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π-dual Baer Modules and π-dual Baer Rings

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Abstract. Let R be a ring and let M be an R-module with $\mathbf{S} = \mathrm{End}_R(M)$. A submodule N of M is said to be *projection invariant* in M (denoted $N \trianglelefteq_p M$) if $eN \subseteq N$ for all $e = e^2 \in \mathbf{S}$. We call M *π-dual Baer*, if for each $N \trianglelefteq_p M$ there exists $e^2 = e \in \mathbf{S}$ such that $\{f \in \mathbf{S} \mid f(M) \subseteq N\} = e\mathbf{S}$. A characterization of π-dual Baer modules is provided. We show that the class of π-dual Baer modules lies strictly between the classes of dual Baer modules and quasi-dual Baer modules. It is also shown that in general, the class of π-dual Baer modules is neither closed under direct sums nor closed under direct summands. The structure of π-dual Baer modules over Dedekind domains is completely determined. We conclude the paper by studying right π-dual Baer rings. We call a ring R *right π-dual Baer* if the right R-module R_R is right π-dual Baer. A characterization of this class of rings is provided. We also investigate the transfer between a base ring R and many of its extensions (for example, full matrix rings over R or $R[x]$ or $R[[x]]$). In addition, we characterize the 2-by-2 generalized triangular right π-dual Baer matrix rings.

Key Words: dual Baer module; quasi-dual Baer module; π-dual Baer module; endomorphism rings; projection invariant submodule.
2010 MSC: Primary 16D10, 16S50; Secondary 16D80.

Dedicated to the memory of Professor Muhammad Zafrullah

1 Introduction

Throughout this paper R will always be an associative ring with unity and any module will be a unital right R-module unless stated otherwise. Let M be an R-module. By $\mathbf{S} = \mathrm{End}_R(M)$ and $\mathbf{I}_M$, we denote the endomorphism ring of the module M and the subring of $\mathbf{S}$ generated by the idempotents of $\mathbf{S}$, respectively. For a ring R, we use $\mathbf{I}(R)$ to denote the subring of R generated by idempotents. The notations $N \subseteq M$, $N \leq M$ and $N \leq_d M$ mean that N is a subset of M, N is a submodule of M and N is a direct summand of M, respectively. Let $N \leq M$. Then N is called a *fully invariant* submodule of M (denoted $N \trianglelefteq M$) if $f(N) \subseteq N$ for all $f \in \mathbf{S}$, and N is called a *projection invariant* submodule of M (denoted $N \trianglelefteq_p M$) if $e(N) \subseteq N$ for all $e^2 = e \in \mathbf{S}$. Note that every fully invariant submodule is projection invariant and the projection invariant submodules of a module M form a complete sublattice of the lattice of submodules of M. One may observe that if N is fully (projection) invariant in M, then there exists a ring homomorphism $\alpha : \mathbf{S} \to \mathrm{End}_R(N)$ ($\beta : \mathbf{I}_M \to \mathbf{I}_N$) defined by $\alpha(h) = h|_N$ ($\beta(h) = h|_N$) for all $h \in \mathbf{S}$ ($h \in \mathbf{I}_M$) (see [7]). Note that a right ideal I of a ring R is called *projection invariant* in R_R (denoted $I_R \trianglelefteq_p R_R$) if $eI \subseteq I$ for all $e^2 = e \in R$. Moreover, fully invariant right ideals of R coincide with two-sided ideals of R.

The notions of Baer modules and quasi-Baer modules were introduced in 2004 (see [16]). In 2010 (see [13]), Keskin Tütüncü and Tribak dualized the notion of Baer modules. A module M is said to be *dual Baer* if for every submodule N of M, there exists an idempotent $e \in \mathbf{S}$ such that $\{f \in \mathbf{S} \mid f(M) \subseteq N\} = e\mathbf{S}$. The right ideal $\{f \in \mathbf{S} \mid f(M) \subseteq N\}$ of $\mathbf{S}$ will be denoted by $D_{\mathbf{S}}(N)$. For a subset X in $\mathbf{S}$ and a submodule N of M, let $X(N)$ denote the submodule $\sum_{f\in X} f(N)$ of M. Note that a module M is dual Baer if and only if for every subset A of $\mathbf{S}$, $A(M)$ is a direct summand of M if and only if for every right ideal A of $\mathbf{S}$, $A(M)$ is a direct summand of M (see [13, Theorem 2.1]). In 2013 (see [3]), Amouzegar and Talebi introduced the notion of quasi-dual Baer modules by dualizing the notion of quasi-Baer modules. A module M is said to be *quasi-dual Baer* if for every fully invariant submodule N of M, there exists an idempotent $e \in \mathbf{S}$ such that $D_{\mathbf{S}}(N) = e\mathbf{S}$. In [18], the authors continued the study of quasi-dual Baer modules. They showed that a module M is quasi-dual Baer if and only if for every left ideal I of $\mathbf{S}$, $I(M)$ is a direct summand of M (see [18, Proposition 2.4]).

In 2020 (see [7]), Birkenmeier, Kara and Tercan introduced the notion of π-endo Baer (π-e.Baer for short) modules. According to [7, Definition 3.3], a module M is called *π-e.Baer*, if for each $\emptyset \neq X \subseteq M$ such that $j(X) \subseteq X$ for all $j^2 = j \in \mathbf{S}$ there exists $e^2 = e \in \mathbf{S}$ such that $l_{\mathbf{S}}(X) = \{s \in \mathbf{S} \mid s(X) = 0\} = \mathbf{S}e$. By [7, Lemma 3.4], a module M is π-e.Baer if and only if for each $N \trianglelefteq_p M$, there exists $f^2 = f \in \mathbf{S}$ such that $l_{\mathbf{S}}(N) = \mathbf{S}f$ if and only if for each ${}_{\mathbf{S}}Y \trianglelefteq_p {}_{\mathbf{S}}\mathbf{S}$, there exists $e^2 = e \in \mathbf{S}$ such that $\bigcap_{g\in Y} \operatorname{Ker} g = eM$. Later in 2021, this notion was dualized by Kara (see [12]) by introducing the following definition.

Definition 1.1. A module M is called *dual π-endo Baer*, if for each $N \trianglelefteq_p M$, there exists $e^2 = e \in \mathbf{S}$ such that $D_{\mathbf{S}}(N) = e\mathbf{S}$.

Note that in [4] and [7], the authors used the terminology *endomorphism Baer* module, denoted briefly by e-Baer, for the Baer modules defined by Rizvi and Roman in [16]. The rings R for which the right R-module R_R is π-e.Baer were studied in 2018 (see [6]). It was shown in [6, Proposition 2.1] that the π-e.Baer property is left-right symmetric for any ring R. Then (right) π-e.Baer rings were called *π-Baer* rings in [6, Definition 2.2].

Motivated by all these research works ([3], [7], [12] and [13]), we continue to study dual π-endo Baer modules, but under the name *π-dual Baer modules* in this paper. We also study π-dual Baer rings. A ring R is said to be *right* (left) *π-dual Baer* if the right (left) R-module R_R (${}_RR$) is π-dual Baer. The aim of this paper is to show that some results of π-e.Baer modules and π-Baer rings have corresponding duals for π-dual Baer modules and right π-dual Baer rings. In addition, we will obtain the π-dual Baer analogues of certain results appearing in [6] or in [18].

Section 2 is devoted to the study of some basic properties of π-dual Baer modules. We provide some equivalent formulations of being a π-dual Baer module (Theorem 2.4). We show that for an indecomposable $\mathbb{Z}$-module M, M is dual Baer if and only if M is π-dual Baer if and only if M is quasi-dual Baer if and only if $M \cong \mathbb{Q}$ or $M \cong \mathbb{Z}(p^\infty)$ or $M \cong \mathbb{Z}/p\mathbb{Z}$, where p is a prime number (Proposition 2.12). We construct some examples showing that the π-dual Baer condition is strictly between the dual Baer and quasi-dual Baer conditions (Example 2.14).

In Section 3, we investigate direct sums and direct summands of π-dual Baer modules. We first provide examples showing that, in general, the π-dual Baer condition is neither preserved under direct sums nor preserved under direct summands (Examples 3.1 and 3.5). Then we prove that any projection invariant direct summand of a π-dual Baer module inherits the property (Theorem 3.6). It is also shown that if a module $M = \oplus_{i\in I} M_i$ such that $M_i \trianglelefteq_p M$ for all $i \in I$, then M is π-dual Baer if and only if M_i is π-dual Baer for all $i \in I$ (Theorem 3.8). We conclude this section by describing the structure of π-dual Baer modules over Dedekind domains (Theorem 3.15).

In Section 4, we deal with right π-dual Baer rings. We show that the class of right π-dual Baer rings lies strictly between the classes of dual Baer rings and quasi-dual Baer rings (Remark 4.14). We provide a characterization of right π-dual Baer rings (Theorem 4.15). In addition, we study the transfer of the right π-dual Baer property between a base ring R and several extensions. For example, full matrix rings over R or $R[x]$ or $R[[x]]$ (see Propositions 4.19 and 4.21, Examples 4.20 and 4.22).

We conclude the paper by characterizing the 2-by-2 generalized triangular right π-dual Baer matrix rings (Theorem 4.24).

Throughout this paper, by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ we denote the ring of integer numbers, ring of rational numbers and the Prüfer p-group, respectively where p is a prime number.

2 Some results on π-dual Baer modules

Definition 2.1. A module M is called *π-dual Baer*, if for each $N \trianglelefteq_p M$, there exists $e^2 = e \in \mathbf{S}$ such that $D_{\mathbf{S}}(N) = e\mathbf{S}$.

Example 2.2. (i) Clearly, every semisimple module is π-dual Baer.

(ii) Let M be an indecomposable module. Then 0 and 1 are the only idempotents of $\mathbf{S}$. This implies that all submodules of M are projection invariant. Therefore M is dual Baer if and only if M is π-dual Baer.

(iii) Let R be a commutative ring. Using [13, Corollary 2.9], we see that the R-module R is dual Baer if and only if it is π-dual Baer if and only if it is quasi-dual Baer if and only if R is semisimple.

Recall that an idempotent $e \in R$ is called *left semicentral* if $xe = exe$ for all $x \in R$. The set of left semicentral idempotents of R is denoted by $S_l(R)$. We begin with the following lemma which is taken from [12, Lemmas 2.1 and 2.2] and [7, Lemma 3.1(iii)]. This lemma will be used throughout the paper.

Lemma 2.3. *Let M be a module with $\mathbf{S} = \mathrm{End}_R(M)$.*

(i) *If $N \trianglelefteq_p M$, then $D_{\mathbf{S}}(N) \trianglelefteq_p \mathbf{S}_{\mathbf{S}}$.*

(ii) *If $I_{\mathbf{S}} \trianglelefteq_p \mathbf{S}_{\mathbf{S}}$, then $I(M) \trianglelefteq_p M$.*

(iii) *If I is a right ideal of $\mathbf{S}$, then $D_{\mathbf{S}}(I(M))(M) = I(M)$.*

(iv) *If $N \leq M$, then $D_{\mathbf{S}}(D_{\mathbf{S}}(N)(M)) = D_{\mathbf{S}}(N)$.*

(v) *Let $e = e^2 \in \mathbf{S}$. Then $(eM)_R \trianglelefteq_p M_R$ if and only if $(eM)_R \trianglelefteq M_R$ if and only if $e \in S_l(\mathbf{S})$.*

The following characterization of π-dual Baer modules will be used later to obtain other results in this study.

Theorem 2.4. Let M be a module. Then the following are equivalent:

(i) M is π-dual Baer;

(ii) For each $I_{\mathbf{S}} \trianglelefteq_p \mathbf{S}_{\mathbf{S}}$, $I(M)$ is a (projection invariant) direct summand of M;

(iii) For each $N \trianglelefteq_p M$, there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \leq N$, $M_1 \trianglelefteq_p M$ and $\mathrm{Hom}_R(M, N \cap M_2) = 0$;

(iv) For each $N \trianglelefteq_p M$, there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \leq N$, $M_1 \trianglelefteq M$ and $\mathrm{Hom}_R(M, N \cap M_2) = 0$;

(v) For each $N \trianglelefteq_p M$, there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \leq N$ and $\mathrm{Hom}_R(M, N \cap M_2) = 0$.

Proof. (i) $\Leftrightarrow$ (ii) This follows from [12, Proposition 2.4] and Lemma 2.3(ii).

(i) $\Rightarrow$ (iii) This implication follows by adapted the proof of [18, Proposition 2.1((i) $\Rightarrow$ (ii))] and using Lemma 2.3.

(iii) $\Rightarrow$ (iv) This follows from Lemma 2.3(v) (see also [1, Proposition 3.1(4)]).

(iv) $\Rightarrow$ (v) This is evident.

(v) $\Rightarrow$ (i) The proof of this implication is similar to that of [18, Proposition 2.1((ii) $\Rightarrow$ (i))]. □

Example 2.5. Let M be a module such that $\mathrm{Hom}_R(M,N) = 0$ for every projection invariant proper submodule N of M. Then M is π-dual Baer by Theorem 2.4. For example, the Prüfer p-group $\mathbb{Z}(p^\infty)$ and the group of rational numbers $\mathbb{Q}$ are π-dual Baer $\mathbb{Z}$-modules, where p is any prime number.

As applications of Theorem 2.4, we obtain the following corollaries.

Corollary 2.6. *Let M be a π-dual Baer module and $N \trianglelefteq_p M$. Then the following are equivalent:*

(i) *$N \leq_d M$;*

(ii) *$D_{\mathbf{S}}(N)(M) = N$.*

Proof. (i) $\Rightarrow$ (ii) Let $\pi : M \to N$ be the projection map and $i : N \to M$ be the inclusion map. Then $i\pi \in D_{\mathbf{S}}(N)$ and $i\pi(M) = N$. Hence $D_{\mathbf{S}}(N)(M) = N$.

(ii) $\Rightarrow$ (i) Since $N \trianglelefteq_p M$, $D_{\mathbf{S}}(N) \trianglelefteq_p \mathbf{S}_{\mathbf{S}}$ by Lemma 2.3(i). Applying Theorem 2.4, we get $D_{\mathbf{S}}(N)(M) \leq_d M$. Therefore $N \leq_d M$ by (ii). □

Corollary 2.7. *Let M be a module such that every projection invariant submodule of M is a direct summand of M. Then M is π-dual Baer.*

Proof. Let $I_{\mathbf{S}} \trianglelefteq_p \mathbf{S}_{\mathbf{S}}$. Then by Lemma 2.3(ii), $I(M) \trianglelefteq_p M$. So, by hypothesis, $I(M) \leq_d M$. From Theorem 2.4, it follows that M is a π-dual Baer module. □

Corollary 2.8. *Let M be an indecomposable module. Then the following are equivalent:*

(i) *M is a π-dual Baer module;*

(ii) *For every proper submodule N of M, $\mathrm{Hom}_R(M,N) = 0$.*

Proof. Since M is indecomposable, the set of all idempotents of $\mathbf{S}$ is $\{0,1\}$. Therefore all submodules of M are projection invariant.

(i) $\Rightarrow$ (ii) Let N be a proper submodule of M. By Theorem 2.4, $\mathrm{Hom}_R(M,N) = 0$.

(ii) $\Rightarrow$ (i) Let $N \trianglelefteq_p M$ with $N \neq M$. Since $\mathrm{Hom}_R(M,N) = 0$, $D_{\mathbf{S}}(N) = 0$ is a direct summand of $\mathbf{S}_{\mathbf{S}}$. If $N = M$, then $D_{\mathbf{S}}(N) = \mathbf{S}$ is again a direct summand of $\mathbf{S}_{\mathbf{S}}$. This completes the proof. □

Next, we compare the notions of dual Baer, π-dual Baer and quasi-dual Baer modules. From the definitions of these three notions, we infer the following remark.

Remark 2.9. (see also [12, Theorem 2.6]) It is easily seen that the following implications hold for a module M:

M is a dual Baer module $\Rightarrow$ M is a π-dual Baer module $\Rightarrow$ M is a quasi-dual Baer module.

Next, we provide some sufficient conditions under which these three notions coincide. Recall that a ring R is called a *right duo ring* if every right ideal of R is a two-sided ideal.

Example 2.10. Let M be a module such that $\mathbf{S} = \mathrm{End}_R(M)$ is a right duo ring. By [18, Remark 2.8], M is quasi-dual Baer if and only if M is dual Baer. Therefore from Remark 2.9, it follows that M is dual Baer if and only if M is π-dual Baer if and only if M is quasi-dual Baer.

Proposition 2.11. *Let R be a local ring with maximal right ideal m and $M = R/m$. Assume that* $\text{Rad}(E(M)) \neq E(M)$*. Then the following are equivalent:*

(i) *$E(M)$ is a dual Baer R-module;*

(ii) *$E(M)$ is a π-dual Baer R-module;*

(iii) *$E(M)$ is a quasi-dual Baer R-module;*

(iv) *R is a division ring.*

Proof. This follows directly from Remark 2.9 and [18, Corollary 2.14]. □

Proposition 2.12. *Let M be an indecomposable $\mathbb{Z}$-module. Then the following are equivalent:*

(i) *M is dual Baer;*

(ii) *M is π-dual Baer;*

(iii) *M is quasi-dual Baer;*

(iv) *$M \cong \mathbb{Q}$ or $M \cong \mathbb{Z}(p^\infty)$ or $M \cong \mathbb{Z}/p\mathbb{Z}$, where p is a prime number.*

Proof. This is clear by Remark 2.9 and [18, Corollary 3.7]. □

Combining Remark 2.9 and [18, Corollary 3.9], we obtain the following proposition.

Proposition 2.13. *Let M be a nonzero module over a commutative perfect ring R. Then the following conditions are equivalent:*

(i) *M is dual Baer;*

(ii) *M is π-dual Baer;*

(iii) *M is quasi-dual Baer;*

(iv) *M is a semisimple module.*

Next, we present some examples to show that the class of π-dual Baer modules lies properly between the class of dual Baer modules and that of quasi-dual Baer modules (see Remark 2.9).

Example 2.14. (i) Let S be a simple ring and let ${}_SN_S$ be an S-S-bimodule. Consider the generalized matrix ring $R = \begin{bmatrix} S & N \\ N & S \end{bmatrix}$ and the right R-module $M = N \oplus S$. Assume that S is a domain that is not a division ring. We know from [15, p. 1278] that $\text{End}_R(M) \cong S$ (as rings). Then $\text{End}_R(M)$ is a domain and hence M is indecomposable. Therefore all submodules of M are projection invariant. By [18, Example 2.9(ii)], M is a quasi-dual Baer module which is not dual Baer. This implies that M is a quasi-dual Baer module which is not π-dual Baer by [12, Proposition 2.8(ii)].

(ii) Let R be a ring which is a finite product of simple rings such that R is not semisimple. Then R_R is a quasi-dual Baer module by [18, Proposition 2.10]. Let F be a free R-module with a finite rank $n > 1$. Using [3, Theorem 2.7], we conclude that F is a quasi-dual Baer module. Thus F is π-dual Baer by the proof of [12, Corollary 2.9]. On the other hand, the module F is not dual Baer, since otherwise R will be semisimple by [13, Corollaries 2.5 and 2.9].

In the following result, we characterize the class of rings R for which every finitely cogenerated right R-module is π-dual Baer.

Proposition 2.15. *The following conditions are equivalent for a ring R:*

(i) *Every finitely cogenerated right R-module is π-dual Baer;*

(ii) *Every finitely cogenerated right R-module is quasi-dual Baer;*

(iii) *R is a right V-ring.*

Proof. (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (iii) Assume that R has a simple right R-module S which is not injective. Then $E(S) \neq S$. Let $M = A \oplus B$ be a right R-module such that $A \cong S$ and $B \cong E(S)$. Let $S_1 = \mathrm{Soc}(B)$. Clearly, $S_1 \cong S$. Note that $N = \mathrm{Soc}(M) = A \oplus S_1$ is an essential submodule of M that is fully invariant in M. By [18, Proposition 2.1], there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq N$ and $\mathrm{Hom}_R(M, N \cap M_2) = 0$. Since $N \neq M$, we have $M_2 \neq 0$ and hence $N \cap M_2 \neq 0$. Therefore $N \cap M_2$ contains a simple submodule S_2 with $S_2 \cong S \cong A$. It follows that $\mathrm{Hom}_R(M, N \cap M_2) \neq 0$, a contradiction. This proves that R is a right V-ring.

(iii) $\Rightarrow$ (i) This follows from the fact every finitely cogenerated right module over a right V-ring is semisimple. □

3 Direct sums and direct summands of π-dual Baer modules

A direct sum of π-dual Baer modules may not be π-dual Baer as we see in the following example. Another example is provided in [12, Example 2.13].

Example 3.1. Let L be a simple R-module such that the injective hull of L has no maximal submodules. It is shown in [18, Example 2.17] that the module $M = E(L) \oplus L$ is not quasi-dual Baer. Thus M is not π-dual Baer (see Remark 2.9). Now let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and quotient field K. It is well known that $K/R \cong E(R/\mathfrak{m})$. Therefore the R-module $(K/R) \oplus (R/\mathfrak{m})$ is not π-dual Baer. On the other hand, note that both K/R and $R/\mathfrak{m}$ are π-dual Baer by [13, Theorem 3.4].

Next, we deal with a special case of direct sums of π-dual Baer modules. First, we include the following lemma which will be useful to our work in this paper.

Lemma 3.2. [7, Lemma 3.1]

(i) *Let $X_R \leq N_R \leq M$. Then $X \trianglelefteq_p N \trianglelefteq_p M$ implies that $X \trianglelefteq_p M$.*

(ii) *Let $M = \oplus_{i \in I} M_i$ and $X_R \trianglelefteq_p M_R$. Then $X = \oplus_{i \in I}(X \cap M_i)$ and $X \cap M_i \trianglelefteq_p M_i$ for all $i \in I$.*

Theorem 3.3. Let M be a π-dual Baer module. Then every direct sum of copies of M is a π-dual Baer module.

Proof. Let $N = \oplus_{i \in I} M_i$ such that $M_i \cong M$ for all $i \in I$. Let $X \trianglelefteq_p N$. By Lemma 3.2(ii), we have $X = \oplus_{i \in I}(X \cap M_i)$ and $X \cap M_i \trianglelefteq_p M_i$ for all $i \in I$. Fix $i \in I$. Since M_i is π-dual Baer, there exists a decomposition $M_i = K_i \oplus L_i$ with $K_i \leq X \cap M_i$ and $\mathrm{Hom}_R(M_i, X \cap L_i) = 0$ by Theorem 2.4. Put $K = \oplus_{i \in I} K_i$ and $L = \oplus_{i \in I} L_i$. Clearly, $M = K \oplus L$ and $K \subseteq X$. Moreover, we have $X \cap L = \oplus_{i \in I}(X \cap L_i)$. Now assume that $\mathrm{Hom}_R(M, X \cap L) \neq 0$. Then there exist $i, j \in I$ such that $\mathrm{Hom}_R(M_i, X \cap L_j) \neq 0$. But $M_j \cong M_i$. So $\mathrm{Hom}_R(M_j, X \cap L_j) \neq 0$, a contradiction. Hence $\mathrm{Hom}_R(M, X \cap L) = 0$. Applying again Theorem 2.4, it follows that N is a π-dual Baer module. □

The following corollary is an immediate consequence of Theorem 3.3.

Corollary 3.4. *Let R be a ring such that R_R is a right π-dual Baer R-module. Then all free right R-modules are π-dual Baer.*

Note that both the class of dual Baer modules and the class of quasi-dual Baer modules are closed under direct summands (see [13, Corollary 2.5] and [18, Corollary 2.5]). However, the following example illustrates that being π-dual Baer is not preserved by taking direct summands.

Example 3.5. Let R be a simple ring which is a domain but not a division ring. From [18, Proposition 2.10], we infer that R_R is a quasi-dual Baer R-module. On the other hand, R_R is not a π-dual Baer module by [12, Proposition 2.8(ii)] and [13, Corollary 2.9]. Now consider a free right R-module $F_R = \oplus_{i=1}^n R_i$ for some integer $n > 1$, where $R_i \cong R$ for all $1 \le i \le n$. Note that F is quasi-dual Baer by [3, Theorem 2.7]. Then F is π-dual Baer by [12, Corollary 2.9].

As an application of Theorem 2.4, we can improve and generalize Proposition 2.11 of [12] as follows. The proof and the techniques used are different from those of [12, Proposition 2.11].

Theorem 3.6. Let $M = M_1 \oplus M_2$ be a π-dual Baer module for some submodules M_1 and M_2 of M. If $M_1 \trianglelefteq_p M$, then M_1 and M_2 are π-dual Baer.

Proof. Let us first prove that M_1 is π-dual Baer. Take $N_1 \trianglelefteq_p M_1$. Then $N_1 \trianglelefteq_p M$ by Lemma 3.2(i). Since M is π-dual Baer, there exists a decomposition $M = K_1 \oplus K_2$ with $K_1 \le N_1$ and $\mathrm{Hom}_R(M, N_1 \cap K_2) = 0$ (see Theorem 2.4). By modularity, we have $M_1 = K_1 \oplus (K_2 \cap M_1)$. Moreover, $N_1 \cap (K_2 \cap M_1) = N_1 \cap K_2$. It is clear that $\mathrm{Hom}_R(M_1, N_1 \cap K_2) = 0$. Using Theorem 2.4, we deduce that M_1 is π-dual Baer. To show that M_2 is π-dual Baer, take $N_2 \trianglelefteq_p M_2$. Then $N = M_1 \oplus N_2 \trianglelefteq_p M$ by [5, Lemma 4.13]. So there exist submodules K and L of M such that $M = K \oplus L$, $K \subseteq N$, $K \trianglelefteq_p M$ and $\mathrm{Hom}_R(M, N \cap L) = 0$ (see Theorem 2.4). Note that $K = (K \cap M_1) \oplus (K \cap M_2)$ by Lemma 3.2(ii). Hence $M = (K \cap M_1) \oplus (K \cap M_2) \oplus L$ and so $M_2 = (K \cap M_2) \oplus [((K \cap M_1) \oplus L) \cap M_2]$. In addition, it is clear that $K \cap M_2 = K \cap N_2 \subseteq N_2$ as $K \subseteq N$. Thus $N_2 = (K \cap N_2) \oplus [((K \cap M_1) \oplus L) \cap N_2]$. Moreover, since $M = (K \cap M_1) \oplus (K \cap N_2) \oplus L$, it follows that $N = (K \cap M_1) \oplus (K \cap N_2) \oplus (N \cap L)$ by modularity. Therefore $N_2 = (K \cap N_2) \oplus [((K \cap M_1) \oplus (N \cap L)) \cap N_2]$. Note that $((K \cap M_1) \oplus (N \cap L)) \cap N_2 \subseteq ((K \cap M_1) \oplus L) \cap N_2$. Then $((K \cap M_1) \oplus (N \cap L)) \cap N_2 = ((K \cap M_1) \oplus L) \cap N_2$. Now assume that $\mathrm{Hom}_R(M_2, N_2 \cap [((K \cap M_1) \oplus L) \cap M_2)]) \neq 0$ and let $f : M_2 \to ((K \cap M_1) \oplus (N \cap L)) \cap N_2$ be a nonzero homomorphism. Let $\pi : (K \cap M_1) \oplus (N \cap L) \to N \cap L$ be the projection map. It is easy to check that $0 \neq \pi f \in \mathrm{Hom}_R(M_2, N \cap L)$. This contradicts the fact that $\mathrm{Hom}_R(M, N \cap L) = 0$. From Theorem 2.4, we infer that M_2 is a π-dual Baer module. □

Proposition 3.7. *Let $M = M_1 \oplus M_2$ for some submodules M_1 and M_2 of M. If M is a π-dual Baer module with $\mathbf{I}_{M_1} = \mathrm{End}_R(M_1)$, then M_1 is π-dual Baer.*

Proof. By Remark 2.9, M is quasi-dual Baer. So M_1 is quasi-dual Baer by [18, Corollary 2.5]. Therefore M_1 is π-dual Baer by [12, Proposition 2.8(iv)]. □

Combining [12, Theorem 2.14] and Lemma 2.3(v), we obtain the following theorem. By using Theorem 2.4, we next provide another proof of this result.

Theorem 3.8. Let $M = \oplus_{i \in I} M_i$, where $M_i \trianglelefteq_p M$ for all $i \in I$. Then M is π-dual Baer if and only if M_i is π-dual Baer for all $i \in I$.

Proof. Assume that M is π-dual Baer. By Theorem 3.6, each M_i $(i \in I)$ is π-dual Baer. Conversely, assume that each M_i is π-dual Baer. By Lemma 2.3(v), $M_i \trianglelefteq M$ for all $i \in I$. So, $\mathrm{Hom}_R(M_i, M_j) = 0$ for all $i \neq j \in I$. Let $N \trianglelefteq_p M$. Thus $N = \oplus_{i \in I}(N \cap M_i)$ and $N \cap M_i \trianglelefteq_p M_i$ for all $i \in I$ by Lemma 3.2(ii). Fix $i \in I$. By Theorem 2.4, there exists a decomposition $M_i = K_i \oplus L_i$ with $K_i \subseteq N \cap M_i$ and $\mathrm{Hom}_R(M_i, N \cap L_i) = 0$. Set $K = \oplus_{i \in I} K_i$ and $L = \oplus_{i \in I} L_i$. Clearly, $M = K \oplus L$ and $K \subseteq N$. Moreover, it is easy to see that $N \cap L = \oplus_{i \in I}(N \cap L_i)$. Combining the facts that $\mathrm{Hom}_R(M_i, M_j) = 0$ for all $i \neq j \in I$ and $\mathrm{Hom}_R(M_i, N \cap L_i) = 0$ for all $i \in I$, we conclude that $\mathrm{Hom}_R(M, N \cap L) = 0$. Using Theorem 2.4, it follows that M is π-dual Baer. □

Let M be a module. The radical of M will be denoted by Rad(M). Note that Rad(M) is a fully invariant submodule of M by [2, Proposition 9.14]. Clearly, if M is semisimple, then $\mathrm{Rad}(M) = 0$.

Corollary 3.9. *Let an R-module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 such that $\mathrm{Rad}(M_1) = M_1$, M_2 is semisimple. If M is π-dual Baer, then M_1 is π-dual Baer. The converse holds when $\mathrm{Hom}_R(M_2, M_1) = 0$.*

Proof. Note that $\mathrm{Rad}(M) = \mathrm{Rad}(M_1) \oplus \mathrm{Rad}(M_2) = M_1 \trianglelefteq M$.

($\Rightarrow$) This follows by Theorem 3.6.

($\Leftarrow$) Since $\mathrm{Hom}_R(M_2, M_1) = 0$, $M_2 \trianglelefteq M$. Now the result follows from Theorem 3.8. □

For the proof of the implication (i) $\Rightarrow$ (ii) in the following proposition, we mainly follow the proof of [18, Proposition 2.15((i) $\Rightarrow$ (ii))].

Proposition 3.10. *Let an R-module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 such that $\mathrm{Rad}(M_1) = M_1$ and M_2 is semisimple. Then the following are equivalent:*

(i) *M is π-dual Baer;*

(ii) *M_1 is π-dual Baer and $I(M_2) \cap M_1 \subseteq I(M_1)$ for all $I_{\mathbf{S}} \trianglelefteq_p \mathbf{S}_{\mathbf{S}}$.*

Proof. (i) $\Rightarrow$ (ii) By Corollary 3.9, M_1 is π-dual Baer. Now we will prove that $I(M_2) \cap M_1 \subseteq I(M_1)$ for all $I_{\mathbf{S}} \trianglelefteq_p \mathbf{S}_{\mathbf{S}}$. Let $I_{\mathbf{S}} \trianglelefteq_p \mathbf{S}_{\mathbf{S}}$. By Lemma 2.3(ii), $I(M_1) + I(M_2) = I(M) \trianglelefteq_p M$. Hence $I(M) = (I(M) \cap M_1) \oplus (I(M) \cap M_2)$ by Lemma 3.2(ii). As $M_1 \trianglelefteq M$, we have $I(M_1) \subseteq M_1$. By modularity, $M_1 \cap I(M) = M_1 \cap (I(M_1) + I(M_2)) = I(M_1) + (M_1 \cap I(M_2))$. Since $M_1 \cap I(M_2)$ is semisimple, there exists a semisimple submodule N of $M_1 \cap I(M_2)$ such that $I(M_1) + (M_1 \cap I(M_2)) = I(M_1) \oplus N$. Therefore $I(M) = (I(M) \cap M_1) \oplus (I(M) \cap M_2) = I(M_1) \oplus N \oplus (I(M) \cap M_2)$. Now by Theorem 2.4, $I(M) = I(M_1) \oplus N \oplus (I(M) \cap M_2) \leq_d M$. Thus $N \leq_d M_1$ and so $\mathrm{Rad}(N) = N \cap \mathrm{Rad}(M_1) = N \cap M_1 = N$. On the other hand, we have $\mathrm{Rad}(N) = 0$ since N is semisimple. Therefore $N = 0$. This implies that $I(M_1) + (M_1 \cap I(M_2)) = I(M_1)$. Consequently, $I(M_2) \cap M_1 \subseteq I(M_1)$.

(ii) $\Rightarrow$ (i) Let $N \trianglelefteq_p M$. Then $N = (N \cap M_1) \oplus (N \cap M_2)$ and $N \cap M_1 \trianglelefteq_p M_1$ (see Lemma 3.2(ii)). Since M_1 is π-dual Baer, there exist submodules K_1 and L_1 of M_1 such that $M_1 = K_1 \oplus L_1$, $K_1 \subseteq N \cap M_1$ and $\mathrm{Hom}_R(M_1, N \cap L_1) = 0$ (see Theorem 2.4). Since M_2 is semisimple, there exists a submodule $L_2 \leq M_2$ such that $M_2 = (N \cap M_2) \oplus L_2$. Put $K = K_1 \oplus (N \cap M_2)$ and $L = L_1 \oplus L_2$. Then $M = K \oplus L$ with $K \subseteq N$. It is easily seen that $N \cap L = (N \cap L_1) \oplus (N \cap L_2)$. But $N \cap L_2 = 0$, so $N \cap L = N \cap L_1$. Applying Theorem 2.4, it remains to prove that $\mathrm{Hom}_R(M, N \cap L_1) = 0$. Let $f \in \mathrm{Hom}_R(M, N \cap L_1)$ and consider the ideal $I = \mathbf{S} f \mathbf{S}$ of $\mathbf{S}$. By (ii), $I(M_2) \cap M_1 \subseteq I(M_1)$. Note that $f(M_1) = 0$ as $\mathrm{Hom}_R(M_1, N \cap L_1) = 0$. Since $M_1 \trianglelefteq M$, we have $I(M_1) = 0$. Therefore $I(M_2) \cap M_1 = 0$ and hence $f(M_2) \cap M_1 = f(M_2) = 0$. It follows that $f = 0$, as desired. □

Next, we provide a characterization of π-dual Baer modules over a commutative semilocal ring. But first we need a lemma.

Lemma 3.11. *Let M be a π-dual Baer module over a commutative ring R. Then $M\mathfrak{a}$ is a direct summand of M for any ideal $\mathfrak{a}$ of R.*

Proof. This follows from Remark 2.9 and [18, Proposition 3.3]. □

Proposition 3.12. *Let M be a nonzero module over a commutative semilocal ring R. Then the following are equivalent:*

(i) *M is π-dual Baer;*

(ii) *$M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2 such that* $\mathrm{Rad}(M_1) = M_1$ *is π-dual Baer and M_2 is semisimple, and $I(M_2) \cap M_1 \subseteq I(M_1)$ for every $I_{\mathbf{S}} \trianglelefteq_p \mathbf{S}_{\mathbf{S}}$.*

Proof. (i) $\Rightarrow$ (ii) By Lemma 3.11 and the proof of [18, Theorem 3.8], the module M has a decomposition $M = M_1 \oplus M_2$ such that $\mathrm{Rad}(M_1) = M_1$ and M_2 is semisimple. The result now follows from Proposition 3.10.

(ii) $\Rightarrow$ (i) This is clear by Proposition 3.10. □

In the remainder of this section we assume that R is a Dedekind domain with quotient field Q such that $Q \neq R$. Let M be an R-module. The set $T(M) = \{x \in M \mid xr = 0$ for some nonzero $r \in R\}$ is a submodule of M which is called the *torsion submodule* of M. The module M is said to be *torsion* (resp., *torsion-free*) if $T(M) = M$ (resp., $T(M) = 0$). Let $\mathbb{P}$ denote the set of all nonzero prime ideals of R. For any $0 \neq \mathfrak{p} \in \mathbb{P}$, let $T_{\mathfrak{p}}(M)$ denote the set $\{x \in M \mid \mathfrak{p}^n x = 0$ for some integer $n \geq 0\}$ which is called the $\mathfrak{p}$-*primary component* of M. The module M is called $\mathfrak{p}$-*primary* if $T_{\mathfrak{p}}(M) = M$. It is well known that if M is a torsion R-module, then M is a direct sum of its $\mathfrak{p}$-primary components. The $\mathfrak{p}$-primary component of the torsion R-module Q/R will be denoted by $R(\mathfrak{p}^\infty)$.

Next, we aim to describe the structure of quasi-dual Baer modules and π-dual Baer modules over Dedekind domains. First, we prove the following needed lemmas.

Lemma 3.13. *Let M be a nonzero torsion-free R-module. If M is quasi-dual Baer, then M is an injective module.*

Proof. Assume that M is quasi-dual Baer and let $0 \neq s \in R$. By [18, Proposition 3.3], there exists a submodule K of M such that $M = sM \oplus K$. Hence $sK = 0$. Therefore $K = 0$ since M is torsion-free. Thus $M = sM$. Hence M is a divisible R-module. By [17, Proposition 2.7], it follows that M is injective. □

Lemma 3.14. *Let M be a torsion R-module. Assume that M is quasi-dual Baer. Then $M = E \oplus F$ is a direct sum of an injective submodule E and a semisimple submodule F.*

Proof. By [18, Corollary 2.5], every primary component $T_{\mathfrak{p}}(M)$ is quasi-dual Baer. Note that every direct sum of injective R-modules is injective since R is a noetherian ring. So without loss of generality we can assume that $M = T_{\mathfrak{p}}(M)$ for some nonzero prime ideal $\mathfrak{p}$ of R. Since $\mathfrak{p}M \trianglelefteq M$, there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq \mathfrak{p}M$ and $\mathrm{Hom}_R(M, \mathfrak{p}M \cap M_2) = 0$ (see [18, Proposition 2.1]). Then $\mathfrak{p}M = M_1 \oplus (\mathfrak{p}M \cap M_2)$ by modularity. Moreover, we have $\mathfrak{p}M = \mathfrak{p}M_1 \oplus \mathfrak{p}M_2$. Therefore $\mathfrak{p}M_1 = M_1$ and $\mathfrak{p}M \cap M_2 = \mathfrak{p}M_2$. Thus $\mathrm{Hom}_R(M_2, \mathfrak{p}M_2) = 0$. This implies that $rM_2 = 0$ for all $r \in \mathfrak{p}$, that is, $\mathfrak{p}M_2 = 0$. Hence M_2 is a semisimple module. Moreover, we have $M_1 = \mathfrak{p}M = \mathrm{Rad}(M)$ and $M = \mathfrak{p}M \oplus M_2$. It follows that $\mathfrak{p}M = \mathfrak{p}(\mathfrak{p}M)$. This yields $\mathrm{Rad}(M) = \mathrm{Rad}(\mathrm{Rad}(M))$. Since R is a Dedekind domain, we see that $\mathrm{Rad}(M) = M_1$ is injective. This completes the proof. □

For an R-module M, we will denote the sum of all divisible (injective) submodules of M by $d(M)$. It is well known that $d(M)$ is an injective fully invariant submodule of M. It is shown in [11, Theorem 7] that every injective R-module is a direct sum of copies of Q and $R(\mathfrak{p}^\infty)$ for various nonzero prime ideals $\mathfrak{p}$. An R-module M is said to be *reduced* if M has no divisible submodules (that is $d(M) = 0$).

Theorem 3.15. Let R be a Dedekind domain with quotient field Q such that $Q \neq R$. Then the following assertions are equivalent for an R-module M:

(i) M is dual Baer;

(ii) M is π-dual Baer;

(iii) M is quasi-dual Baer;

(iv) M is a direct sum of copies of Q, $(R(\mathfrak{p}_i^{\infty}))_{i\in I}$ and $(R/\mathfrak{q})_{j\in J}$, where $(\mathfrak{p}_i)_{i\in I}$ and $(\mathfrak{q})_{j\in J}$ are nonzero prime ideals of R with $\mathfrak{p}_i \neq \mathfrak{q}_j$ for every couple $(i,j) \in I \times J$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) See Remark 2.9.

(iii) $\Rightarrow$ (iv) Since $d(M)$ is injective, it follows that $M = d(M) \oplus L$ for some reduced submodule L of M. Note that $d(M)$ and L are quasi-dual Baer by [18, Corollary 2.5]. Since $T(L) \trianglelefteq L$, there exists a decomposition $L = N \oplus K$ with $N \subseteq T(L)$ and $\mathrm{Hom}_R(L, T(L) \cap K) = 0$ (see [18, Proposition 2.1]). But $T(L) \cap K = T(K)$. Then $\mathrm{Hom}_R(L, T(K)) = 0$. Now assume that $T(K) \neq 0$. Then K has a direct summand K_0 which is isomorphic to $R/\mathfrak{p}^n$ for some nonzero prime ideal $\mathfrak{p}$ of R and some positive integer n (see [11, Theorem 9]). Since $K_0 \subseteq T(K)$, we have $\mathrm{Hom}_R(K, T(K)) \neq 0$. Hence $\mathrm{Hom}_R(L, T(K)) \neq 0$, a contradiction. Therefore $T(K) = 0$ and so $T(L) = N$. Using again [18, Corollary 2.5], we infer that N and K are quasi-dual Baer. Now taking into account Lemmas 3.13 and 3.14, we conclude that $K = 0$ and $N = L$ is semisimple. Note that $d(M)$ is a direct sum of copies of Q and $R(\mathfrak{p}^{\infty})$ for various nonzero prime ideals $\mathfrak{p}$. Moreover, for each nonzero prime ideal $\mathfrak{p}$ of R, the R-module $R(\mathfrak{p}^{\infty}) \oplus R/\mathfrak{p}$ is not quasi-dual Baer by [18, Example 2.17]. Now (iv) follows from the fact that the class of quasi-dual Baer modules is closed under direct summands (see [18, Corollary 2.5]).

(iv) $\Rightarrow$ (i) This follows from [13, Theorem 3.4]. □

4 π-dual Baer Rings

We will call a ring R a *right π-dual Baer* (resp., *right dual Baer*) ring if the right R-module R_R is π-dual Baer (resp., *dual Baer*). Following [18], a ring R is called a *right quasi-dual Baer* ring if the right R-module R_R is a quasi-dual Baer module. Left π-dual Baer rings, left dual Baer rings and left quasi-dual Baer rings are defined similarly. It was shown in [13, Corollary 2.9] and [18, Corollary 2.11] that dual Baer and quasi-dual Baer properties are left-right symmetric for any ring R. Moreover, the dual Baer rings are exactly the semisimple rings and the class of quasi-dual Baer rings is precisely the class of finite product of simple rings. This implies that a commutative ring R is (right) π-dual Baer if and only if R is semisimple. We begin by characterizing right π-dual Baer rings in some special cases.

Recall that a ring R is called *Abelian* if every idempotent of R is central.

Remark 4.1. (i) Let R be an Abelian ring. By [12, Proposition 2.8(iii)], we infer that R is a right π-dual Baer ring if and only if R is a left π-dual Baer ring if and only if R is a semisimple ring.

(ii) Let R be a ring with $\mathbf{I}(R) = R$. Combining [12, Proposition 2.8(iv)] with [18, Proposition 2.10], we conclude that R is a right π-dual Baer ring if and only if R is a left π-dual Baer ring if and only if R is a quasi-dual Baer ring if and only if R is a finite product of simple rings.

Recall that a ring R is called *projection invariant Baer* (or π-Baer) if for each ${}_RY \trianglelefteq_p {}_RR$, there exists $c^2 = c \in R$ such that $r_R(Y) = \{r \in R \mid Yr = 0\} = cR$ (see [6, Definition 2.2]). It is proven in [6] that π-Baer condition for a ring is left-right symmetric. Therefore R is π-Baer if and only if for each $Y_R \trianglelefteq_p R_R$, there exists $c^2 = c \in R$ such that $l_R(Y) = \{r \in R \mid rY = 0\} = Rc$.

Next, we compare the class of right π-dual Baer rings and that of π-Baer rings.

Remark 4.2. From [12, Proposition 3.1], it follows that every right or left π-dual Baer ring R is a π-Baer ring.

Remark 4.3. It was shown in [6, Corollary 2.2(ii)] that if R is a π-Baer ring and S is a subring of R with $\mathbf{I}(R) \subseteq S$, then S is π-Baer. The analogue of this fact is not true, in general, for right π-dual Baer rings. To see this, consider the ring $\mathbb{Q}$ which is (right) π-dual Baer. However, since the subring $\mathbb{Z}$ of $\mathbb{Q}$ is not semisimple, the ring $\mathbb{Z}$ is not (right) π-dual Baer even if $\mathbf{I}(\mathbb{Q}) = \mathbb{Z}$ (see Remark 4.1(i)).

Note that a ring R is a domain if and only if it is π-Baer and 0 and 1 are its only idempotents. In the following example, we present some rings which are π-Baer, but not right π-dual Baer.

Example 4.4. Let R be a π-Baer ring such that R is not semisimple and the right R-module R_R is indecomposable. Then R cannot be right π-dual Baer by Remark 4.1(i). Explicit examples are:

(i) Let R be the free ring $\mathbb{Z} < x, y >$. Since R is a domain, R is a π-Baer ring (see [6, Example 2.1]). On the other hand, the ring R is not semisimple.

(ii) Let A be a prime ring such that $Z(A_A) \neq 0$, $Z(A_A) \neq A$ and A_A is a uniform module (see specific examples in [8, Example 4.3]). Thus A is not a domain and $\{0, 1\}$ is the set of all idempotent elements of A. Therefore A is not a π-Baer ring. Now let $R = \mathbf{Mat}_n(A)$ be the n-by-n full matrix ring over A for some integer $n > 1$. It is well known that $\mathbf{I}(R) = R$. Moreover, by [6, Example 2.2], R is a π-Baer ring. On the other hand, suppose that the ring R is right π-dual Baer. Then R is quasi-dual Baer (see Remark 4.1(ii)). Hence A is also quasi-dual Baer (see Proposition 4.23 below). Using [18, Proposition 2.10], we deduce that A is a simple ring since A_A is indecomposable. This contradicts the fact that $Z(A_A) \neq 0$ and $Z(A_A) \neq A$. This proves that R is not a right π-dual Baer ring.

Lemma 4.5. *Let e be a central idempotent in a ring R. Then eR is π-dual Baer as a right R-module if and only if eR is π-dual Baer as a right eR-module.*

Proof. This follows directly from Theorem 2.4. □

Proposition 4.6. *Assume that R is a right π-dual Baer ring and let $e^2 = e \in R$. If $eR \trianglelefteq_p R_R$, then e and $1-e$ are central idempotents. Moreover, $eR = eRe$ and $(1-e)R = (1-e)R(1-e)$ are right π-dual Baer rings.*

Proof. Note that R is quasi-dual Baer. Thus R is a semiprime ring by the proof of [18, Proposition 2.10((iii) $\Rightarrow$ (iv))]. Since $eR \trianglelefteq_p R_R$, eR is a two-sided ideal of R by Lemma 2.3(v). Now using [10, Lemma 3.1], it follows that e is central. So $1-e$ is also central. The last assertion follows directly by applying Theorem 3.6 and Lemma 4.5. □

Proposition 4.7. *For a ring R, the following are equivalent:*

(i) *R is a right π-dual Baer ring;*

(ii) *Every projection invariant right ideal of R is a direct summand of R_R;*

(iii) *Every projection invariant right ideal of R is a two-sided ideal of R and R is a quasi-dual Baer ring.*

Proof. Given $a \in R$, let $\varphi_a : R \to R$ be the R-endomorphism of R_R defined by $\varphi_a(x) = ax$ for all $x \in R$.

(i) $\Rightarrow$ (ii) Let $I_R \trianglelefteq_p R_R$. Define the set $\mathcal{I} = \{\varphi_a : a \in I\}$. It is not hard to see that $\mathcal{I}$ is a right ideal of $\mathbf{S} = \mathrm{End}_R(R_R)$. Moreover, $\mathcal{I}_{\mathbf{S}} \trianglelefteq_p \mathbf{S}_{\mathbf{S}}$. To see this, let $e^2 = e \in \mathbf{S}$. Then $e = \varphi_{e(1)}$ and $e(1)$ is an idempotent in R. Hence $e(1)I \subseteq I$. Now let $\varphi_b \in \mathcal{I}$, where $b \in I$. Then $\varphi_{e(1)}\varphi_b = \varphi_{e(1)b} \in \mathcal{I}$. Therefore $e\mathcal{I} \subseteq \mathcal{I}$. It follows that $\mathcal{I}_{\mathbf{S}} \trianglelefteq_p \mathbf{S}_{\mathbf{S}}$. Now by Theorem 2.4, $\mathcal{I}(R_R) = \sum_{a \in I} \varphi_a(R) = \sum_{a \in I} aR = I \leq_d R_R$.

(ii) $\Rightarrow$ (iii) Note that every two-sided ideal of R is a direct summand of R_R. Thus R is a quasi-dual Baer ring by [18, Proposition 2.10]. Let $I_R \trianglelefteq_p R_R$. By (ii), $I \leq_d R_R$. Hence there exists an idempotent $e \in R$ such that $I = eR$. By Lemma 2.3(v), I is fully invariant in R_R and hence I is a two-sided ideal of R.

(iii) $\Rightarrow$ (i) Let $I_R \trianglelefteq_p R_R$. By (iii), I is a two-sided ideal of R. Therefore $I \leq_d R_R$ by [18, Proposition 2.10]. Hence R is a right π-dual Baer ring by Corollary 2.7. □

Proposition 4.8. *Let $\{R_i : i \in I\}$ be a family of rings. Then the direct product $R = \prod_{i \in I} R_i$ is a right π-dual Baer ring if and only if the indexing set I is finite and each R_i is right π-dual Baer.*

Proof. Using Theorem 3.8 and Lemma 4.5, we are reduced to proving that if R is right π-dual Baer, then I is a finite set. Suppose that R is right π-dual Baer. Assume that I is not finite. Note that $A = \oplus_{i\in I} R_i$ is a two-sided ideal of the ring R. Hence the right ideal A is a direct summand of R_R by Proposition 4.7. Therefore $R_R = A \oplus X$ for some proper right ideal X of R. This is impossible. It follows that I is a finite set. □

To obtain another characterization of right π-dual Baer rings, we introduce the following type of rings which is a stronger form of simple rings.

Definition 4.9. A ring R is said to be a *right* (*left*) *π-simple ring* if 0 and R are the only projection invariant right (left) ideals in R.

It is clear that any right π-simple ring is a simple ring which is right π-dual Baer.

Lemma 4.10. *Let R be a simple ring. Then the following conditions are equivalent:*

(i) *R is a right π-dual Baer ring;*

(ii) *R is a right π-simple ring.*

Proof. (i) ⇒ (ii) Let $I_R \trianglelefteq_p R_R$. By Proposition 4.7, I is a two-sided ideal of R. Since R is a simple ring, it follows that $I = 0$ or $I = R$.

(ii) ⇒ (i) This is immediate. □

In the next example, we exhibit some right π-simple rings.

Example 4.11. Let R be a simple ring such that $\mathbf{I}(R) = R$. Then R is a right and left π-dual Baer ring by Remark 4.1(ii). Therefore R is a right and left π-simple ring by Lemma 4.10. For example, if R' is a simple ring and $n > 1$ is a positive integer, then $\mathbf{Mat}_n(R')$ is a simple ring by [14, Theorem 3.1]. Moreover, we have $\mathbf{I}(\mathbf{Mat}_n(R')) = \mathbf{Mat}_n(R')$. It follows that $\mathbf{Mat}_n(R')$ is a right and left π-simple ring.

Proposition 4.12. *Let R be a right π-simple ring. Then either R is a division ring or R has a non-trivial idempotent element.*

Proof. Assume that R has no idempotent element except 0 and 1. Then clearly every right ideal of R is projection invariant. Since R is right π-simple, it follows that R is a division ring. □

Next, we present some simple rings which are not right π-simple.

Example 4.13. Let R be a simple ring that is not a division ring which has no idempotent element except 0 and 1. Then R is not a right π-simple ring by Proposition 4.12. As explicit examples, we can take:

(a) Weyl algebras, $A_n(F)$, over a field F of characteristic zero (see [14, Corollary 3.17]), or

(b) the Zalesskii-Neroslavskii example (see, for example [9, Example 14.17]).

Remark 4.14. By Remark 2.9, the following implications hold for any ring R:

R is a (right) dual Baer ring ⇒ R is a right π-dual Baer ring ⇒ R is a (right) quasi-dual Baer ring.

The following examples show that these implications are not reversible, in general:

(i) Let R be a simple ring which is not semisimple (see [14]) and let $n > 1$ be a positive integer. Then $\mathbf{Mat}_n(R)$ is a right π-dual Baer ring by Lemma 4.10 and Example 4.11. Let e be the matrix unit E_{11} in $\mathbf{Mat}_n(R)$. Then the rings $e\mathbf{Mat}_n(R)e$ and R are isomorphic (see [14, Example 21.14]). Now using [14, Corollary 21.13], we see that the ring $\mathbf{Mat}_n(R)$ is not semisimple. Hence $\mathbf{Mat}_n(R)$ is not a (right) dual Baer ring by [13, Corollary 2.9].

(ii) Using [18, Proposition 2.10] and Lemma 4.10, it follows easily that the rings given in Example 4.13(a)-(b) are quasi-dual Baer, but not right π-dual Baer.

Theorem 4.15. For a ring R, the following are equivalent:

(i) R is a right π-dual Baer ring;

(ii) R is a finite product of right π-simple rings.

Proof. (i) $\Rightarrow$ (ii) Assume that R is a right π-dual Baer ring. Then R is a (right) quasi-dual Baer ring by Remark 4.14. By [18, Proposition 2.10], there exist nonzero two-sided ideals $R_1, \ldots, R_n$ of R for some positive integer n such that $R = R_1 \oplus \cdots \oplus R_n$ and each R_i $(1 \le i \le n)$ is a simple ring. By [2, Proposition 7.6], there exist pairwise orthogonal central idempotents $e_1, \ldots, e_n \in R$ with $1 = e_1 + \cdots + e_n$, and $R_i = e_i R$ for every $i = 1, \ldots, n$. From Proposition 4.6, it follows that each R_i $(1 \le i \le n)$ is a right π-dual Baer ring. Now using Lemma 4.10, we infer that each R_i $(1 \le i \le n)$ is a right π-simple ring.

(ii) $\Rightarrow$ (i) This follows from Proposition 4.8 and Lemma 4.10. □

Remark 4.16. It would be desirable to investigate if the property of being a π-dual Baer ring is left-right symmetric but we have not been able to do this. Note that from Theorem 4.15, it follows that the π-dual Baer ring property is left-right symmetric if and only if so is the π-simple ring property.

Let R be a ring. For each $A \subseteq R$, the right annihilator of A in R is

$$r_R(A) = \{r \in R \mid ar = 0 \text{ for all } a \in A\}.$$

In the next proposition, we provide a necessary condition for a ring to be right π-simple.

Proposition 4.17. *Let R be a right π-simple ring. Then for every nonzero projection invariant left ideal I of R, we have $r_R(I) = 0$.*

Proof. Note that R is a right π-dual Baer ring by Theorem 4.15. Then R is a π-Baer ring by Remark 4.2. Let $0 \neq {}_RI \trianglelefteq_p {}_RR$. Then $r_R(I) \trianglelefteq_p R_R$ by [6, Lemma 2.1]. Since R is right π-simple, we have $r_R(I) = 0$ or $r_R(I) = R$. But $I \neq 0$. So $r_R(I) = 0$. □

Proposition 4.18. *Let R be a ring with* $\mathrm{Soc}(R_R)$ *essential in R_R. Then the following are equivalent:*

(i) *R is a dual Baer ring;*

(ii) *R is a right π-dual Baer ring;*

(iii) *R is a quasi-dual Baer ring;*

(iv) *R is a semisimple ring.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear by Remark 4.14.

(iii) $\Rightarrow$ (iv) Note that $\mathrm{Soc}(R_R)$ is a two-sided ideal of R. Then $\mathrm{Soc}(R_R)$ is a direct summand right ideal of R by [18, Proposition 2.10]. Hence $R = \mathrm{Soc}(R_R)$ since $\mathrm{Soc}(R_R)$ is essential in R_R.

(iv) $\Rightarrow$ (i) is clear. □

Next, we investigate the transfer of the right π-dual Baer condition between a base ring R and several extensions. We begin with $R[x]$ and $R[[x]]$.

Proposition 4.19. *Let R be a ring satisfying one of the following conditions:*

(i) *$R[x]$ is a right π-dual Baer ring;*

(ii) *$R[[x]]$ is a right π-dual Baer ring.*

Then R is a right π-dual Baer ring.

Proof. (i) Suppose that $R[x]$ is a right π-dual Baer ring and let I be a projection invariant right ideal of R. By [6, Lemma 4.1(iv)], $I[x]$ is a projection invariant right ideal of $R[x]$. This implies that $I[x] = e(x)R[x]$ for some idempotent $e(x) = e_0 + e_1 x + \cdots + e_n x^n \in R[x]$ (see Proposition 4.7). Let us show that $I = e_0 R$. Since $e(x) \in I[x]$, we have $e_0 \in I$ and so $e_0 R \subseteq I$. Now let $a \in I$. Therefore $a \in I[x] = e(x)R[x]$. Hence $a = e(x)f(x)$ for some $f(x) = f_0 + f_1 x + \cdots + f_m x^m \in R[x]$. It follows that $a = e_0 f_0 \in e_0 R$. This proves that $I = e_0 R$. Therefore R is a right π-dual Baer ring by Proposition 4.7.

(ii) This follows by the same method as in (i). □

The next example shows that polynomial extensions of right π-dual Baer rings need not be right π-dual Baer.

Example 4.20. Let F be a field. Clearly, F is a right π-dual Baer ring. On the other hand, it is well known that both $F[x]$ and $F[[x]]$ are integral domains, but they are not semisimple. From Remark 4.1(i), it follows that neither $R[x]$ nor $R[[x]]$ is right π-dual Baer.

We conclude this paper by investigating when full or generalized triangular matrix rings are right π-dual Baer.

Proposition 4.21. *Let R be a quasi-dual Baer ring (in particular if R is a right π-dual Baer ring). Then $\mathbf{Mat}_n(R)$ is a right and left π-dual Baer ring for every positive integer $n > 1$.*

Proof. By [18, Proposition 2.10], there exists a positive integer t such that $R = \prod_{i=1}^{t} R_i$ is a finite product of simple rings R_i $(1 \le i \le t)$. Let $n > 1$ be a positive integer. Note that $A = \mathbf{Mat}_n(R) \cong \prod_{i=1}^{t} \mathbf{Mat}_n(R_i)$ (as rings). By [14, Theorem 3.1], each $\mathbf{Mat}_n(R_i)$ $(1 \le i \le t)$ is a simple ring. Since $\mathbf{I}(A) = A$, it follows from Remark 4.1(ii) that A a right and left π-dual Baer ring. □

The next example illustrates the fact that the right π-dual Baer property is not Morita invariant.

Example 4.22. It is well known that for any ring R and any positive integer m, the rings R and $\mathbf{Mat}_m(R)$ are Morita equivalent (see [2, Corollary 22.6]). Let R be a simple ring which is not right π-simple (see Example 4.13). Then R is not right π-dual Baer by Lemma 4.10. On the other hand, for every positive integer $n > 1$, $\mathbf{Mat}_n(R)$ is a right π-dual Baer ring by Proposition 4.21.

Proposition 4.21 and Example 4.22 should be compared with the following proposition.

Proposition 4.23. *Let R be a ring. Then the following statements are equivalent:*

(i) *R is a quasi-dual Baer ring;*

(ii) *$\mathbf{Mat}_n(R)$ is a quasi-dual Baer ring for every positive integer n;*

(iii) *$\mathbf{Mat}_n(R)$ is a quasi-dual Baer ring for some positive integer $n > 1$.*

Proof. (i) ⇒ (ii) This follows from Remark 4.14 and Proposition 4.21.

(ii) ⇒ (iii) This is immediate.

(iii) ⇒ (i) Let $n > 1$ be a positive integer such that $A = \mathbf{Mat}_n(R)$ is a quasi-dual Baer ring. Then A is a semiprime ring (see the proof of [18, Proposition 2.10]). Let e be the matrix unit E_{11} in A. Clearly, e is an idempotent in A. Moreover, $eAe = \{aE_{11} \mid a \in R\}$ and R are isomorphic rings (see [14, Example 21.14]). Let us show that eAe is a quasi-dual Baer ring. Take a two-sided ideal U of eAe. Then AUA is a two-sided ideal of A. Thus AUA is a direct summand of A_A by [18, Proposition 2.10]. This implies that $AUA = fA$ for some $f^2 = f \in A$. Since A is a semiprime ring, it follows from [10, Lemma 3.1] that f is a central idempotent in A. Now [14, Theorem 21.11(2)] gives that $U = e(AUA)e$. Therefore $U = e(fA)e$. Hence $U = e^2(fAe) = efe(eAe)$ as f is central. Moreover, it is clear that efe is an idempotent in the ring eAe. It follows that U is a direct summand of eAe_{eAe}. Consequently, eAe is a quasi-dual Baer ring by [18, Proposition 2.10]. □

Next, we characterize right π-dual Baer 2-by-2 generalized triangular matrix rings.

Theorem 4.24. Let $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ denote a 2-by-2 generalized upper triangular matrix ring where R and S are rings and M is an (R,S)-bimodule. Then the following statements are equivalent:

(i) T is a right π-dual Baer ring;

(ii) R and S right π-dual Baer rings and $M = 0$.

Proof. (i) $\Rightarrow$ (ii) It is well known that $\mathrm{Rad}(T) = \begin{bmatrix} \mathrm{Rad}(R) & M \\ 0 & \mathrm{Rad}(S) \end{bmatrix}$ is a two-sided ideal of T and hence it is a direct summand of T_T by Proposition 4.7. But $\mathrm{Rad}(T)$ is small in T_T. Then $\begin{bmatrix} \mathrm{Rad}(R) & M \\ 0 & \mathrm{Rad}(S) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This yields $M = 0$. It follows that $T = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix} \cong R \times S$ (as rings). Now from Proposition 4.8, we infer that R and S are right π-dual Baer rings.

(ii) $\Rightarrow$ (i) This follows by using again Proposition 4.8. $\square$

Remark 4.25. From the previous theorem, it follows that for any nonzero ring R, the 2-by-2 upper triangular matrix ring over R is never a right π-dual Baer ring.

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