterize the quasi local rings by the help of our new concept. Furthermore, we examine the attitudes of the concept in a cartesian product of modules, in the localization of modules, and under homomorphisms. In Section 3, we introduce the *minimal 1-absorbing prime submodules*, *the radical*$_1$ of ideals and submodules. In Theorem 3.2 and Corollary 3.3, we obtain some famous results for 1-absorbing prime submodules. Afterwards, the Section 4 aims to give two characterizations of the concept in multiplication modules. By the help of some main proven results, we obtain two characterizations, see Theorem 4.1 and Theorem 4.10. Finally, the last section is dedicated to the Prime Avoidance Theorem for 1-absorbing prime submodules. After proving some propositions, we obtain 1-Prime Avoidance Theorem for submodules and 1-Prime Avoidance Theorem for cosets, see Theorem 5.2 and Theorem 5.6, respectively.

2 Properties of 1-absorbing prime submodules

Definition 2.1. For all non-units element $r,s \in A$ and $x \in M$, if $rsx \in L$, either $rs \in (L:M)$ or $x \in L$, then L is called **1-absorbing prime**.

Example 2.2. Assume $(A,\mathfrak{X})$ is a local ring with $\mathfrak{X}^2 = (0_A)$ and M is a A-module. Then every proper submodule in M is 1-absorbing prime. To see this, choose non-units $r,s \in A$ and $x \in M$ such that $rsx \in L$. Since $rs \in \mathfrak{X}^2 = (0_A)$, we have $rs \in (L:M)$, which implies L is 1-absorbing prime.

Proposition 2.3. *{Prime submodules} ⊆ {1-absorbing prime submodules} ⊆ {2-absorbing submodules}.*

Proof. Suppose L is a prime submodule of M. Take non-unit elements $r,s \in A$; $x \in M$ such that $rsx \in L$. Since L is prime, $rs \in (L:M)$ or $x \in L$, as desired. Suppose L is 1-absorbing prime. Take any $r,s \in A$ and $x \in M$ such that $rsx \in L$. Then we must obtain that $rs \in (L:M)$ or $rx \in L$ or $sx \in L$. If r,s are non-units, we have $rs \in (L:M)$ or $x \in L$, as required. Without loss generality, let r be unit. Thus one can see $rsx \in L$ yields $sx \in L$, as desired. □

Example 2.4. **(1-absorbing prime submodule that is not prime)** Consider the submodule $L =< X >$ of $\mathbb{Z}_4$-module $\mathbb{Z}_4[X]$. By previous example, L is a 1-absorbing prime submodule. However, L is not a prime submodule.

Example 2.5. **(2-absorbing submodule that is not 1-absorbing prime)** Let consider $\mathbb{Z}$-module $\mathbb{Z}_{30}$. Suppose that N is the cyclic submodule of $\mathbb{Z}$-module $\mathbb{Z}_{30}$ generated by $\overline{6}$, that is, $N =< \overline{6} >$. It is clear that $< \overline{6} >$ is a 2-absorbing submodule of $\mathbb{Z}$-module $\mathbb{Z}_{30}$, but $< \overline{6} >$ is not 1-absorbing prime. Indeed, $2 \cdot 2 \cdot \overline{3} \in < \overline{6} >$ but $\overline{4} \notin (N : \mathbb{Z}_{30})$ and $3 \notin < \overline{6} >$.

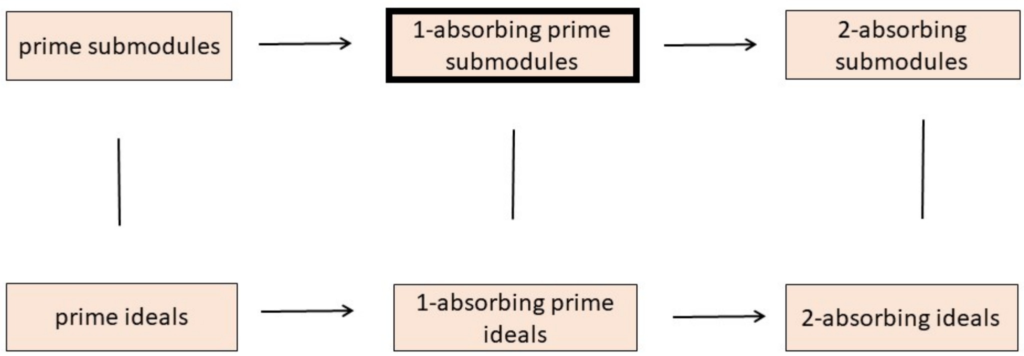

Figure 1: 1-absorbing prime submodules (ideals) vs other classical submodules (ideals)

Proposition 2.6. *Let L be a 1-absorbing prime submodule in M.*

1. *$(L : M)$ is a 1-absorbing prime ideal in A, hence $\sqrt{(L : M)}$ is prime.*
2. *$(L : x)$ is 1-absorbing prime, hence $\sqrt{(L : x)}$ is a prime ideal of A for every $x \in M \backslash L$.*

Proof. (1) Choose non-units $r, s, t \in A$ with $rst \in (L : M)$. For all $x \in M$ then $rstx \in L$. By our hypothesis, $rs \in (L : M)$ or $tx \in L$. This implies that $t \in (L : M)$ or $rs \in (L : M)$. Consequently, $(L : M)$ is 1-absorbing prime. Also, since $(L : M)$ is 1-absorbing prime, we conclude $\sqrt{(L : M)}$ is prime with the help of Theorem 2.3 in [19].
(2) Similar to (1). □

The next example displays that when $(L : M)$ is 1-absorbing prime, one can not say that L is 1-absorbing prime.

Example 2.7. Let $M = \mathbb{Z}\times\mathbb{Z}$ and $A = \mathbb{Z}$. Consider $L =< (3,0) >= \mathbb{Z}(3,0)$. Then it is clear that $(L:M) = (0)$. Then $(L:M)$ is a prime ideal of $\mathbb{Z}$, so 1-absorbing prime ideal in $\mathbb{Z}$. But $\mathbb{Z}(3,0)$ is not 1-absorbing prime. Indeed, choose $(1,0)\in\mathbb{Z}\times\mathbb{Z}$ and $3,2\in\mathbb{Z}$, thus $3\cdot 2\cdot(1,0)\in\mathbb{Z}(3,0)$. However, $6\notin(L:M) = (0)$ and $(1,0)\notin\mathbb{Z}(3,0)$. Thus $\mathbb{Z}(3,0)$ is not 1-absorbing prime.

Now, we give a characterization of the concept of 1-absorbing submodules of an A-module M.

Theorem 2.8. The items are equivalent:

1. L is a 1-absorbing prime submodule of M.
2. $(L:ab)\subseteq L$ for all non-units $a,b\in A$ such that $ab\notin(L:M)$.
3. For all non-units $a,b\in A$; a submodule K of M, $abK\subseteq L$ implies either $ab\in(L:M)$ or $K\subseteq L$.
4. If $IJK\subseteq L$, then $K\subseteq L$ or $IJ\subseteq(L:M)$ for any two proper ideals I,J and a proper submodule K of M.

Proof. (1) $\Rightarrow$ (2) Choose $x\in(L:_M ab)$, that is, $abx\in L$. By hypothesis, either $ab\in(L:M)$ or $x\in L$. The first one contradicts with our assumption, so that we conclude $(L:_M ab)\subseteq L$.

(2) $\Rightarrow$ (3) Let $a,b\in A$ be non-units and K be a submodule of M with $abK\subseteq L$, i.e., $K\subseteq(L:_M ab)$. Assume $ab\notin(L:M)$. By the item (2), we obtain $K\subseteq(L:_M ab)\subseteq L$, as needed.

(3) $\Rightarrow$ (4) Choose any two proper ideals I,J and a proper submodule K of M such that $IJK\subseteq L$. Suppose $IJ\nsubseteq(L:M)$. Then there are non-units $a,b\in A$ such that $a\in I$, $b\in J$ and $ab\in IJ\backslash(L:M)$. Also, $IJK\subseteq L$ implies that $abK\subseteq L$. Thus, we have $K\subseteq L$ by the item (3).

(4) $\Rightarrow$ (1) Choose non-unit $x,y\in A$ and $m\in M$ with $xym\in L$. Assume $m\notin L$. This means that $< m >\nsubseteq L$. Consider $I=(x), J=(y)$ and $K =< m >$. Since $IJK\subseteq L$ and $K\nsubseteq L$, by our hypothesis, $IJ\subseteq(L:M)$. Consequently, it means that $xy\in(L:M)$, as desired. □

Note that if A has exactly one maximal ideal, then A is called a *quasilocal ring*. In the following theorem, we prove a result on 1-absorbing prime submodules over quasilocal rings.

Theorem 2.9. If N is a 1-absorbing prime submodule in M which is not a prime submodule, then A is a quasilocal ring.

Proof. Assume that N is 1-absorbing prime that is not prime. Then there exist a non-unit $r\in A$; $m\in M$ which $rm\in N$ but $r\notin(N:M)$ and $m\notin N$. Choose a non-unit element $s\in A$. Hence we have that $rsm\in N$ and $m\notin N$. Because N is 1-absorbing prime, $rs\in(N:M)$. Let us take a unit element $u\in A$. We claim that $s+u$ is a unit element of A. To see this, assume $s+u$ is non-unit. Then $r(s+u)m\in N$. As N is 1-absorbing prime, $r(s+u)\in(N:M)$. This means that $ru\in(N:M)$, i.e., $r\in(N:M)$, which is a contradiction. Thus for any non-unit element s and unit element u in A, we have $s+u$ is a unit element. Similar to the proof of Theorem 2.4 in [19], we obtain A is a quasilocal ring. □

Corollary 2.10. *Assume M is a A-module, where A is not a quasilocal ring. L is 1-absorbing prime necessary and sufficient condition L is prime.*

Proof. It follows from previous theorem. □

Proposition 2.11. *Let $\{N_i\}_{i\in\Delta}$ be a chain of 1-absorbing prime submodules of A-module M. Then the followings hold:*

1. *$\bigcap_{i\in\Delta} N_i$ is 1-absorbing prime.*

2. *Assume that M is finitely generated. Then $\bigcup_{i\in\triangle} N_i$ is 1-absorbing prime.*

Proof. (1) Take non-unit $r,s \in A$ and $x \in M$ such that $rsx \in \bigcap_{i\in\triangle} N_i$. Assume that $x \notin \bigcap_{i\in\triangle} N_i$, so there exists $i \in\triangle$ such that $x \notin N_i$. Since N_i is 1-absorbing prime, we get $rs \in (N_i : M)$. For any $j \in\triangle$, we two cases.
Case 1: If $N_i \subseteq N_j$, then $(N_i : M) \subseteq (N_j : M)$, that is, $rs \in (N_j : M)$.
Case 2: If $N_j \subset N_i$, we obtain that $rs \in (N_j : M)$ since $x \notin N_j$ and N_j is 1-absorbing prime. As a consequence, we have $rs \in (\bigcap_{i\in\triangle} N_i : M)$.

(2) Since M is finitely generated, $\bigcup_{i\in\triangle} N_i$ is a proper submodule of M. Choose non-unit $r,s \in A$ and $x \in M$ such that $rsx \in \bigcup_{i\in\triangle} N_i$ and $x \notin \bigcup_{i\in\triangle} N_i$. Thus for $i \in\triangle$, $rsx \in N_i$ and $x \notin N_i$. This gives us $rs \in (N_i : M) \subseteq (\bigcup_{i\in\triangle} N_i : M)$, which completes the proof. □

Proposition 2.12. *Let $g : M \to M'$ be a homomorphism of A-module M and M'. Then the followings hold:*

1. *If L' is 1-absorbing prime in M' with $g^{-1}(L') \neq M$, $g^{-1}(L')$ is 1-absorbing prime in M.*

2. *Assume g is an epimorphism. If L is a 1-absorbing prime submodule of M with $Ker(g) \subseteq L$, $g(L)$ is a 1-absorbing prime submodule of M'.*

Proof. (1) Take non-units $r,s \in A$ and $x \in M$ such that $rsx \in g^{-1}(L')$. This means that $rsg(x) = g(rsx) \in L'$. Since L' is 1-absorbing prime, one can see $rs \in (L' : M')$ or $g(x) \in L'$. Then either $rs \in (g^{-1}(L') : M)$ or $x \in g^{-1}(L')$.

(2) Choose non-units $r,s \in A$ and $x' \in M'$ such that $rsx' \in g(L)$. By assumption there exists $x \in M$ such that $x' = g(x)$ and so $g(rsx) \in g(L)$. Then $rsx \in g^{-1}(g(L)) \subseteq L$, as $Ker(g) \subseteq L$. This implies that either $rs \in (L : M)$ or $x \in L$. If $rs \in (L : M)$, then $rsM \subseteq L$, that is, $rsg(M) = rsM' \subseteq g(L)$. Thus $rs \in (g(L) : M')$, it is done. If $x \in L$, then $x' = g(x) \in g(L)$, as required. □

One can easily obtain the following result by previous proposition.

Corollary 2.13. *Let $K \subset L$ be submodules of M. If L is a 1-absorbing prime submodule of M, then L/K is a 1-absorbing prime submodule of M/K.*

Theorem 2.14. Let $\emptyset \neq S \subseteq A$ be a multiplicatively closed subset and $S^{-1}L \neq S^{-1}M$. If L is 1-absorbing prime in M, $S^{-1}L$ is 1-absorbing prime in $S^{-1}A$-module $S^{-1}M$.

Proof. Choose two non-units $\frac{a}{x}, \frac{b}{y} \in S^{-1}A$ and $\frac{m}{z} \in S^{-1}M$ such that $\frac{a}{x}\frac{b}{y}\frac{m}{z} \in S^{-1}L$. Then there is $v \in S$ with $vabm \in L$. As L is 1-absorbing prime, one can see either $ab \in (L : M)$ or $vm \in L$. This result implies that either $\frac{ab}{xy} \in S^{-1}(L : M) \subseteq (S^{-1}L : S^{-1}M)$ or $\frac{vm}{vz} = \frac{m}{z} \in S^{-1}L$, which completes the proof. □

Theorem 2.15. Let $M_1 \times M_2$ be a module over $A_1 \times A_2$, where A_1 and A_2 are two commutative rings with nonzero identities. For two proper submodules L_1 of M_1 and L_2 of M_2, if $L_1 \times L_2$ is a 1-absorbing prime submodule in $M_1 \times M_2$, L_1 and L_2 are 1-absorbing prime.

Proof. Take two non-units $r,s \in A_1$; $x \in M_1$ with $rsx \in L_1$. Then consider $(r,0)(s,0)(x,0) \in L_1 \times L_2$. As $L_1 \times L_2$ is 1-absorbing prime, either $(rs,0) \in (L_1 \times L_2 : M_1 \times M_2)$ or $(x,0) \in L_1 \times L_2$. This implies that $rs \in (L_1 : M_1)$ or $x \in L_1$, that is, L_1 is 1-absorbing prime. Similarly, one can obtain L_2 is 1-absorbing prime. □

3 The radical_1 of ideals and submodules

Definition 3.1. Let P be a 1-absorbing prime submodule of M which $L \subsetneq P$. If there isn't a 1-absorbing prime P' such that $L \subseteq P' \subset P$, then P is called a **minimal 1-absorbing prime submodule of** L.

Theorem 3.2. If P is a 1-absorbing prime submodule of M with $L \subsetneq P$, there exists a minimal 1-absorbing prime submodule in L that it is contained in P.

Proof. Let define $\Lambda := \{P_i \in S(M) : P_i$ is a 1-absorbing prime submodule in M with $L \subseteq P_i \subseteq P\}$. Since $L \subsetneq P$, we get $\Lambda \neq \emptyset$. Consider $(\Lambda, \supseteq)$. Let us take a chain $\{P_i\}_{i\in\Delta}$ in Λ. By Proposition 2.11(1), since $\bigcap_{i\in\Delta} P_i$ is a 1-absorbing prime, we can use Zorn's Lemma. Thus, there exists a maximal element $K \in \Lambda$. Then K is 1-absorbing prime and $L \subseteq K \subsetneq P$. Now, we shall prove K is minimal 1-absorbing prime. For the contrary, assume that there exists a 1-absorbing prime submodule K' which $L \subseteq K' \subseteq K$. Then $K' \in \Lambda$ and $K \subseteq K'$. This implies $K = K'$. It is done. ☐

Corollary 3.3. *For a proper submodule L in M, the statements hold:*

1. *Each 1-absorbing prime submodule contains at least one minimal 1-absorbing prime submodule of M.*
2. *Suppose that M is finitely generated. Every proper submodule of M has at least one minimal 1-absorbing prime submodule of M.*
3. *If M is finitely generated, then there exists a 1-absorbing prime submodule of M which contains L.*

Proof. (1) Obvious by Theorem 3.2.

(2) Let M be finitely generated. Then there exists a prime submodule P such that $L \subseteq P$, see [13]. Then P is 1-absorbing prime. Thus, by (1), it is done.

(3) By the claim in (2), it is clear. ☐

Definition 3.4. For any $I \in Id(A)$, we define

$$\Omega := \{I_i \in Id(A) : \ I_i \text{ is a 1-absorbing prime ideal with } I \subseteq I_i\}.$$

The intersection of all elements in Ω is called the **radical_1 of I**, and we denote it as

$$rad_1(I) := \bigcap_{I_i\in\Omega} I_i \text{ and if } \Omega = \emptyset \text{ or } I = A, \text{ we define } rad_1(I) := A.$$

Remark 3.5. One can easily see $rad_1(I) \subseteq \sqrt{I} = rad(I)$, since every prime ideal is a 1-absorbing prime ideal.

Definition 3.6. For any $N \in S(M)$, we define

$$\Omega := \{P_i \in S(M) : \ P_i \text{ is a 1-absorbing prime submodule such that } N \subseteq P_i\}.$$

Then the intersection of all elements in Ω is called the **radical_1 of N**, and we denote it as

$$rad_1(N) := \bigcap_{P_i\in\Omega} P_i \text{ and if } \Omega = \emptyset \text{ or } N = M, \text{ we define } rad_1(N) := M.$$

Remark 3.7. It is clear that $rad_1(N) \subseteq rad(N)$, since every prime submodule is a 1-absorbing prime submodule.

Proposition 3.8. *Let N, L be two submodules of M. Then the following statements hold:*

1. $N \subseteq rad_1(N)$.
2. $rad_1(rad_1(N)) \subseteq rad_1(N)$.
3. $rad_1(N \cap L) \subseteq rad_1(N) \cap rad_1(L)$.
4. $rad_1(IM) \subseteq rad_1(\sqrt{I}M)$.
5. $rad_1(L : M) \subseteq (rad_1(L) : M)$.

Proof. The first four items are elementary.

(5) If $rad_1(L) = M$, it is trivial. Let $rad_1(L) \neq M$. Then there is a 1-absorbing prime submodule K such that $L \subseteq K$. Thus $(K : M)$ is 1-absorbing prime with $(L : M) \subseteq (K : M)$. Hence $rad_1(L : M) \subseteq (K : M)$, that is, $rad_1(L : M)M \subseteq K$. The containment is held for all 1-absorbing prime submodules K_i with $L \subseteq K_i$. This implies that $rad_1(L : M)M \subseteq rad_1(L)$, i.e., $rad_1(L : M) \subseteq (rad_1(L) : M)$. □

Proposition 3.9. *Let M be finitely generated. Then $rad_1(L) = M$ if and only if $L = M$.*

Proof. Let $rad_1(L) = M$. Suppose that $L \neq M$. By Corollary 3.3(3), there exists a 1-absorbing prime submodule L' of M which $L \subseteq L'$. Hence, we conclude that $rad_1(L) = M \subseteq L'$, a contradiction. The other way of the claim is clear. □

Theorem 3.10. Let M be finitely generated and K, L be two submodules in M. Then $K + L = M$ if and only if $rad_1(K) + rad_1(L) = M$.

Proof. Assume that $K + L = M$. We know that $K \subseteq rad_1(K)$ and $L \subseteq rad_1(L)$. Thus $M = K + L \subseteq rad_1(K) + rad_1(L)$, it is done. Conversely, suppose that $K + L \neq M$. Since M is finitely generated, there is a maximal submodule T of M such that $K + L \subseteq T$, see [13]. Furthermore, T is prime (thus, 1-absorbing prime). As $rad_1(K) \subseteq T$ and $rad_1(L) \subseteq T$, we have $rad_1(K) + rad_1(L) \subseteq T$, that is, $M \subseteq T$. This contradicts with our assumption. Consequently, it must be $K + L = M$. □

4 Characterizations of 1-absorbing prime submodules of multiplication modules

Now, our aim is to give two characterizations of the new concept in multiplication modules. For the integrity of our study, we will remind some knowledge about multiplication modules. An A-module M is said to be *multiplication* if each submodule L of M has the form JM for an ideal J of A, see [9]. It is clear that one can write $L = JM \subseteq (L : M)M \subseteq L$. Then if M is multiplication, $L = (L : M)M$.

For the first characterization examine the following result:

Theorem 4.1. Assume M is a faithful finitely generated multiplication A-module. Then L is 1-absorbing prime in M if and only if for each submodules K_1, K_2, K_3 of M if $K_1K_2K_3 \subseteq L$, either $K_1K_2 \subseteq L$ or $K_3 \subseteq L$.

Proof. Let L be a 1-absorbing prime submodule of M. Suppose $K_1K_2K_3 \subseteq L$. Since M is a multiplication module, there are some ideals I_1, I_2, I_3 of A such that $K_1 = I_1M, K_2 = I_2M$, and $K_3 = I_3M$. Consider $I_1I_2I_3M \subseteq L$. By Theorem 2.8, it must be either $I_1I_2 \subseteq (L : M)$ or $I_3M \subseteq L$. This implies that $K_1K_2 \subseteq L$ or $K_3 \subseteq L$ because M is multiplication. For the converse, let I_1, I_2 be two ideals in A and choose a proper submodule K of M which $I_1I_2K \subseteq L$. As M is a multiplication module, there is an ideal I_3 of A such that $K = I_3M$. Then $I_1I_2I_3M \subseteq L$, by our hypothesis, $I_1I_2M \subseteq L$ or $I_3M \subseteq L$. The second option means $K \subseteq L$, it is done by Theorem 2.8. If $I_1 I_2 M \subseteq L = (L:M)M$, we conclude $I_1 I_2 \subseteq (L:M) + Ann(M)$ by Corollary in page 231 of [16]. Since M is faithful, $I_1 I_2 \subseteq (L:M)$, which completes the proof with Theorem 2.8. □

Recall from [9], a multiplication module can be represented by a maximal ideal. Assume P is a maximal ideal of A. To give the characterization, let us define the submodule $T_P(M) = \{x \in M :$ there is a $p \in P$ such that $(1-p)x = 0_M\}$. Whenever $T_P(M) = M$, M is said to be a *P-torsion module*. Also, if there is $p \in P$; $x \in M$ with $(1-p)M \subseteq Ax$, M is called a *P-cyclic module*. In Theorem 1.2 of [9], the authors proved that M is a multiplication A-module if and only if for any maximal ideal P of A, M is a P-cyclic or M is a P-torsion. For more information about multiplication modules, we refer the reader to [1], [6].

To obtain the second characterization, firstly we need the following results.

Theorem 4.2. Let M be a faithful multiplication A-module. Let J be a 1-absorbing prime ideal of A. Then $abx \in JM$ implies $ab \in J$ or $x \in JM$ for all non-units $a, b \in A$; $x \in M$.

Proof. Take $x \in M$; non-units $a, b \in A$ with $abx \in JM$. Suppose $ab \notin J$. Let us define $J' := \{r \in A : rx \in JM\}$. In case $J' = A$, there is nothing to prove. So, $J' \neq A$. Then there is a maximal ideal P of A, which $J' \subseteq P$. Now, we will prove $x \notin T_P(M)$. If $x \in T_P(M)$, there is $p \in P$ with $(1-p)x = 0_M$. This yields $1 - p \in J' \subseteq P$, a contradiction. Hence, $T_P(M) \neq M$. As M is multiplication, M is P-cyclic by the help of Theorem 1.2 in [9]. So, there is $p' \in P$ and $x' \in M$ which $(1-p')M \subseteq Ax'$. Then $(1-p')x \in Ax'$, so that there exists $s \in A$ with $(1-p')x = sx'$. Then $(1-p')abx = sabx' \in JM$ and $(1-p')abx \in Ax'$. Thus, there are $a' \in J$ such that $(1-p')abx = a'x'$. Since $sabx' = a'x'$, we obtain $abs - a' \in Ann(x')$. Moreover, $(1-p')M \subseteq Ax'$ gives us $(1-p')Ann(x')M \subseteq AAnn(x')x' = 0_M$, i.e., $(1-p')Ann(x') \subseteq Ann(M)$. Then $(1-p')Ann(x') = 0_A$, because M is faithful. This implies $(1-p')(abs - a') = 0_A$. Hence, one can see $abs(1-p') = a'(1-p') \in J$. Then $abs(1-p') \in J$. Now, there are two cases for $s \in A$:

Case 1: Assume s is unit. Then $ab(1-p') \in J$. If $1-p'$ is unit, then $ab \in J$. This contradicts our assumption $ab \notin J$. Suppose $1-p'$ is non-unit. As J is 1-absorbing prime, $ab \in J$ (which gives a contradiction) or $1-p' \in J$. If $1-p' \in J$, then we have $(1-p')x \in JM$, that is, $1-p' \in J' \subseteq P$, a contradiction.

Case 2: Assume s is non-unit. Now, we have two possibilities for $1-p'$. If $1-p'$ is a unit element of A, then $sab \in J$. Since J is 1-absorbing prime, $ab \in J$ (again, it is not possible) or $s \in J$. Then $sx' \in JM$. Since $sx' = (1-p')x$, we have $(1-p')x \in JM$. So, $1-p' \in J' \subseteq P$, which is not possible. If $1-p'$ is non-unit, because J is 1-absorbing prime, either $abs \in J$ or $1-p' \in J$. Again, since it is 1-absorbing prime, we have $ab \in J$ or $s \in J$ or $1-p' \in J$. Every probability concludes a contradiction by the help of the previous explications.

Consequently, $J' = A$, i.e., $x \in JM$. □

Corollary 4.3. *Suppose M is a faithful multiplication A-module. Let J be an ideal of A such that $JM \neq M$. If J is a 1-absorbing prime ideal of A, then JM is 1-absorbing prime.*

Proof. Take non-unit elements $x, y \in A$ and $m \in M$ such that $xym \in JM$. Suppose $xy \notin (JM : M)$. Then $xy \notin J$. By Theorem 4.2, it must be $m \in JM$, as required. □

Theorem 4.4. Let M be a faithful finitely generated multiplication A-module. Let I be an ideal of A such that $IM \neq M$. Then IM is a 1-absorbing prime submodule of M if and only if I is a 1-absorbing prime of A.

Proof. Let IM be 1-absorbing prime. Take non-unit $x, y, z \in A$ such that $xyz \in I$. Suppose $xy \notin I$. By Theorem 10 in [16], we have $I = (IM : M)$. This implies that $xy \notin (IM : M)$ and $xyzM \subseteq IM$. Since IM is 1-absorbing prime, it must be either $xy \in (IM : M)$ or $zM \subseteq IM$. The first one contradicts with

$xy \notin I$. Thus, the second one implies $z \in (IM : M) = I$, as needed. The other way of the claim is obvious from Corollary 4.3. □

The proof of the next result is omitted since it is straightforward by Theorem 4.4.

Corollary 4.5. *Let M be a faithful finitely generated multiplication A-module and N be a proper submodule of M. Then*

1. *N is a 1-absorbing prime submodule of M if and only if $(N : M)$ is 1-absorbing prime ideal of A.*
2. *N is a 1-absorbing prime submodule of M if and only if $N = IM$ for some 1-absorbing prime ideal I of A.*

Proposition 4.6. *Let T, Q be some ideals in A with $T \subseteq Q$. If Q is 1-absorbing prime, Q/T is 1-absorbing prime.*

Proof. Choose non-unit $x+T, y+T, z+T$ in A/T such that $xyz+T \in Q/T$. This implies that $xyz \in Q$. Since $\{r+T : r \in U(A)\} \subseteq U(A/T)$, one can see x, y, z are non-units. As Q is 1-absorbing prime, either $xy \in Q$ or $z \in Q$. This means $xy+T \in Q/T$ or $z+T \in Q/T$. □

Definition 4.7. Let Q be an ideal of A. If the following equation $U(A/Q) = \{r+Q : r \in U(A)\}$ holds, then we say A satisfies the **good unit element property for** Q.

For the other way of Proposition 4.6, we need to the good unit element property:

Remark 4.8. (Corollary 2.17 in [19]) Let Q be an ideal of A such that $T \subseteq Q$ and $U(A/T) = \{r+T : r \in U(A)\}$. Then Q is 1-absorbing prime if and only if Q/T is a 1-absorbing prime ideal in A/T.

Proposition 4.9. *Let A satisfy the good unit element property for $Ann(M)$. When L is a 1-absorbing prime submodule in M over $A/Ann(M)$, L is a 1-absorbing prime submodule in M over A.*

Proof. Choose $x \in M$; non-unit $a, b \in A$ with $abx \in L$. We must show that either $ab \in (L :_A M)$ or $x \in L$. Consider $(a+Ann(M))(b+Ann(M))x = abx+Ann(M)x \in L$. By our hypothesis, one can say $a+Ann(M)$ and $b+Ann(M)$ are non-unit elements in $A/Ann(M)$. Since L is a 1-absorbing prime submodule over the ring $A/Ann(M)$, we obtain $ab+Ann(M) \in (L :_{A/Ann(M)} M)$ or $x \in L$. If the second one holds, it is done. The first one implies $abM \subseteq L$, i.e., $ab \in (L :_A M)$, as required. □

As a final result in this section, we give the second characterization of 1-absorbing prime submodules of multiplication modules in the next theorem.

Theorem 4.10. Let A satisfy the good unit element property for $Ann(M)$, where M is a multiplication A-module. Then the followings are equivalent:

1. L is a 1-absorbing prime submodule of M.
2. $(L : M)$ is a 1-absorbing prime ideal of A.
3. For a proper ideal P of A such that $Ann(M) \subseteq P$ and P is 1-absorbing prime, then $L = PM$.

Proof. (1) ⇒ (2) By Proposition 2.6.
(2) ⇒ (3) Consider $P = (L : M)$.
(3) ⇒ (1) By page 759 of [9], as M is a multiplication A-module, it is also a faithful multiplication $A/Ann(M)$-module. Moreover, because P is a 1-absorbing prime ideal in A, $P/Ann(M)$ is a 1-absorbing prime ideal in $A/Ann(M)$ by Proposition 4.6. Then $[P/Ann(M)]M$ is a 1-absorbing prime submodule in $A/Ann(M)$-module M with the help of Corollary 4.3. Then Proposition 4.9 implies that $[P/Ann(M)]M$ is 1-absorbing prime. As $[P/Ann(M)]M = L$, it is done. □

5 The 1-absorbing Prime Avoidance Theorem

Our aim for the part is to demonstrate the Prime Avoidance Theorem for 1-absorbing prime submodules of M. Firstly, we need the following proposition.

Now, recall from [12], a covering $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ is efficient if N is not contained in the union of any $(n-1)$ of the submodules $N_1, N_2, \ldots, N_n$. Similary, $N = N_1 \cup N_2 \cup \cdots \cup N_n$ is said to be an efficient union if none of the N_i may be excluded, where $i = 1, 2, \ldots, n$. Also, note that a covering consists of two submodules can not be efficient.

Proposition 5.1. *Let $N, N_1, N_2, \ldots, N_n$ be submodules of a A-module M such that $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ is an efficient covering ($n > 2$). If $\sqrt{(N_j : M)} \not\subseteq \sqrt{(N_k : m)}$ for $\forall\ m \notin N_k$ and $\forall\ j \neq k$, then no N_k is a 1-absorbing prime submodule of M, where $k = 1, 2, \ldots, n$.*

Proof. It is clear that $N = (N_1 \cap N) \cup (N_2 \cap N) \cup \cdots \cup (N_n \cap N)$ is an efficient union. Hence, for every $k \leq n$, there is $e_k \in N - N_k$. Note that $\bigcap_{j \neq k}^{n} N_j \cap N \subseteq N \cap N_k$ by Lemma 2.1 in [12]. Now, without losing the generality assume that N_1 is 1-absorbing prime. It must be $\sqrt{(N_j : M)} \not\subseteq \sqrt{(N_1 : m)}$ for $\forall\ m \notin N_1$ and $\forall\ j = 2, 3, \ldots, n$, by our hypothesis. Thus there is $s_j \in \sqrt{(N_j : M)}$ and $s_j \notin \sqrt{(N_1 : m)}$. This implies that there exists $n_j \in \mathbb{N}$ such that $s_j^{n_j} \in (N_j : M)$. Let $\beta = max\,\{n_j\}_{j=2,3,\ldots,n.}$ Consider $s = (s_2 s_3 \cdots s_n)^{\beta} \in (N_j : M)$. Then clearly $se_1 \in sM \subseteq N_j$ for $\forall\ j = 2, 3, \ldots, n$. This implies that $se_1 \in \bigcap_{j=2}^{n} N \cap N_j$. Here, note that since $e_1 \notin N_1$ and N_1 is 1-absorbing prime, by Proposition 2.6(2), $(N_1 : e_1)$ is a 1-absorbing prime ideal of R. Moreover, $\sqrt{(N_1 : e_1)}$ is a prime ideal by Theorem 2.3 in [19]. Now, we claim that $se_1 \notin N_1$. Indeed, if $s \in (N_1 : e_1)$, we would have $s_2 s_3 \cdots s_n \in \sqrt{(N_1 : e_1)}$, this gives us $s_j \in \sqrt{(N_1 : e_1)}$ for some j. Since $s_j \notin \sqrt{(N_1 : m)}$ for $\forall\ m \notin N_1$ and $e_1 \notin N_1$, we would obtain a contradiction. Hence, we conclude $se_1 \in (\bigcap_{j=2}^{n} N \cap N_j) - N \cap N_1$, a contradiction. Consequently, N_1 is not 1-absorbing prime. □

Theorem 5.2. **(1-absorbing Prime Avoidance Theorem for Submodules)** Let $N_1, N_2, \ldots, N_n$ be a finite number of submodules of a A-module M and N be a submodule in M which $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$. Suppose that at most two of the N_i's are not 1-absorbing prime and $\sqrt{(N_j : M)} \not\subseteq \sqrt{(N_k : m)}$ for $\forall$ $m \notin N_k$ and $\forall\ j \neq k$. Then $N \subseteq N_k$ for some $k = 1, 2, \ldots, n$.

Proof. By using the containment $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$, we can find $N \subseteq N_{j_1} \cup N_{j_2} \cup \cdots \cup N_{j_t}$, which is an efficient covering. Then $1 \leq t \leq n$ and $t \neq 2$. If $t > 2$, there is at least one L_{j_i}, which is 1-absorbing prime. On the other hand, by the help of Proposition 5.1, we conclude a contradiction with $\sqrt{(N_j : M)} \not\subseteq \sqrt{(N_k : m)}$ for $\forall\ m \notin N_k$ and $\forall\ j \neq k$. Thus $t = 1$, that is, $N \subseteq N_k$ for some $k = 1, 2, \ldots, n$. □

As a final conclusion of our study, we will present "1-absorbing Prime Avoidance Theorem for Cosets". For this reason, we need the followings.

Let $N, N_1, N_2, \ldots, N_n$ be submodules of a A-module M and $N_1 + m_1, N_2 + m_2, \ldots, N_n + m_n$ be cosets in M. Then a covering $N \subseteq (N_1 + m_1) \cup (N_2 + m_2) \cup \cdots \cup (N_n + m_n)$ is said to be efficient if N is not contained in the union of any $(n-1)$ of the cosets, see [12].

Remark 5.3. Consider the above efficient covering. If $m_j = m$ for every $j \in \{1, 2, \ldots, n\}$, then the covering equals to $N - m \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$. Thus, $N - m$ is a coset efficiently covered by a union of submodules, see [12].

In order not to lose the entireness of the article, let us notice the following:

Lemma 5.4. *(Lemma 2.4 in [12]) Let $N \subseteq (N_1 + m_1) \cup (N_2 + m_2) \cup \cdots \cup (N_n + m_n)$ be an efficient covering of a submodule of N by cosets, where $n \geq 2$. Then $N \cap (\bigcap_{j \neq k}^{n} N_j) \subseteq N_k$ and $N \nsubseteq N_k$ for all k.*

Proposition 5.5. *Let $N + m \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ be an efficient covering for $m \in M$ with $n \geq 2$. If $\sqrt{(N_j : M)} \nsubseteq \sqrt{(N_k : M)}$ for $\forall\ j \neq k$. Then no N_k is 1-absorbing prime in M, where $k = 1, 2, \ldots, n$.*

Proof. Assume $N + m \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ is an efficient covering for $m \in M$ with $n \geq 2$. Then by Remark 5.3, we can apply Lemma 5.4. Thus, we conclude that $N \cap (\bigcap_{j \neq k}^{n} N_j) \subseteq N_k$ and $N \nsubseteq N_k$ for all k. Without losing the generality, let $k = 1$. For the contradictory, suppose N_1 is 1-absorbing prime. Consider the ideal $I = (\bigcap_{j=2}^{n} N_j : M)$. This implies that $I^2 N \subseteq IN \subseteq N \cap (\bigcap_{j=2}^{n} N_j) \subseteq N_1$. Since N_1 is 1-absorbing prime, either $I^2 \subseteq (N_1 : M)$ or $N \subseteq N_1$ by Theorem 2.8. The second one gives us a contradiction. Assume $I^2 \subseteq (N_1 : M)$. Then $\sqrt{I} = \sqrt{I^2} \subseteq \sqrt{(N_1 : M)}$. As $\sqrt{(\bigcap_{j=2}^{n} N_j : M)} = \sqrt{\bigcap_{j=2}^{n} (N_j : M)} = \bigcap_{j=2}^{n} \sqrt{(N_j : M)}$, then $\bigcap_{j=2}^{n} \sqrt{(N_j : M)} \subseteq \sqrt{(N_1 : M)}$. Note that $\sqrt{(N_1 : M)}$ is a prime ideal by Proposition 2.6(1). This result gives us $\sqrt{(N_j : M)} \subseteq \sqrt{(N_1 : M)}$ for some j, which contradicts with our assumption. Consequently, N_1 is not a 1-absorbing prime submodule. □

Finally, by the help of Remark 5.3, we can express the following result.

Theorem 5.6. **(1-absorbing Prime Avoidance Theorem for Cosets)** Let $N + m \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ be a covering for $m \in M$. Suppose that at most one submodule N_i is not 1-absorbing prime. If $\sqrt{(N_j : M)} \nsubseteq \sqrt{(N_k : M)}$ for $\forall\ j \neq k$, then the submodule $N + \langle m \rangle \subseteq N_k$ for some $k = 1, 2, \ldots, n$.

Proof. By using the covering $N + m \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$, we can find an efficient covering, $N + m \subseteq N_{j_1} \cup N_{j_2} \cup \cdots \cup N_{j_t}$. Then $1 \leq t \leq n$. It follows from Proposition 5.5 that $t = 1$. Thus, we conclude $N + m \subseteq N_k$ for some $k = 1, 2, \ldots, n$. It is clear that $N + \langle m \rangle \subseteq N_k$ since $m \in N + m \subseteq N_k$. □

References

[1] D. D. Anderson, T. Arabaci, U. Tekir and S. Koc, *On S-multiplication modules*, Comm. Algebra 48(8), 3398–3407, 2020.

[2] D. F. Anderson and A. Badawi, *On (m,n)-closed ideals of commutative rings*, J. Algebra Appl. 16(1), 1750013(1–21), 2017.

[3] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*, CRC Press, 1969.

[4] A. Badawi, M. Issoual and N. Mahdou, *On n-absorbing ideals and (m,n)-closed ideals in trivial ring extensions of commutative rings*, J. Algebra Appl. 18(7), 1950123(1–19), 2019.

[5] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc. 75, 417–429, 2007.

[6] A. Barnard, *Multiplication modules*, J. Algebra 71, 174–178, 1981.

[7] A. Y. Darani and F. Soheilnia, *On 2-absorbing and weakly 2-absorbing submodules*, Thai J. Math. **9**, 577–584, 2011.

[8] J. Dauns, *Prime modules*, J. reine Angew. Math. 2, 156–181, 1978.

[9] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. Algebra 16(4), 755–779, 1988.

[10] S. Koc, U. Tekir and E. Yildiz, *On weakly 1-absorbing prime ideals*, Ricerche mat 1–16, 2021.

[11] C. P. Lu, *Prime submodules of modules*, Comm. Math. Univ. Sancti Pauli. 33, 61–69, 1984.

[12] C. P. Lu, *Unions of prime submodules*, Houston J. Math. 23(2), 203–213, 1997.

[13] R. L. McCasland and M. E. Moore, *Prime submodules*, Comm. Algebra 20, 1803–1817, 1992.

[14] H. Mostafanasab, U. Tekir, E. Y. Celikel, E. A. Ugurlu, G. Ulucak and A. Y. Darani, *Generalizations of 2-absorbing and 2-absorbing primary submodules* Hacet. J. Math. Stat. 48(4), 1001–1016, 2019.

[15] E. S. Sevim, U. Tekir and S. Koc, *S-Artinian rings and finitely S-cogenerated rings*, J. Algebra Appl. 19(3), 2050051(1–16), 2020.

[16] P. F. Smith, *Some remarks on Multiplication modules*, Arch Math. 50, 223–235, 1988.

[17] U. Tekir, *On the sheaf of modules*, Comm. Algebra 33(8), 2557–2562, 2005.

[18] G. Ulucak, U. Tekir and S. Koc, *On S-2-absorbing submodules and vn-regular modules*, An. St. Univ. Ovidius Constanta 28(2), 239–257, 2020.

[19] A. Yassine and M. J. Nikmehr, R. Nikandish, *On 1-absorbing prime ideals of commutative rings*, J. Algebra Appl. 20, 2150175(1–12), 2021.