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What can we say about the Pólya Group of a Bicyclic Biquadratic Number Field?

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Abstract. The Pólya group $\mathcal{P}o(K)$ of a finite Galois extension K of $\mathbb{Q}$ is the subgroup of the class group of K formed by the strong ambiguous classes of K. In this paper, we state a general formula which gives the order of $\mathcal{P}o(K)$ when K is a bicyclic biquadratic number field by means of classical indices, namely, the unit index of K, the number of ramified primes, and the number of fundamental units with norm 1 of the quadratic subfields of K. Then we study separately the imaginary case and the real case.

Key Words: Pólya group, Pólya field, Biquadratic number field, Ambiguous ideal.
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1 Introduction

Recall first some definitions, notations, and results.

Definition 1.1. [2, Definition II.3.8] and [25, § 1]

1. The *Pólya-Ostrowski group* or simply the *Pólya group* of an algebraic number field K is the subgroup $\mathcal{P}o(K)$ of the class group $\mathcal{C}l(K)$ of K generated by the classes of the products of all the maximal ideals of the ring of integers $\mathcal{O}_K$ with the same norm.

2. A *Pólya field* is a number field whose Pólya group is trivial.

Notation. We classicaly denote by $\Pi_q(K)$ the product of all the maximal ideals of $\mathcal{O}_K$ with norm q. If q is the norm of no maximal ideal, we set $\Pi_q(K) = \mathcal{O}_K$. Obviously, if $\Pi_q(K) \neq \mathcal{O}_K$, then q is a prime power: $q = p^f$.

The Galois Case. When $K/\mathbb{Q}$ is a Galois extension, that is in the case that interests us here, denoting by e_p and f_p the ramification index and the inertial degree of p in the extension $K/\mathbb{Q}$, we have

$$\Pi_{p^{f_p}}(K) = \prod_{\mathfrak{m}\in \mathrm{Max}(\mathcal{O}_K),\, \mathfrak{m}|p} \mathfrak{m} \quad \text{and} \quad p\mathcal{O}_K = \Pi_{p^{f_p}}(K)^{e_p}. \tag{1}$$

Consequently, following Ostrowski [20], if p in not ramified in the Galois extension $K/\mathbb{Q}$, $\Pi_{p^{f_p}}(K)$ is principal generated by p. Thus, the Pólya group $\mathcal{P}o(K)$ is generated by the classes of the $\Pi_{p^{f_p}}(K)$ where p is ramified in $K/\mathbb{Q}$, so that, $\mathcal{P}o(K)$ is nothing else than the group of strongly ambiguous classes of K.

There are already contributions of several authors who studied the Pólya group of a bicyclic biquadratic number field, for instance: Chattopadhyay and Saikia [6], Heidaryan and Rajaei ([10] and

[11]), Leriche ([16] and [17]), Maarefparvar ([18] and [19]), Taous and Zekhnini ([22], [23] and [26]), and Tougma [24].

We begin in the next section with some preliminaries, in fact some well known facts that will be useful for our study of the bicyclic biquadratic number fields K. Then, in the following section, we establish a formula giving the order of the Pólya group of K by means of several indices and we deduce upper bounds for the number s_K of ramified primes for K to be a Pólya field. In the fourth section, we consider the imaginary case and end, in the last section, with the real case.

2 Preliminaries

General Notation

For every algebraic number field K, we denote by

$\mathcal{O}_K$ the ring of integers of K,

$\mathcal{O}_K^\times$ the units of $\mathcal{O}_K$,

$\mathcal{I}_K$ the group of nonzero fractional ideals of $\mathcal{O}_K$,

$\mathcal{P}_K$ the subgroup of $\mathcal{I}_K$ formed by the principal ideals,

$\mathcal{C}l(K) = \mathcal{I}_K/\mathcal{P}_K$ the class group of K.

Moreover, when $K/\mathbb{Q}$ is a Galois extension, we denote by

$G = \mathrm{Gal}(K/\mathbb{Q})$ the Galois group of $K/\mathbb{Q}$,

$\mathcal{I}_K^G$ the subgroup of $\mathcal{I}_K$ formed by the ambiguous ideals,

$\mathcal{P}_K^G = \mathcal{P}_K \cap \mathcal{I}_K^G$ the subgroup of $\mathcal{P}_K$ formed by the principal ambiguous ideals,

$\mathcal{P}o(K) = \mathcal{I}_K^G/\mathcal{P}_K^G$ the Pólya group of K,

$\mathcal{I}_K^G/\mathcal{P}_\mathbb{Q}$ and $\mathcal{P}_K^G/\mathcal{P}_\mathbb{Q}$ the quotient of $\mathcal{I}_K^G$ and $\mathcal{P}_K^G$ by the extension of $\mathcal{P}_\mathbb{Q}$ in $\mathcal{I}_K^G$,

e_p and f_p the ramification index and the inertial degree of p in $K/\mathbb{Q}$.

Some Exact Sequences

Proposition 2.1. *If $K/\mathbb{Q}$ is a Galois extension with Galois group G, the following sequence of abelian groups is exact*:

$$0 \to \mathcal{P}_K^G/\mathcal{P}_\mathbb{Q} \to \oplus_{p\in\mathbb{P}} \mathbb{Z}/e_p\mathbb{Z} \to \mathcal{P}o(K) \to 0. \tag{2}$$

Proof. The containments $\mathcal{P}_\mathbb{Q} \subseteq \mathcal{P}_K^G \subseteq \mathcal{I}_K^G$ lead to the obvious exact sequence:

$$0 \to \mathcal{P}_K^G/\mathcal{P}_\mathbb{Q} \to \mathcal{I}_K^G/\mathcal{P}_\mathbb{Q} \to \mathcal{I}_K^G/\mathcal{P}_K^G \to 0.$$

We already said that by definition $\mathcal{P}o(K) = \mathcal{I}_K^G/\mathcal{P}_K^G$. It remains to show that $\mathcal{I}_K^G/\mathcal{P}_\mathbb{Q} \simeq \oplus_{p\in\mathbb{P}} \mathbb{Z}/e_p\mathbb{Z}$. The group $\mathcal{I}_K^G$ of ambiguous ideals is the free group generated by the ideals $\Pi_{p^{f_p}}(K)$. Thus we can consider the natural isomorphism

$$\mathrm{I} = \prod_{p\in\mathbb{P}} (\Pi_{p^{f_p}}(K))^{k_p(\mathrm{I})} \in \mathcal{I}_K^G \mapsto (k_p(\mathrm{I}))_{p\in\mathbb{P}} \in \oplus_{p\in\mathbb{P}} \mathbb{Z},$$

which induces a surjective morphism

$$\mathrm{I} \in \mathcal{I}_K^G \mapsto (\overline{k_p(\mathrm{I})})_{p\in\mathbb{P}} \in \oplus_{p\in\mathbb{P}} \mathbb{Z}/e_p\mathbb{Z}.$$

The kernel of this last morphism is clearly formed by the ideals I such that, for each $p \in \mathbb{P}$, $k_p(\mathrm{I}) = e_p m_p$ with $m_p \in \mathbb{Z}$, that is, the ideals $\mathrm{I} = (\prod_p p^{m_p})\mathcal{O}_K$, in other words, the kernel is the image by extension of $\mathcal{P}_\mathbb{Q}$ in $\mathcal{I}_K^G$. □

The exact sequence (2) should be compared with the well known following cohomological exact sequence.

Proposition 2.2. [25, p. 163] *If $K/\mathbb{Q}$ is a Galois extension with Galois group G, the following sequence of abelian groups is exact*:

$$0 \to H^1(G,\mathcal{O}_K^\times) \to \oplus_{p\in\mathbb{P}} \mathbb{Z}/e_p\mathbb{Z} \to \mathcal{P}o(K) \to 0\,. \tag{3}$$

Sequences (2) and (3) together show that

$$\mathcal{P}_K^G/\mathcal{P}_\mathbb{Q} \simeq H^1(G,\mathcal{O}_K^\times) \qquad \text{(Iwasawa [13])}. \tag{4}$$

It is not a surprise, indeed, from the short exact sequence

$$1 \to \mathcal{O}_K^\times \to K^* \to \mathcal{P}_K \to 1,$$

the left exactness of the functor $U \mapsto U^G$ on the abelian category of G-modules leads, with Hilbert 90, to:

Proposition 2.3. [1, Lemma 2.1] *If $K/\mathbb{Q}$ is a Galois extension with Galois group G, the following sequence of abelian groups is exact*:

$$1 \to \mathbb{Q}^*/\{\pm 1\} \to \mathcal{P}_K^G \to H^1(G,\mathcal{O}_K^\times) \to 1\,. \tag{5}$$

Quadratic Fields

The first important result about Pólya groups is due to Hilbert who was interested in the ambiguous ideals.

Proposition 2.4. (Hilbert [12, Theorem 105–106]) *Let $k = \mathbb{Q}(\sqrt{d})$ be a quadratic number field where d is a square-free integer. If s_k denotes the number of ramified prime numbers in the extension $k/\mathbb{Q}$, then*

$$|\mathcal{P}o(k)| = \begin{cases} 2^{s_k-2} & \text{if } k \text{ is real and } N_{k/\mathbb{Q}}(\mathcal{O}_K^\times) = \{1\} \\ 2^{s_k-1} & \text{else.} \end{cases} \tag{6}$$

Let us recall that the reason of this formula comes from the fact that the group of classes of ambiguous ideals is generated by the classes of the ramified prime ideals of k. But, when $d \neq -1$, the principal ambiguous ideal $\sqrt{d}\mathcal{O}_K$ leads to a relation between the previous generators. Moreover, it is known that there is another relation induced by another principal ambiguous ideal if and only if k is real and the norm of the fundamental unit ε is $+1$. In this case, there are $\alpha \in \mathcal{O}_k$ and $a \in \mathbb{Z}$ such that $\alpha^2 = \varepsilon a$ and $a|2d$.

3 The order of the Pólya group of a biquadratic number field

Let us introduce some more notations with some indices.

Notation. From now on, K denotes a bicyclic biquadratic number field, that is, $K = \mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$ where d_1 and d_2 are two distinct square-free integers. The three quadratic subfields of K are

$$k_1 = \mathbb{Q}(\sqrt{d_1}),\, k_2 = \mathbb{Q}(\sqrt{d_2}) \text{ and } k_3 = \mathbb{Q}(\sqrt{d_3}) \text{ where } d_3 = \frac{d_1 d_2}{(\gcd(d_1,d_2))^2}\,.$$

- We denote by s_K the number of ramified primes in the extension $K/\mathbb{Q}$ and analogously by s_i ($i = 1,2,3$) the number of ramified primes in the extension $k_i/\mathbb{Q}$.
- We set $i_2 = 1$ or 0 depending on whether 2 is, or is not, totally ramified in $K/\mathbb{Q}$.
- If k_i is real, $\varepsilon_i > 1$ denotes the fundamental unit, $\mathcal{O}_{k_i}^\times = \{\pm\varepsilon_i^t \mid t \in \mathbb{Z}\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$.

• If k_i is imaginary, $\mathcal{O}_{k_i}^\times = \mu_{k_i}$ where μ_{k_i} denotes the group of roots of unity.
• For $1 \le i \le 3$, we let $\nu_i = 1$ or 0 according to the fact that k_i is real and the norm $N_{k_i/\mathbb{Q}}(\varepsilon_i)$ is equal to 1 or not.
• Now we introduce two 'global' indices:

$$\nu_K = \nu_1 + \nu_2 + \nu_3$$

and the unit index

$$q_K = (\mathcal{O}_K^\times : \mathcal{O}_{k_1}^\times \mathcal{O}_{k_2}^\times \mathcal{O}_{k_3}^\times).$$

An odd prime number is ramified in k_i if and only if it divides d_i, thus an odd prime number which is ramified in one of the subfields is ramified in exactly two subfields. The prime 2 is ramified in k_i if and only if $d_i \equiv 2$ or $3 \pmod 4$, so that, if 2 is ramified in one of the subfields, then it is ramified in at least two subfields. Moreover, 2 is totally ramified in K if and only if it is ramified in each subfields k_i. Consequently, we have the relation

$$s_1 + s_2 + s_3 = 2s_K + i_2 . \tag{7}$$

The prime 2 is not totally ramified in $K/\mathbb{Q}$ if and only if at least one of the three integers d_i is congruent to 1 modulo 4. Equivalently, 2 is totally ramified in $K/\mathbb{Q}$ if and only if two of the integers d_i are even and the third one is congruent to 3 modulo 4.

From the natural morphisms

$$j_{k_i}^K : \mathfrak{a}_i \in \mathcal{I}_{k_i} \mapsto \mathfrak{a}_i \mathcal{O}_K \in \mathcal{I}_K \quad (1 \le i \le 3),$$

we deduce a morphism:

$$(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3) \in \mathcal{I}_{k_1} \times \mathcal{I}_{k_2} \times \mathcal{I}_{k_3} \mapsto \mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{a}_3 \mathcal{O}_K \in \mathcal{I}_K,$$

which itself induces a morphism:

$$\varphi_K : \mathcal{C}l(k_1) \times \mathcal{C}l(k_2) \times \mathcal{C}l(k_3) \to \mathcal{C}l(K).$$

It is easy to see that the Pólya groups behave well by extensions when all the considered fields are Galois extensions of $\mathbb{Q}$ (see [3, Proposition 3.4]). Thus, φ_K itself induces by restriction a natural morphism:

$$\psi_K : \mathcal{P}o(k_1) \times \mathcal{P}o(k_2) \times \mathcal{P}o(k_3) \to \mathcal{P}o(K).$$

Proposition 3.1. *The natural morphism $\psi_K : \mathcal{P}o(k_1) \times \mathcal{P}o(k_2) \times \mathcal{P}o(k_3) \to \mathcal{P}o(K)$ is surjective either if 2 is not totally ramified or if $\Pi_2(K)$ is principal. Else, the quotient $\mathcal{P}o(K)/\mathrm{Im}(\psi_K)$ has order 2.*

Proof. Let p be a ramified prime number which is odd or equal to 2 if 2 is not totally ramified in K. There is a quadratic subfield k_i of K in which p is ramified, and then, $\Pi_p(k_i)$ is not ramified in the extension K/k_i, else p would be totally ramified in $K/\mathbb{Q}$. Thus, we have $\Pi_p(k_i)\mathcal{O}_K = \Pi_p(K)$ or $\Pi_{p^2}(K)$ according to the fact that $\Pi_p(k_i)$ is decomposed or inert in the extension K/k_i.

Assume now that 2 is totally ramified. Then, whatever the quadratic subfield k_i, we have $\Pi_2(k_i)\mathcal{O}_K = (\Pi_2(K))^2$. Thus, denoting by $\mathrm{Im}(\psi_K)$ the image of ψ_K, we have the isomorphism

$$\mathcal{P}o(K)/\mathrm{Im}(\psi_K) \simeq \langle \overline{\Pi_2(K)} / \overline{\Pi_2(K)^2} \rangle$$

which allows us to conclude. $\square$

Finally, following Kubota, we have:

Proposition 3.2. [14, Satz 4] *With the previous notation*

$$|\mathrm{Ker}(\psi_K)| = \begin{cases} \frac{1}{q_K}\prod_p e_p(K/\mathbb{Q}) & \text{if } K \text{ is real and } \nu_K = 0 \\ \frac{1}{2q_K}\prod_p e_p(K/\mathbb{Q}) & \text{else.} \end{cases} \tag{8}$$

We are now able to state a formula for the order of the Pólya group of K.

Theorem 3.3. The order of the Pólya group $\mathcal{P}o(K)$ of K is equal to

$$|\mathcal{P}o(K)| = \begin{cases} q_K \times 2^{s_K+j_2-2-\max(1,\nu_K)} & \text{if } K \text{ is real} \\ q_K \times 2^{s_K+j_2-2-\nu_K} & \text{if } K \text{ is imaginary} \end{cases} \tag{9}$$

where $j_2 = 1$ if 2 is totally ramified and $\Pi_2(K)$ is not principal and $j_2 = 0$ else.

Proof. By Proposition 2.4,

$$|\mathcal{P}o(k_1)\times\mathcal{P}o(k_2)\times\mathcal{P}o(k_3)| = 2^{\sum_{i=1}^3(s_i-1-\nu_i)} = 2^{s_1+s_2+s_3-3-\nu_K}.$$

By Kubota's result,

$$|\mathrm{Im}(\psi_K)| = \frac{|\mathcal{P}o(k_1)\times\mathcal{P}o(k_2)\times\mathcal{P}o(k_3)|}{|\mathrm{Ker}(\psi_K)|} = \frac{2^{s_1+s_2+s_3-3-\nu_K}}{2^{s_K+i_2}}\times q_K\times(1 \text{ or } 2).$$

Proposition 3.1 says that $\frac{|\mathcal{P}o(K)|}{|\mathrm{Im}(\psi_K)|} = 2^{j_2}$. We may conclude with Formula (7). □

Remark 3.4. It is known (Hasse [8]) that

$$q_K = (\mathcal{O}_K^\times : \mathcal{O}_{k_1}^\times\mathcal{O}_{k_2}^\times\mathcal{O}_{k_3}^\times) = \begin{cases} 1,2,4 & \text{if } K \text{ is real} \\ 1,2 & \text{if } K \text{ is imaginary.} \end{cases} \tag{10}$$

We will prove the two cases in the next sections by means of elementary remarks. Let us start here with a rough result: q_K divides 8 in the real case and divides 4 in the imaginary case.

Letting $\mathrm{Gal}(k_i/\mathbb{Q}) = \langle\sigma_i\rangle$, then

$$\forall x \in \mathcal{O}_K^\times \quad x^2 N_{K/\mathbb{Q}}(x) = x^3\sigma_1(x)\sigma_2(x)\sigma_3(x) \in \mathcal{O}_{k_1}^\times\mathcal{O}_{k_2}^\times\mathcal{O}_{k_3}^\times.$$

Thus,

$$(\mathcal{O}_K^\times)^{(2)} \subseteq \mathcal{O}_{k_1}^\times\mathcal{O}_{k_2}^\times\mathcal{O}_{k_3}^\times \subseteq \mathcal{O}_K^\times.$$

In the real case, $\mathcal{O}_K^\times \simeq \{\pm1\}\times\mathbb{Z}^3$, and hence, $\mathcal{O}_K^\times/(\mathcal{O}_K^\times)^{(2)} \simeq (\mathbb{Z}/2\mathbb{Z})^4$, as moreover $-1 \notin (\mathcal{O}_K^\times)^{(2)}$, $q_K|8$. In the imaginary case, $\mathcal{O}_K^\times \simeq \mu_K\times\mathbb{Z}$, and hence, $\mathcal{O}_K^\times/(\mathcal{O}_K^\times)^{(2)} \simeq (\mathbb{Z}/2\mathbb{Z})^2$, consequently, $q_K|4$.

From Formula (9) we easily deduce an upper bound for the number of ramified primes in a bicyclic biquadratic Pólya field.

Corollary 3.5. *If K is a bicyclic biquadratic Pólya field, then*

$$s_K \leq \begin{cases} 2+\nu_K & \text{if } K \text{ is imaginary} \\ 3+\max\{0,\nu_K-1\} & \text{if } K \text{ is real} \end{cases} \tag{11}$$

In particular, we always have $s_K \leq 5$ and, if K is imaginary, $s_K \leq 3$.

Remark 3.6. Zantema [25, §4] proved that this bound $s_K \leq 5$ is sharp since the field $\mathbb{Q}(\sqrt{5.7},\sqrt{3.127})$ where $2,3,5,7,127$ are ramified is a Pólya field. Note that, by Corollary 4.7 below, the bound $s_K \leq 3$ is sharp in the imaginary case.

Corollary 3.5 gives necessary conditions on s_K for K to be a Pólya field. On the other hand, s_K can provide sufficient conditions for K to be Pólya.

Proposition 3.7. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ and assume that 2 is not totally ramified.*
(i) If $s_K \leq 2$, then K is a Pólya field.
(ii) If $d_2 \nmid d_1$ and $\nu_1 = 1$, then $s_K \leq 3$ implies that K is a Pólya field.
(iii) If $(d_1, d_2) = 1$ and $\nu_1 = \nu_2 = 1$, then $s_K \leq 4$ implies that K is a Pólya field.

Proof. (i) Assume that $s_K \leq 2$. Note first that, as $s_i \geq 1$ for each $i = 1, 2, 3$, Formula (7) leads to $3 \leq 2s_K + i_2 \leq 4 + i_2$. Under the hypothesis $i_2 = 0$, $s_K = 1$ is impossible. Thus, $s_K = 2$ and, for instance, $s_1 = s_2 = 1$ and $s_3 = 2$. Since, $s_1 = s_2 = 1$ implies that k_1 and k_2 are Pólya fields, we may conclude with Proposition 3.8 below.

(ii) As $\nu_1 = 1$, there are $a_1 \in \mathbb{Z}$ and $\alpha_1 \in k_1$ such that $\alpha_1^2 = \varepsilon_1 a_1$ and $a_1 | 2d_1$. Assume that $\alpha_1 \mathcal{O}_K = \sqrt{d_2}\mathcal{O}_K$, and hence, that $d_2\mathcal{O}_K = a_1\mathcal{O}_K$.

Assume also that a_1 does not divide d_1. Then, a_1 is even, d_1 is odd, and 2 is ramified in $\mathbb{Q}(\sqrt{d_1})$. As $d_2\mathcal{O}_K = a_1\mathcal{O}_K$, 2 divides d_2, and hence, d_3. The prime 2 would be totally ramified. Thus, a_1 divides (strictly) d_1 and the hypothesis $d_2 \nmid d_1$ implies that $a_1 \neq \pm d_2$, and hence, $\alpha_1\mathcal{O}_K \neq \sqrt{d_2}\mathcal{O}_K$.

The group $\mathcal{P}_K^G$ contains obviously the ideals of $\mathcal{O}_K$ generated by $\sqrt{d_1}$, $\sqrt{d_2}$, and α_1. These principal ideals are not congruent modulo $\mathcal{P}_\mathbb{Q}$ because their norms in the extension $K/\mathbb{Q}$, namely d_1^2, d_2^2, and a_1^2, are not congruent modulo z^4 for any $z \in \mathbb{Q}^*$ thanks to the fact that $d_2' = \frac{d_2}{(d_1,d_2)} \neq \pm 1$ is coprime to d_1 and a_1. Thus, 2^3 divides the order of the group $\mathcal{P}_K^G/\mathcal{P}_\mathbb{Q}$. It follows then from the exact sequence (2) that the order of $\mathcal{P}o(K)$ divides $2^{s_K - 3}$.

(iii) Since $\nu_1 = \nu_2 = 1$, for $i = 1, 2$, there are $a_i \in \mathbb{Z}$ and $\alpha_i \in k_i$ such that $\alpha_i^2 = \varepsilon_i a_i$ where $a_i | 2d_i$. It follows from the proof of assertion (ii) that a_i is a strict divisor of d_i. Consequently, the group $\mathcal{P}_K^G$ contains in particular the ideals of $\mathcal{O}_K$ generated by $\sqrt{d_1}$, $\sqrt{d_2}$, α_1, and α_2. These ideals are not congruent modulo $\mathcal{P}_\mathbb{Q}$ because their norms in the extension $K/\mathbb{Q}$, namely d_1^2, d_2^2, a_1^2, and a_2^2, cannot be congruent modulo z^4 for any $z \in \mathbb{Q}^*$. Thus, 2^4 divides the order of the group $\mathcal{P}_K^G/\mathcal{P}_\mathbb{Q}$ and, it follows from the exact sequence (2) that the order of $\mathcal{P}o(K)$ divides $2^{s_K - 4}$. □

Another way to obtain sufficient conditions for K to be Pólya comes from the idea that if two number fields are Pólya, then the field they generate is likely to be too. This is not completely wrong, nor completely true according to the following proposition.

Proposition 3.8. [16, Proposition 4.3] *The compositum K of two quadratic Pólya fields is a Pólya field if and only the ideal $\Pi_2(K)$ is principal. This is the case in particular if 2 is not totally ramified.*

Proof. Let k_i $(1 \leq i \leq 3)$ be the three quadratic subfields of K. If p be a prime such that $e_p(K/\mathbb{Q}) = 2$, then p is ramified in exactly two of the three subfields. By hypothesis, at least two subfields k_i are Pólya fields. Clearly, there is a Pólya subfield k_i such that p is ramified in $k_i/\mathbb{Q}$. Thus, on the one hand, p is not ramified in K/k_i and, on the other hand, $\Pi_p(k_i)$ is principal. Consequently, $\Pi_{p^{f_p(K/\mathbb{Q})}}(K) = \Pi_p(k_i)\mathcal{O}_K$ is principal. □

4 Imaginary Bicyclic Biquadratic Number Fields

The biquadratic field K is imaginary, that is, is not real, if one, and only one, of the three integers d_i is > 0. We introduce some specific notation for the imaginary case.

Notation. In this section, $K = \mathbb{Q}(\sqrt{n}, \sqrt{-m})$ where n and m are two square-free positive integers. The real quadratic subfield $\mathbb{Q}(\sqrt{n})$ is denoted by K^+ and its fundamental unit by ε_{K^+}. We let $\nu_K = \nu_{K^+} = 1$ or 0 according to the fact that $N_{K^+/\mathbb{Q}}(\varepsilon_{K^+}) = 1$ or -1.

While the integer n is uniquely determined, we can replace m by $\frac{m\times n}{\gcd(m,n)^2}$. Most often we choose a priori the smaller of the two, but not always.

The Unit Index. If $K \neq \mathbb{Q}(\sqrt{2},\sqrt{-1}) = \mathbb{Q}(\zeta_8)$, we always have the equality $\mu_K \mathcal{O}_{K^+}^\times = \mathcal{O}_{k_1}^\times \mathcal{O}_{k_2}^\times \mathcal{O}_{k_3}^\times$, and therefore, q_K is equal to Hasse's unit index

$$q_K = (\mathcal{O}_K^\times : \mu_K \mathcal{O}_{K^+}^\times) \qquad (K \neq \mathbb{Q}(\zeta_8)).$$

We already said that $q_K = 1$ or 2 [8, Satz 14]. Indeed, recall the containments

$$(\mathcal{O}_K^\times)^{(2)} \subseteq \mathcal{O}_{k_1}^\times \mathcal{O}_{k_2}^\times \mathcal{O}_{k_3}^\times \subseteq \mathcal{O}_K^\times.$$

If the first inclusion was an equality, $\sqrt{-1} = i$ would be in some k_j, then $\zeta_8 = \sqrt{i} = \frac{1}{2}(1+i)\sqrt{2}$ would be in K and $\sqrt{2}$ would be in some k_l, and finally, $K = \mathbb{Q}(\sqrt{-1},\sqrt{2}) = \mathbb{Q}(\zeta_8)$. But the fundamental unit $1+\sqrt{2}$ of $\mathbb{Q}(\sqrt{2})$ does not belong to $(\mathcal{O}_{\mathbb{Q}(\zeta_8)}^\times)^{(2)}$. Thus, q_K is a strict divisor of 4.

Following Zantema [25], being a cyclotomic field, $\mathbb{Q}(i,\sqrt{2}) = \mathbb{Q}(\zeta_8)$ is a Pólya field, where 2 is totally ramified. This is a very particular case as shown by the following lemma.

Lemma 4.1. *Let K be an imaginary bicyclic biquadratic number field distinct from $\mathbb{Q}(\zeta_8)$. If 2 is totally ramified in the extension $K/\mathbb{Q}$, the ideal $\mathfrak{P}$ lying over 2 is not principal and K is not a Pólya field.*

Proof. Let $K = \mathbb{Q}(\sqrt{n},\sqrt{-m})$ and assume that 2 is totally ramified in $K/\mathbb{Q}$. Denote by $\mathfrak{P}$ and $\mathfrak{p}$ the prime ideals lying over 2 of respectively K and $\mathbb{Q}(\sqrt{-m})$. As $K \neq \mathbb{Q}(\sqrt{-1},\sqrt{2})$, either $m > 2$ or $\frac{mn}{(m,n)^2} > 2$. Thus, we may assume that $m \geq 3$, and then there is no element of $\mathbb{Q}(\sqrt{-m})$ with norm ± 2. Consequently, $\mathfrak{p}$ is not principal and Lemma 4.2 below shows that $\mathfrak{P}$ cannot be principal. As $\Pi_2(K) = \mathfrak{P}$, K is not a Pólya field. □

Lemma 4.2. *Let L be a number field and p be a prime number which is totally ramified in $L/\mathbb{Q}$. Assume that the ideal $\Pi_p(L)$ is principal, and hence, generated by an element y of $\mathcal{O}_L$ such that $N_{L/\mathbb{Q}}(y) = p$ or $-p$. Then, for every subfield K of L, $\Pi_p(K)$ is principal generated by an element $x \in \mathcal{O}_K$ such that $N_{K/\mathbb{Q}}(x) = N_{L/\mathbb{Q}}(y)$ $(= p$ or $-p)$.*

Proof. As p is totally ramified in $L/\mathbb{Q}$, and hence, in $K/\mathbb{Q}$, $\Pi_p(L)$ is the prime of L and $\Pi_p(K)$ the prime of K lying over p. Then we have $N_L^K(\Pi_p(L)) = \Pi_p(K)$. Consequently, if $\Pi_q(L) = y\mathcal{O}_L$ for some $y \in \mathcal{O}_L$, then $\Pi_p(K) = N_{L/K}(y)\mathcal{O}_K$ and $x = N_{L/K}(y)$ satisfies $N_{K/\mathbb{Q}}(x) = N_{L/\mathbb{Q}}(y)$. □

Proposition 4.3. *An imaginary bicyclic biquadratic field K which is the compositum of two quadratic Pólya fields is a Pólya field if and only if, either $K = \mathbb{Q}(\zeta_8)$, or 2 is not totally ramified in $K/\mathbb{Q}$.*

Proof. The necessary condition is an obvious consequence of Lemma 4.1 while Proposition 3.8 implies the sufficient condition. □

Example 4.4. According to Leriche's assertion [16, Proposition 4.3] (our Proposition 3.8), the following fields are Pólya fields.

(a) $K = \mathbb{Q}(\sqrt{2},\sqrt{-q})$ where $q \equiv 3 \pmod 4) : s_K = 2, i_2 = 0, \nu_{K^+} = 0, q_K = 1$.
(b) $K = \mathbb{Q}(\sqrt{p},\sqrt{-2})$ where $p \equiv 1 \pmod 4) : s_K = 2, i_2 = 0, \nu_{K^+} = 0, q_K = 1$.
(c) $K = \mathbb{Q}(\sqrt{2q},\sqrt{-q'})$ where $q \equiv q' \equiv 3 \pmod 4) : s_K = 3, i_2 = 0, \nu_{K^+} = 1, q_K = 1$.

Note that these three examples do not agree with [23, Theorem 3] which describes the Pólya groups of all imaginary bicyclic biquadratic number fields and says that in the three examples $\mathcal{P}o(K) \simeq \mathbb{Z}/2\mathbb{Z}$.

Remark 4.5. There are examples of imaginary biquadratic Pólya fields that are not the compositum of two quadratic Pólya fields: for instance, let p and q be two primes such that $p \equiv 1$ and $q \equiv 3 \pmod 4$, then the field $\mathbb{Q}(\sqrt{q}, \sqrt{-p})$ is a Pólya field (by Proposition 3.7(ii)) which contains only one Pólya quadratic subfield (namely $\mathbb{Q}(\sqrt{q})$) by Formula (6). If moreover $\left(\frac{q}{p}\right) = -1$, then $\mathbb{Q}(\sqrt{-p}, \sqrt{-q})$ is another example [10, Theorem 3.3].

Proposition 4.6. *If $K \neq \mathbb{Q}(\zeta_8)$ is an imaginary bicyclic biquadratic number field, then*

$$|\mathcal{P}o(K)| = q_K 2^{s_K + i_2 - 2 - \nu_{K^+}}. \tag{12}$$

More precisely,

$$\mathcal{P}o(K) \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{s_K + i_2 - 2 - \nu_{K^+} + \log_2 q_K} & if\ \Pi_2(K)^2\ is\ principal, \\ (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^{s_K - 3 - \nu_{K^+} + \log_2 q_K} & else. \end{cases} \tag{13}$$

Note in particular that either when 2 is not totally ramified in $K/\mathbb{Q}$ or when $\Pi_2(k_i)$ is principal for some subfield k_i, then $\Pi_2(K)^2$ is principal.

Proof. When $K \neq \mathbb{Q}(\zeta_8)$, Formula (12) is just Formula (9) where i_2 replaced j_2 thanks to Lemma 4.1. Formula (13) is a consequence of Formula (12) since $\Pi_2(K)$ is the only ambiguous ideal whose class could be of order > 2. □

As a consequence, we obtain Zantema's characterization of the imaginary bicyclic biquadratic number fields that are Pólya fields (which confirms that Examples 4.4 are Pólya fields).

Corollary 4.7. [25, Theorem 4.1] *An imaginary bicyclic biquadratic number field K is a Pólya field if and only if one of the following conditions holds:*

1. *2 is the only ramified prime: $K = \mathbb{Q}(\sqrt{2}, \sqrt{-1}) = \mathbb{Q}(\zeta_8)$,*
2. *there are exactly two primes that are ramified in $K/\mathbb{Q}$ and 2 is not totally ramified,*
3. *there are exactly three primes that are ramified in $K/\mathbb{Q}$, 2 is not totally ramified, $\nu_{K^+} = 1$ and $\mathcal{O}_K^\times = \mu_K \mathcal{O}_{K^+}^\times$.*

Proof. If K is a Pólya field distinct from $\mathbb{Q}(\zeta_8)$, then necessarily $s_K \leq 3$ by Corollary 3.5 and 2 is not totally ramified by Lemma 4.1. If $s_K = 2$, $i_2 = 0$ is sufficient for K to be Pólya by Lemma 3.7 (i). It remains the case where $s_K = 3$ and it follows from Formula (12) that K is Pólya if and only if $q_K = \nu_K = 1$. □

Remark 4.8. Zantema obtained Corollary 4.7 by another way. Indeed, the 4 parameters in Formula (12), namely, s_K, i_2, ν_{K^+}, and q_K, which are needed to compute the order of the Pólya group of K, are also those that are needed to describe the first terms of the exact sequence (3): the first two parameters, s_K and i_2, characterize easily the middle term $\oplus_{p \in \mathbb{P}} \mathbb{Z}/e_p\mathbb{Z}$, while the last two parameters, ν_{k^+} and q_K, characterize the first term $H^1(G, \mathcal{O}_K^\times)$ as shown by the following lemma due to Zantema.

Lemma 4.9. *[25, Lemma 4.3] Let $K \neq \mathbb{Q}(\zeta_8)$ be an imaginary bicyclic biquadratic number field. Then,*

$$H^1(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathcal{O}_K^\times) \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})^3 & if\ \ \nu_{K^+} \times q_K = 1 \\ (\mathbb{Z}/2\mathbb{Z})^2 & else. \end{cases}$$

Since $\mathcal{P}o(K)$ is trivial if and only if $H^1(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathcal{O}_K^\times) \simeq \oplus_{p \in \mathbb{P}} \mathbb{Z}/e_p\mathbb{Z}$, we see once again that 2 cannot be totally ramified if K is a Pólya field (except if $K = \mathbb{Q}(\zeta_8)$). Formulas (12) and (13) describe $\mathcal{P}o(K)$ up to an isomorphism, but when 2 is totally ramified we have to know the order of the class of the prime ideal $\mathfrak{P}$ of K lying over 2. By Lemma 4.1, this order is 2 or 4. Here are some particular results about this order.

Lemma 4.10. *Let $K = \mathbb{Q}(\sqrt{n}, \sqrt{-m})$ be an imaginary bicyclic biquadratic number field such that 2 is totally ramified but distinct from $\mathbb{Q}(\zeta_8)$. Let $\mathfrak{P}$ (resp. $\mathfrak{p}^+$) be the prime ideal of K (resp. of K^+) lying over 2. Then*

1. *If, for some $i \in \{1,2,3\}$, the prime ideal $\mathfrak{P} \cap k_i$ of k_i is principal, then the ideal $\mathfrak{P}^2$ is principal.*
2. *If $m \nmid 2n$, the ideal $\mathfrak{P}^2$ is principal if and only if the ideal $\mathfrak{p}^+$ is principal.*

Recall that $\mathfrak{p}_+$ is principal if and only if, either $2\left(\mathrm{Tr}_{K^+/\mathbb{Q}}(\varepsilon_{K^+})+2\right) \in \mathbb{Z}^{(2)}$, or $2\left(\mathrm{Tr}_{K^+/\mathbb{Q}}(\varepsilon_{K^+})-2\right) \in \mathbb{Z}^{(2)}$. Note that, in particular, the ideal $\mathfrak{P}^2$ is principal if one of the quadratic subfields k_i is a Pólya field. This is the case for $m = 1$, or 2, or any prime $p \equiv 3 \pmod 4$, or for $n = 2, 3, \ldots$

Proof. (1) If, in one of the three quadratic subfields k_i, the ideal $\mathfrak{p}_i = \mathfrak{P} \cap k_i$ is principal, then $\mathfrak{P}^2$ is principal since $\mathfrak{p}_i \mathcal{O}_K = \mathfrak{P}^2$.
(2) Assume that $m \nmid 2n$ and that $\mathfrak{P}^2$ is principal and let us prove that $\mathfrak{p}^+$ is principal. If $n = 2$ or 3, then $\mathcal{O}_{K^+}$ is a principal ideal domain, in particular $\mathfrak{p}^+$ is principal. Thus, we also assume that $n > 3$. If we can prove that the morphism $\varepsilon_{K^+}^K : \mathcal{C}l(K^+) \to \mathcal{C}l(K)$ is injective, then the fact that $\mathfrak{p}^+\mathcal{O}_K = \mathfrak{P}^2$ is principal will imply that $\mathfrak{p}^+$ itself is principal.

The fact that $m \nmid 2n$ implies obviously that $m \neq 1$, which implies that $|\mu_K| \equiv 2 \pmod 4$, indeed it is easy to check that $|\mu_K| \equiv 0 \pmod 4$ if and only if $m = 1$. Moreover, as m is assumed to be square-free, the fact that $m \nmid 2n$ implies the existence of some odd prime number p dividing m and not n. Such a p is not ramified in $K^+/\mathbb{Q}$ but is ramified in the extension $K^+(\sqrt{-m})/K^+$, which is then 'essentially ramified' in Hasse's sense [8, Chapter III] or in Lemmermeyer' sense [15, § 1]. Finally, the injectivity of $\varepsilon_{K^+}^K$ follows from Lemma 4.11 below. □

Lemma 4.11. (Hasse [8, Satz 17] or Lemmermeyer [15, Theorem 1 (i)]) *If $|\mu_K| \equiv 2 \pmod 4$ and the extension K/K^+ is essentially ramified, then $q_K = 1$ and $\varepsilon_{K^+}^K : \mathcal{C}l(K^+) \to \mathcal{C}l(K)$ is injective.*

Noticing that the hypotheses of Lemma 4.11 are satisfied when $m \nmid 2n$, we are able to describe the group $\mathcal{P}o(K)$ in this case.

Proposition 4.12. [5, Proposition V.30] *Let $K = \mathbb{Q}(\sqrt{n}, \sqrt{-m})$ be an imaginary bicyclic biquadratic number field where 2 is totally ramified. Assume that $m \nmid 2n$. Then $\mathcal{P}o(K)$ is isomorphic to*

1. *$(\mathbb{Z}/2\mathbb{Z})^{s_K-1}$ if $n = 2$,*
2. *$(\mathbb{Z}/2\mathbb{Z})^{s_K-3} \times (\mathbb{Z}/4\mathbb{Z})$ if $n \neq 2$ and $\nu_{K^+} = 0$,*
3. *$(\mathbb{Z}/2\mathbb{Z})^{s_K-4} \times (\mathbb{Z}/4\mathbb{Z})$ if $\nu_{K^+} = 1$ and $\mathcal{O}_{K^+}$ has no element with norm ± 2,*
4. *$(\mathbb{Z}/2\mathbb{Z})^{s_K-2}$ if $n \neq 2$ and $\mathcal{O}_{K^+}$ has an element with norm ± 2.*

Proof. By Lemma 4.11, $q_K = 1$ and, by hypothesis, $i_2 = 1$. Thus, $|\mathcal{P}o(K)| = 2^{s_K - 1 - \nu_{K^+}}$. Let $\mathfrak{P}$ (resp. $\mathfrak{p}^+$) be the prime of K (resp. of K^+) lying over 2. By Lemma 4.10, the class of $\mathfrak{P}$ is of order 2 or 4 depending on whether $\mathfrak{p}^+$ is principal or not. In order to use Lemma 4.9 we just have to know whether $\nu_{K^+} = 1$ or 0. Note also that, if $\mathfrak{p}^+$ is principal, then $\nu_{K^+} = 1$ or $n = 2$. □

Propositions 4.6 and 4.12 characterize $\mathcal{P}o(\mathbb{Q}(\sqrt{n}, \sqrt{-m}))$ up to an isomorphism. One finds descriptions of $\mathcal{P}o(K)$ for imaginary bicyclic biquadratic number fields based on Corollary 4.7 due to Zantema [25] when the field is of the form $\mathbb{Q}(\sqrt{-1}, \sqrt{n})$ in [22], or of the form $\mathbb{Q}(\sqrt{-2}, \sqrt{n})$ in [26] (note that Formula (13) gives also such descriptions since the class number of the fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-2})$ is one). We already spoke about [23] and its Theorem 3.

5 Real Bicyclic Biquadratic Number Fields

In the real case, the three square-free integers are positive. Let us recall Formula (9) in the real case:

$$|\mathcal{P}o(K)| = q_K \times 2^{s_K+j_2-2-\max(1,\nu_K)}. \tag{14}$$

Luckily, our formula agrees with the different formulas provided by [19, Theorem 3.3]. About the unit index q_K, we proved in Remark 3.4 that, in the real case, $q_K|8$, but said that in fact $q_K|4$ (Formula (10)). Let us see that $q_K \neq 8$ by means of the simple arguments that Jacques Boulanger suggested to me.

Proposition 5.1. *If $\sqrt{\varepsilon_1} \in K$, then either $d_2d_3 = d_1$ or $d_2d_3 = 4d_1$.*

Proof. Since $\sqrt{\varepsilon_1} \in K$, for every $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, $\sigma(\varepsilon_1) = (\sigma(\sqrt{\varepsilon_1}))^2 > 0$, in other words, ε_1 is totally positive, in particular $N_{k_1/\mathbb{Q}}(\varepsilon_1) = 1$. Let $\mathrm{Gal}(k_1/\mathbb{Q}) = \langle\sigma_1\rangle$ and to simplify write tr_1 instead of $Tr_{k_1/\mathbb{Q}}$. Then for $\delta \in \{\pm 1\}$, one see easily that

$$(\varepsilon_1+\delta)(\sigma_1(\varepsilon_1)+\delta) = tr_1(\varepsilon_1)+2\delta \;\; and \;\; \varepsilon_1(\varepsilon_1+\delta)(\sigma_1(\varepsilon_1)+\delta) = (\varepsilon_1+\delta)^2.$$

Consequently,

$$\sqrt{\varepsilon_1} = \frac{\varepsilon_1+\delta}{\sqrt{tr_1(\varepsilon_1)+2\delta}}$$

If $\sqrt{tr_1(\varepsilon_1)+2\delta}$ was a square in $\mathbb{N}$, then $\sqrt{\varepsilon_1}$ would be in $\mathcal{O}_{k_1}$, but this is impossible since ε_1 is a fundamental unit in k_1, thus $\sqrt{\varepsilon_1}$ is of degree 4. Consequently, the integers $tr_1(\varepsilon_1)+2\delta$ are not square in $\mathbb{N}$ and the fields $\mathbb{Q}(\sqrt{tr(\varepsilon_1)+2\delta})$ are quadratic subfields of K. Both subfields are distinct from k_1 because, if one of them was equal to k_1, it would contain $\sqrt{\varepsilon_1}$. They can not be equal to each other since else they would be a quadratic field containing $\sqrt{tr_1(\varepsilon_1)^2-4}$, and then equal to k_1. Indeed, if $\varepsilon_1 = a+b\sqrt{d_1}$, then $\sqrt{tr_1(\varepsilon_1)^2-4} = b\sqrt{d_1}$ since $1 = N_{k_1/\mathbb{Q}}(\varepsilon_1) = a^2-d_1b^2$.

Thus, for instance, $\mathbb{Q}(\sqrt{tr(\varepsilon_1)+2}) = \mathbb{Q}(\sqrt{d_2})$ and $\mathbb{Q}(\sqrt{tr(\varepsilon_1)-2}) = \mathbb{Q}(\sqrt{d_3})$. Consequently, $\gcd(d_2,d_3) \mid \gcd(tr_1(\varepsilon_1)+2, tr_1(\varepsilon_1)-2) \mid 4$. As $\gcd(d_2,d_3)$ is square-free, it is equal to 1 or 2 which means that $d_2d_3 = d_1$ or $d_2d_3 = 4d_1$. □

Corollary 5.2. *If $\sqrt{\varepsilon_1}$ and $\sqrt{\varepsilon_2} \in K$, then $d_3 = 2$.*

Proof. By Proposition 5.1, $d_2d_3 = 4^u d_1$ and $d_1d_3 = 4^v d_2$ where $u,v \in \{0,1\}$. Consequently, $d_3 = 2^{u+v}$, and hence, $d_3 = 2$. □

Corollary 5.3. *The unit index q_K divides 4.*

Proof. The containments

$$(\mathcal{O}_K^\times)^{(2)} \subsetneq \{\pm1\}\times(\mathcal{O}_K^\times)^{(2)} \subseteq \mathcal{O}_{k_1}^\times\mathcal{O}_{k_2}^\times\mathcal{O}_{k_3}^\times \subseteq \mathcal{O}_K^\times$$

together with $|\mathcal{O}_K^\times/(\mathcal{O}_K^\times)^{(2)}| = 16$ show that, the assumption $q_K = 8$ implies that $\{\pm1\}\times(\mathcal{O}_K^\times)^{(2)} = \mathcal{O}_{k_1}^\times\mathcal{O}_{k_2}^\times\mathcal{O}_{k_3}^\times$, and hence, that $\sqrt{\varepsilon_i} \in K$ for $i = 1,2,3$. By Corollary 5.2, this would imply $d_1 = d_2 = d_3 = \sqrt{2}$, this is a contradiction. □

In view of a partial converse of Lemma 4.2, we recall Setzer's following result.

Proposition 5.4. [21, Theorem 4] *Let K be a real bicyclic biquadratic number field. Let $H = H^1(G,\mathcal{O}_K^\times)$ and denote by $H[2]$ the subgroup of H formed by the elements of order ≤ 2. Then, $(H:H[2]) = 1$ or 2. It is 2 if and only if 2 is totally ramified in $K/\mathbb{Q}$ and there exist integers $x_i \in k_i$ such that*

$$N_{k_1/\mathbb{Q}}(x_1) = N_{k_2/\mathbb{Q}}(x_2) = N_{k_3/\mathbb{Q}}(x_3) = +2 \text{ or } -2.$$

Corollary 5.5. *Let K be a real bicyclic biquadratic number field. If 2 is totally ramified in K, the following assertions are equivalent:*

1. *the ideal $\Pi_2(K)$ is principal,*
2. *there exist integers x_i in each quadratic subfields k_i such that*

$$N_{k_1/\mathbb{Q}}(x_1) = N_{k_2/\mathbb{Q}}(x_2) = N_{k_3/\mathbb{Q}}(x_3) = +2 \text{ or } -2. \tag{15}$$

Proof. We already know that (1) implies (2). Assume that (2) holds. By Proposition 5.4, $H^1(G,\mathcal{O}_K^\times) \neq H^1(G,\mathcal{O}_K^\times)[2]$. Now consider $\mathcal{P}_K^G/\mathcal{P}_\mathbb{Q}$ and its subgroup $(\mathcal{P}_K^G/\mathcal{P}_\mathbb{Q})[2]$ formed by the elements of order ≤ 2. By Iwasawa's isomorphism (Formula (7)), $\mathcal{P}_K^G/\mathcal{P}_\mathbb{Q} \simeq H^1(G,\mathcal{O}_K^\times)$, and hence, $(\mathcal{P}_K^G/\mathcal{P}_\mathbb{Q})[2] \simeq H^1(G,\mathcal{O}_K^\times)[2]$. Thus, $(\mathcal{P}_K^G/\mathcal{P}_\mathbb{Q})[2] \neq \mathcal{P}_K^G/\mathcal{P}_\mathbb{Q}$ which means that 2 is totally ramified and that the ideal $\Pi_2(K)$ is principal. □

As a consequence of Corollary 5.5 and Proposition 3.8, we have:

Corollary 5.6. *Let K be the compositum of two real quadratic Pólya fields. The following assertions are equivalent:*

1. *K is a Pólya field,*
2. *the ideal $\Pi_2(K)$ is principal,*
3. *either 2 is not totally ramified, or there exist integers x_i in each quadratic subfields k_i such that $N_{k_1/\mathbb{Q}}(x_1) = N_{k_2/\mathbb{Q}}(x_2) = N_{k_3/\mathbb{Q}}(x_3) = +2$ or -2.*

The study of the composita of two real quadratic Pólya fields is undertaken by Leriche in [16] and [17] and more precise results are then given by Tougma [24].

Remark 5.7. There are Pólya bicyclic biquadratic number fields which are not obtained as such a compositum: the number of ramified primes in the compositum of two Pólya real quadratic number fields is bounded by 4, while, following Zantema [25], there exist bicyclic biquadratic Pólya fields where 5 primes are ramified, namely the number field $\mathbb{Q}(\sqrt{5.7}, \sqrt{3.127})$ which contains only one quadratic Pólya subfield. Maarefparvar [18, Theorem 4.4] generalizes Zantema's example by providing a family of biquadratic Pólya fields with five ramified primes and only one quadratic Pólya subfield. There are other counterexamples with less ramified primes: $\mathbb{Q}(\sqrt{3}, \sqrt{35})$ is a Pólya field with 4 ramified primes and only one quadradic Pólya subfield [11, Theorem C]; $\mathbb{Q}(\sqrt{7}, \sqrt{10})$ is a Pólya field with 3 ramified primes and only one quadradic Pólya subfield. But, following Maarefparvar [18, Theorem 4.1], there are examples of biquadratic Pólya fields with no Pólya quadratic subfield.

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