Moroccan Journal of Algebra and Geometry with Applications
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco

Volume 1, Issue 2 (2022), pp 361-379

ISSN: 2820-7114

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ISSN: 2820-7114

On a realization theorem in plane hyperbolic geometry and a related identity in neutral geometry

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Communicated by Najib Mahdou
(Received 10 May 2022, Revised 20 July 2022, Accepted 25 July 2022)

Abstract. Working in the upper half-plane model of plane hyperbolic geometry, we give a new proof that if $\alpha \leq \beta \leq \gamma$ are positive real numbers such that $\alpha + \beta + \gamma < \pi$, then there exists a hyperbolic triangle whose three (interior) angles have radian measures α, β and γ, respectively. Seeking yet another proof of this realization theorem produces a new identity involving the sines and cosines of the angles of any triangle in hyperbolic or Euclidean geometry. The only prerequisites assumed here are some topics in analytic geometry and trigonometry that are typically covered in a precalculus course and some basic facts from a first course on differential calculus. Thus, much of this paper could be used as enrichment material for a precalculus course or a calculus course, while all of this paper could to used to enrich a course on the classical geometries that features the upper half-plane model.

Key Words: Hyperbolic plane geometry, upper half-plane model, directed angle, tangent line, neutral geometry, identity.
2010 MSC: Primary 51-02; Secondary 33B10, 51N20.

1 Introduction

The discovery of non-Euclidean geometry, specifically that of plane hyperbolic geometry by Bolyai and Lobachevsky independently nearly 200 years ago, was important for many reasons. Perhaps the most important of those was the subsequent development of a deeper interest in the foundations of mathematics, in particular, in its axiomatic nature. For plane Euclidean geometry, several sets of axioms were developed, each sharing the goal of justifying all the planar geometric conclusions in Euclid's *Elements*. All those sets of axioms were equivalent, as it was shown that plane Euclidean geometry is categorical, in the sense that all of its models are isomorphic. Historically, the greatest interest has been placed in an equivalent of one of Euclid's postulates that most readers probably first learned as (Playfair's version of the) "parallel postulate." For centuries, workers had sought to determine whether an equivalent of that postulate was implied by the rest of Euclid's postulates for plane geometry. After removing that kind of postulate from plane Euclidean geometry, Bolyai used the remaining geometric postulates of Euclid to create what he called "absolute geometry." Subsequently, as our understanding of the universe progressed, that name has changed to "neutral geometry." It is known that, up to isomorphism, there are only two kinds of neutral geometry, namely, plane Euclidean geometry and plane hyperbolic geometry, and that each of these geometries is categorical. The above information is developed in a leisurely and lucid manner (in approximately 400 pages) in [2]. Our interest here is in both of these very different kinds of neutral geometry. As explained below, Section 2 will give a new, accessible proof of the realization theorem for plane hyperbolic geometry, while Section 3 will give a trigonometric identity that holds for the angles of any triangle in either Euclidean or hyperbolic geometry.

The reader is surely familiar with studying (plane) Euclidean geometry via the use of analytic geometry and trigonometry in the Euclidean plane $\mathbb{R}^2$. As in [4], our study here of (plane) hyperbolic geometry will focus on the upper half-plane model of plane hyperbolic geometry. The introduction to [4] summarized the salient features of that model and gave some relevant references. For the readers' convenience, the next paragraph summarizes some of that background material. (For the sake of completeness in regard to history, we should probably add the following two comments here. In the interest of brevity, the first sentence of this Introduction has compressed the very long history of other individuals whose work contributed to the emergence of hyperbolic geometry. Also, the logical consistency of plane hyperbolic geometry with the existence of $\mathbb{R}$ as a complete ordered field was essentially established by Beltrami who, in 1868, published two papers which served to provide the first model for hyperbolic plane geometry.)

From now on, most mentions of hyperbolic geometry in this paper will refer to the upper half-plane model of plane hyperbolic geometry. As is well known (cf. [7, Proposition 1], [11, Theorem 4.2.1]), a "line" in hyperbolic geometry is of one of two types: a so-called "straight geodesic," which is the intersection of the upper half-plane (of $\mathbb{R}^2$) with a vertical (Euclidean) line; or a so-called "bowed geodesic," which is the intersection of the upper half-plane with a (Euclidean) circle whose center is on the x-axis. By definition, the (radian) measure of a hyperbolic angle $\angle$ formed at a point of intersection of two hyperbolic "lines" $\mathcal{F}$ and $\mathcal{G}$ is the measure of the corresponding Euclidean angle formed by the corresponding (Euclidean) tangential half-lines to $\mathcal{F}$ and $\mathcal{G}$. An unambiguous 12-step algorithm to measure any such angle can be found in [4, Corollary 2.13]. (An essentially equivalent methodology was given by Millman and Parker earlier in [8].) In particular, the measurement of angles in hyperbolic geometry (whose "sides" are portions of straight or bowed geodesics) comes down to measuring the angles formed at the intersection of a (Euclidean) line with slope m_1 with either a (Euclidean) line with slope m_2 or a (Euclidean) vertical line. Formulas to accomplish that, in turn, were given in [3, Theorem 2.2], which was restated as [4, Lemma 2.6]. In this paper, we will use only the part of that result dealing with an angle formed at a point where a straight geodesic intersects a bowed geodesic. For the readers' convenience, that fragment of [3, Theorem 2.2] is restated as Lemma 2.4 below. For the most part, our interest here will be in angles whose (radian) measure is strictly between 0 and π because of their usefulness in studying triangles. As in [4], any "angle" under consideration here, regardless of whether its study involves a related (Euclidean or hyperbolic) triangle, will be viewed as a *directed angle*, that is, as an angle which arises via a counterclockwise rotation from a designated initial side to a designated terminal side. For more about the time-honored role of (directed) angles in the teaching of geometry and trigonometry, see [4, Remarks 2.4 (a) and 2.11 (a)].

Properties involving possible parallelism do not give the only way to distinguish between the two types of neutral geometry, that is, Euclidean and hyperbolic. The following celebrated results also provide a way to determine which of these options holds for a given neutral geometry $\mathfrak{G}$. Let α, β and γ be the (radian) measures of the three (interior) angles of a triangle Δ of $\mathfrak{G}$. By a result essentially due to Saccheri and Lagrange (cf. [6, Theorem 4.4], [11, Theorem 10.1.1]), $\alpha+\beta+\gamma \leq \pi$; if $\alpha+\beta+\gamma = \pi$, then $\mathfrak{G}$ is Euclidean (for the contrapositive, namely that if $\mathfrak{G}$ is hyperbolic then $\alpha+\beta+\gamma < \pi$, cf. [6, Theorem 6.1], [11, Theorem 7.2.1]); and if $\alpha+\beta+\gamma < \pi$, then $\mathfrak{G}$ is hyperbolic (since it is a classical fact that if $\mathfrak{G}$ is Euclidean, then $\alpha+\beta+\gamma = \pi$). Also, we recommend reading [6, page 99] as to the logical status of using π or its equivalent of 180° in the above role, along with its references to [2] and [9].

The results that were recalled in the last paragraph include the following information. If α, β and γ are the (radian) measures of the three (interior) angles of a triangle in a plane Euclidean geometry (resp., a plane hyperbolic geometry), then $\alpha+\beta+\gamma = \pi$ (resp., then $\alpha+\beta+\gamma < \pi$). In the spirit of obtaining "realizability" results, one can turn matters around and ask the following two questions. First: if $\mathfrak{G}$ is a plane Euclidean geometry and α, β and γ are positive real numbers such that $\alpha+\beta+\gamma = \pi$, does there exist a triangle in $\mathfrak{G}$ whose three (interior) angles have respective measures α, β and γ? Second: if $\mathfrak{G}$ is a plane hyperbolic geometry and α, β and γ are positive real numbers such that $\alpha+\beta+\gamma < \pi$, does there exist a triangle in $\mathfrak{G}$ whose three (interior) angles have respective measures

α, β and γ? The first of these questions is easily answered in the affirmative – in fact, it could likely be answered by most high school students – essentially because of the "homogeneous" nature of the Euclidean plane. This "homogeneity" is related to Euclid's (and Stahl's) propensity for viewing "congruence" of geometric figures as a property that can be proven by using superposition, rather than as an undefined primitive term that must be subjected to appropriate axioms. As can be seen by comparing a number of modern textbooks on geometry – for instance, compare [1] with [9], [10] and [11] – this kind of homogeneity/superposition can play the same kind of role as the SAS ("Side-Angle-Side") axiom in a modern set of axioms for neutral geometry. It is perhaps unfortunate that the upper half-plane model of hyperbolic geometry seems, at first glance, to be far from "homogeneous" in any intuitive sense. Indeed, the open neighborhoods (relative to the hyperbolic metric) "near" two given points in the upper half-plane may not "look alike", especially if the given points have different y-coordinates. (This is due to the hyperbolic metric essentially replacing the Euclidean differentials dx and dy with $(dx)/y$ and $(dy)/y$, respectively: cf. [11, pages 53-54 and 58].) Nevertheless, the answer to the second question is also affirmative, and as one would expect, the proof of that titular realizability result for the hyperbolic case is much harder than the corresponding "child's play" proof for the Euclidean case. Since that realizability result for hyperbolic geometry is already known and Section 2 is devoted to a new proof of it, we will devote Remark 2.7 (d) to addressing the appropriateness and the timeliness of any such new proof, by giving a frank assessment of the proofs of it which appear in [10] and in [11].

In view of the perceived deficiencies in the proofs of the hyperbolic realization theorem that were given in [10] and in [11], we devote Section 2 to an accessible, natural, complete and relatively short proof of the hyperbolic realization theorem. Instead of merely proving the *existence* of a suitable hyperbolic triangle, Corollary 2.6 completes our *construction* of a hyperbolic triangle Δ whose interior angles have the preassigned measures (namely, the three positive real numbers whose sum is less than π). Any such triangle Δ is determined up to congruence, as "Angle-Angle-Angle" is a congruence criterion in hyperbolic geometry: cf. [6, Theorem 6.2], [11, Theorem 7.2.3]. However, Corollary 2.6 (c) shows that the Δ which we construct is actually unique with respect to certain additional properties. Most of the technical details for our constructive proof in Corollary 2.6 (a) are first gathered together as Theorem 2.5. Background material on the cosh and sinh functions will be given as needed in Section 2. The statements of the three classical laws of hyperbolic trigonometry that are used in this paper are collected in Lemma 2.3. Of these, only the law that has no Euclidean counterpart will be used in Section 2.

As noted above, hyperbolic geometry is decidedly non-Euclidean, inasmuch as the hyperbolically open neighborhoods "near" two given points in the upper half-plane do not "look alike" if the given points have significantly different y-coordinates. Nevertheless, the main result in Section 3 (Corollary 3.8) is an identity that holds for the sines and cosines of the angles of *any* triangle in hyperbolic *or* Euclidean geometry. While our proof of this identity depends on the above-mentioned realization theorem, the closing remark of this paper raises the question whether one can find a new direct proof of that identity which does not use any geometric results. Any such direct proof, when coupled with the reasoning in the earlier part of Section 3, would lead to yet another proof of the realization theorem. Remark 3.9 also raises several other open questions and states some related partial results.

In closing, we wish to repeat something from the final two sentences of the abstract: any interested readers are encouraged to use portions of this paper to enrich courses ranging from precalculus to models-based courses on the classical geometries.

2 A new proof of the realization theorem

It will be useful to begin with the following result.

Proposition 2.1. *Let λ, μ and ν be three (not necessarily distinct) positive real numbers such that $\lambda+\mu+\nu \leq \pi$. Then*

$$\cos(\lambda)\cos(\mu)+\cos(\nu) \geq \sin(\lambda)\sin(\mu),$$

with equality if and only if $\lambda+\mu+\nu = \pi$. It follows that

$$\frac{\cos(\lambda)\cos(\mu)+\cos(\nu)}{\sin(\lambda)\sin(\mu)} \geq 1,$$

with equality if and only if $\lambda+\mu+\nu = \pi$.

Proof. Since the hypotheses ensure that $\sin(\lambda) > 0$ and $\sin(\mu) > 0$, it will suffice to prove the first assertion (as the " It follows that" assertion would then be an immediate consequence). To that end, fix λ and μ, and note that $0 < \nu \leq \pi-\lambda-\mu$. Define a function $f : (0, \pi-\lambda-\mu] \to \mathbb{R}$ by

$$f(x) := \frac{\cos(\lambda)\cos(\mu)+\cos(x)}{\sin(\lambda)\sin(\mu)}.$$

As $f^{'}(x) = -\sin(x)/(\sin(\lambda)\sin(\mu)) < 0$ for all x such that $0 < x < \pi-\lambda-\mu$, it follows from the Mean Value Theorem that f is a strictly decreasing function. Thus, since f is continuous and $\cos(\lambda)\cos(\mu)+1 > 0$ ensures that $f(0) > 0$, it will suffice to prove that

$$L := \lim_{x\to(\pi-\lambda-\mu)^-} f(x) = f(\pi-\lambda-\mu) = \frac{\cos(\lambda)\cos(\mu)+\cos(\pi-\lambda-\mu)}{\sin(\lambda)\sin(\mu)}$$

satisfies $L = 1$ or, equivalently, that $\cos(\lambda)\cos(\mu)+\cos(\pi-\lambda-\mu) = \sin(\lambda)\sin(\mu)$. In fact, since $\cos(\pi) = -1$ and $\sin(\pi) = 0$, it follows from the standard identities for expanding $\cos(u \pm v)$ (cf. [5, page 332]) that

$$\cos(\pi-\lambda-\mu) = \cos(\pi-(\lambda+\mu)) = -\cos(\lambda+\mu) =$$

$-\cos(\lambda)\cos(\mu)+\sin(\lambda)\sin(\mu)$. Hence, $\cos(\lambda)\cos(\mu)+\cos(\pi-\lambda-\mu) = \sin(\lambda)\sin(\mu)$, thus completing the proof. □

Before applying Proposition 2.1, we pause to recall the definitions and basic properties of the hyperbolic cosine function (denoted by cosh) and the hyperbolic sine function (denoted by sinh). By definition.

$$\cosh(x) = \frac{e^x+e^{-x}}{2} \text{ and } \sinh(x) = \frac{e^x-e^{-x}}{2} \text{ for all } x \in \mathbb{R}.$$

Observe the identity $\sinh^2(x) = \cosh^2(x)-1$. Thus, for all $a > 0$, the (Cartesian) graph given, for $t \in \mathbb{R}$, by the parametric equations $x = a\cosh(t)$ and $y = a\sinh(t)$ is the rectangular hyperbola $x^2/a^2-y^2/b^2 = 1$. This circumstance is, according to a common belief, the reason that the word "hyperbolic" appears in the names of the functions cosh and sinh and in the name "hyperbolic geometry."

Observe also that cosh and sinh are each differentiable (hence continuous) functions, with $\cosh^{'} = \sinh$ and $\sinh^{'} = \cosh$. Also, by using the facts that the range of the exponential function is $(0,\infty)$, $t+t^{-1} \geq 2$ for all $t > 0$, $\cosh(0) = 1$ and $\lim_{x\to\infty}\cosh(x) = \infty$, one gets via the Intermediate Value Theorem that the range of cosh is $[1,\infty)$. In particular, $\sinh^{'}(x) > 0$ for all $x \in \mathbb{R}$. Thus, by the Mean Value Theorem, sinh is a strictly increasing function. Similarly, by also using the facts that $e^x > e^{-x}$ for all $x > 0$, $\sinh(0) = 0$ and $\lim_{x\to\infty}\sinh(x) = \infty$, one gets via the Intermediate Value Theorem that the range of $\sinh|_{[0,\infty)}$ is $[0,\infty)$ and that $\cosh^{'}(x) > 0$ for all $x > 0$. Thus, by the Mean Value Theorem, the restriction of cosh to the domain $[0,\infty)$ is a strictly increasing function. Consequently, the assignment $d \mapsto \cosh(d)$ sets up a one-to-one correspondence between $[0,\infty)$ and $[1,\infty)$. Remark 2.7 (a) will present a formula for the assignment that induces the inverse one-to-one correspondence.

Corollary 2.2. *Let λ, μ and ν be three (not necessarily distinct) positive real numbers such that $\lambda+\mu+\nu<\pi$. Then there exists a unique $d>0$ such that*

$$\frac{\cos(\lambda)\cos(\mu)+\cos(\nu)}{\sin(\lambda)\sin(\mu)}=\cosh(d).$$

Proof. By the second ("It follows that") assertion in Proposition 2.1, the left-hand side of the displayed equation is in $(1,\infty)$. Therefore, by the last comment before the statement of the present result, there exists exactly one $d\in(0,\infty)$ that satisfies the displayed equation. The proof is complete. □

The functions cosh and sinh are involved in some classical laws of "hyperbolic trigonometry". For reference purposes, the statements of those laws are collected in the next result. Lemma 2.3 (a) is known as the Hyperbolic Law of Cosines (we will abbreviate that name as HLC); and Lemma 2.3 (b) is known as the Hyperbolic Law of Sines (we will abbreviate that to HLS). There is no customary name for Lemma 2.3 (c), as it has no Euclidean counterpart. While all three parts of Lemma 2.3 will be used in Section 3, only its part (c) will be used in Section 2.

Lemma 2.3. *Let $\Delta=\Delta ABC$ be a hyperbolic triangle; let α, β and γ be the (radian) measures of the (interior) angles of Δ at the vertices A, B and C, respectively; and let a, b and c be the hyperbolic lengths of the (straight or bowed geodesic) sides of Δ that are opposite the vertices A, B and C, respectively. Then:*

(*a*) (HLC : cf. *[11, Theorem 8.3.2 (i)], [6, formula (13), page 337]*)

$$\cos(\alpha)=\frac{\cosh(b)\cosh(c)-\cosh(a)}{\sinh(b)\sinh(c)},\cos(\beta)=\frac{\cosh(c)\cosh(a)-\cosh(b)}{\sinh(c)\sinh(a)}$$

$$\text{and}\qquad\cos(\gamma)=\frac{\cosh(a)\cosh(b)-\cosh(c)}{\sinh(a)\sinh(b)}.$$

(*b*) (HLS : cf. *[11, Theorem 8.3.2 (iii)], [6, formula (14), page 337]*)

$$\frac{\sin(\alpha)}{\sinh(a)}=\frac{\sin(\beta)}{\sinh(b)}=\frac{\sin(\gamma)}{\sinh(c)}.$$

(*c*) (Cf. *[11, Theorem 8.3.2 (ii)], [6, formula (15), page 337]*)

$$\cosh(a)=\frac{\cos(\beta)\cos(\gamma)+\cos(\alpha)}{\sin(\beta)\sin(\gamma)},\cosh(b)=\frac{\cos(\gamma)\cos(\alpha)+\cos(\beta)}{\sin(\gamma)\sin(\alpha)}$$

$$\text{and}\ \cosh(c)=\frac{\cos(\alpha)\cos(\beta)+\cos(\gamma)}{\sin(\alpha)\sin(\beta)}.$$

Next, for reference purposes, we restate part of one of our results from an earlier paper.

Lemma 2.4. *([3, Theorem 2.2]) Let L_1 and L_2 be two intersecting non-perpendicular lines in the Euclidean plane such that L_1 is vertical and L_2 has slope m. If $m>0$, then the (radian measures of the) two acute angles formed by L_1 and L_2 at their point of intersection are each given by $\frac{\pi}{2}-\tan^{-1}(m)$, and the (radian measures of the) two obtuse angles formed by L_1 and L_2 at their point of intersection are each given by $\frac{\pi}{2}+\tan^{-1}(m)$. If $m<0$, then the (radian measures of the) two acute angles formed by L_1 and L_2 at their point of intersection are each given by $\frac{\pi}{2}+\tan^{-1}(m)$, and the (radian measures of the) two obtuse angles formed by L_1 and L_2 at their point of intersection are each given by $\frac{\pi}{2}-\tan^{-1}(m)$.*

We will continue to work in the upper half-plane model of plane hyperbolic geometry, but it will be convenient to fix the following **riding hypotheses and notation** from this point on until the end of the proof of Theorem 2.5. Let $0 < \alpha \le \beta \le \gamma$ in $\mathbb{R}$ such that $\alpha + \beta + \gamma < \pi$. Using Corollary 2.2, there exists a uniquely determined positive real number c such that

$$\cosh(c) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)}.$$

Consider the points $A(0,1)$ and $B(0,e^c)$; observe that $A \neq B$ since $c \neq 0$. Using [4, Corollary 2.2], construct (uniquely determined) bowed geodesics $\mathcal{G}$ and $\mathcal{H}$ which pass through A and B, respectively, such that the slope of the tangent line to $\mathcal{G}$ at A is $m_1 := \cot(\alpha)$ (> 0) and the slope of the tangent line to $\mathcal{H}$ at B is $m_2 := -\cot(\beta)$ (< 0). Observe that $\mathcal{G} \neq \mathcal{H}$ (since $A \neq B$). Let C denote the (unique) point of intersection of $\mathcal{G}$ and $\mathcal{H}$. Let Δ denote the hyperbolic triangle ΔABC. Let γ^* be the radian measure of the (interior) angle $\angle BCA$ of Δ.

The main technical facts about the above data are given in the next result. For the sake of brevity, we will let Q_1 denote the first quadrant of $\mathbb{R}^2$.

Theorem 2.5. Given the above riding hypotheses and notation (α, β, γ, c, A, B, C, $\mathcal{G}$, $\mathcal{H}$, m_1, m_2, Δ, γ^* and Q_1), then:
(a) The hyperbolic distance from A to B is c.
(b) C is in Q_1.
(c) The (radian) measure of the (interior) angle $< CAB$ in Δ is α.
(d) The (radian) measure of the (interior) angle $< ABC$ in Δ is β.
(e) $\cos(\gamma^*) = \cos(\gamma)$.
(f) $\sin(\gamma^*) = \sin(\gamma)$.
(g) $\gamma^* = \gamma$; that is, the (radian) measure of the (interior) angle $< BCA$ in Δ is γ.

Proof. (a) By [11, Theorem 2.1 and Proposition 4.1.3], the hyperbolic distance from $A(0,1)$ to $B(0,e^c)$ is the hyperbolic length of the straight geodesic going from A to B, namely, $\ln(e^c/1) = c$, as asserted.

(b) By [4, Theorem 2.2], a Cartesian equation for $\mathcal{G}$ is $x^2 - 2m_1x + y^2 = 1$ and a Cartesian equation for $\mathcal{H}$ is $x^2 - 2m_2e^cx + y^2 = e^{2c}$. To find the x-coordinate of the point of intersection C of $\mathcal{G}$ and $\mathcal{H}$, subtract one of these equations from the other and then solve the resulting linear equation for x, getting

$$x = \frac{e^{2c} - 1}{2(m_1 - m_2e^c)}.$$

Note that $e^{2c} - 1 > 0$ since $2c > 0$. Thus, C is in Q_1 if and only if $m_1 - m_2e^c > 0$. This latter condition *does* hold because of our construction of m_1 and m_2. Indeed, since $0 < \alpha \le \beta < \pi/2$, we have $m_1 := \cot(\alpha) > 0$ and $m_2 := -\cot(\beta) < 0$, whence $m_1 > 0 > m_2e^c$ and $m_1 - m_2e^c > 0$, as desired.

(c) Let $\mathcal{L}$ be the (Euclidean) tangential half-line to $\mathcal{G}$ at A that emanates from A and goes toward C. As the slope of $\mathcal{L}$ is $m_1 = \cot(\alpha) > 0$ and C is in Q_1 by (b), it follows that $\mathcal{L}$ points into Q_1 (rather than into the second quadrant). (The last assertion will likely seem intuitively clear to many readers; for a formal way to identify $\mathcal{L}$, see the final assertion of [4, Lemma 2.12 (b)].) Therefore, the (interior) angle $\angle CAB$ of Δ is an acute angle. It remains to show that the (radian) measure of this acute angle is α. By the first assertion in Lemma 2.4, that measure is

$$\pi/2 - \tan^{-1}(m_1) = \pi/2 - \tan^{-1}(\cot(\alpha)) = \pi/2 - \tan^{-1}(\tan(\pi/2 - \alpha)).$$

Since $0 < \alpha < \pi/2$, the displayed expression simplifies to

$$\pi/2 - (\pi/2 - \alpha) = \alpha,$$

as asserted.

(d) We can adapt the proof of (c) by making some small, but important, changes. For the sake of completeness, we provide the details. Let L be the (Euclidean) tangential half-line to $\mathcal{H}$ at B that emanates from B and goes toward C. As the slope of L is $m_2 = -\cot(\beta) < 0$ and C is in Q_1 by (b), it follows that L points into Q_1 (rather than into the second quadrant), and so the (interior) angle $\angle ABC$ of Δ is an acute angle. It remains to show that the (radian) measure of this acute angle is β. By the second set of assertions in Lemma 2.4, that measure is

$$\pi/2 + \tan^{-1}(m_2) = \pi/2 + \tan^{-1}(-\cot(\beta)) = \pi/2 - \tan^{-1}(\cot(\beta)),$$

where the last step held because $\tan^{-1}$ is an odd function. By using a cofunction identity, we can simplify the last displayed expression to $\pi/2 - \tan^{-1}(\tan(\pi/2 - \beta))$. This, in turn, can be simplified, by adapting the reasoning in the final sentence of the proof of (c), to β, as asserted.

(e) Consider $q := (\cos(\alpha)\cos(\beta) + \cos(\gamma))/(\sin(\alpha)\sin(\beta))$. By the choice of c, we have that $\cosh(c) = q$. However, by applying the law of hyperbolic trigonometry in Lemma 2.3 (c) to the data in the hyperbolic triangle Δ, we also have, thanks to (a), (c) and (d), that

$$\cosh(c) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma^*)}{\sin(\alpha)\sin(\beta)}.$$

Hence,

$$\frac{\cos(\alpha)\cos(\beta) + \cos(\gamma^*)}{\sin(\alpha)\sin(\beta)} = \cosh(c) = q = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)}.$$

It follows that $\cos(\gamma^*) = \cos(\gamma)$, as asserted.

(f) If $0 \leq \theta \leq \pi$, then $\sin(\theta) = \sqrt{1 - \cos^2(\theta)}$. Applying this principle by taking θ to be first γ^* and then γ, we conclude, with the help of (e), that

$$\sin(\gamma^*) = \sqrt{1 - \cos^2(\gamma^*)} = \sqrt{1 - \cos^2(\gamma)} = \sin(\gamma),$$

as asserted.

(g) The assertion follows immediately by combining (e) and (f). The proof is complete. □

We can now complete our proof of the hyperbolic realization theorem. As Corollary 2.6 is the main result of Section 2, its statement does *not* presuppose the riding hypotheses and notation that were in effect for Theorem 2.5.

Corollary 2.6. *Let $\alpha \leq \beta \leq \gamma$ be three (not necessarily distinct) positive real numbers such that $\alpha + \beta + \gamma < \pi$. Then:*

(*a*) *There exists a hyperbolic triangle $\Delta = \Delta ABC$ (in the upper half-plane model) such that the (radian) measure of the (interior) angle of Δ with vertex A (resp., with vertex B; resp., with vertex C) is α (resp., β; resp., γ). The following is one way to construct a hyperbolic triangle Δ with these properties. Take c to be the uniquely determined positive real number such that*

$$\cosh(c) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)};$$

put $m_1 := \cot(\alpha)$ and $m_2 := -\cot(\beta)$; take A to be the point $(0,1)$ and take B to be the point $(0, e^c)$; let $\mathcal{G}$ (resp., $\mathcal{H}$) be the (bowed) geodesic with Cartesian equation $x^2 - 2m_1x + y^2 = 1$ (resp., with Cartesian equation $x^2 - 2m_2e^cx + y^2 = e^{2c}$); and take C to be the (unique) intersection point of $\mathcal{G}$ and $\mathcal{H}$.

(*b*) *The hyperbolic triangle Δ that was constructed in (a) is unique up to congruence.*

(*c*) *The hyperbolic triangle Δ that was constructed in (a) is unique with respect to being a hyperbolic* *triangle* $\Delta = \Delta ABC$ *having all four of the following properties:*

(*i*) *The* (*radian*) *measure of the* (*interior*) *angle of* Δ *with vertex* A (*resp., with vertex* B*; resp., with vertex* C) *is* α (*resp.,* β*; resp.,* γ)*;*

(*ii*) A *is the point* $(0,1)$*;*

(*iii*) B *lies above* A *on the* y*-axis of the Euclidean* (*half-*)*plane;*

(*iv*) C *is in the first quadrant of the Euclidean* (*half-*)*plane.*

Proof. (a) It suffices to combine the proof of part (b) of Theorem 2.5 with the statements of parts (c), (d) and (g) of Theorem 2.5.

(b) This assertion is known, as the AAA (the Angle-Angle-Angle) property is a congruence criterion in plane hyperbolic geometry (cf. [6, Theorem 6.2], [11, Theorem 7.2.3]). Part (b) is being included here in order to motivate, and to stand in contrast to, part (c).

(c) It follows from the running hypotheses and notation which were in effect for Theorem 2.5 that the hyperbolic triangle that was constructed in (a) has the properties (ii) and (iii). Moreover, by the proof of (a), that triangle has property (i); and that triangle has property (iv) by Theorem 2.5 (b).

Conversely, suppose that $\Delta = \Delta ABC$ is a hyperbolic triangle having all four of the properties (i)-(iv). By (ii) and (iii), B has Cartesian coordinates $(0,k)$ for some $k > 1$. Hence, by [11, Theorem 2.1 and Proposition 4.1.3], the hyperbolic distance from $A(0,1)$ to $B(0,k)$ is the hyperbolic length of the straight geodesic going from A to B, namely, $\ln(k/1) = \ln(k)$. We claim that this hyperbolic distance is c. Applying the law of hyperbolic trigonometry in Lemma 2.3 (c) to the hyperbolic triangle Δ and using property (i), we get

$$\cosh(\ln(k)) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)},$$

and so $\cosh(\ln(k)) = \cosh(c)$. By the final comment prior to the statement of Corollary 2.2, this proves the above claim that $\ln(k) = c$. Hence $k = e^c$, and so B does indeed have Cartesian coordinates $(0, e^c)$. It remains only to prove that C is the intersection point of the bowed geodesics $\mathcal{G}$ and $\mathcal{H}$ whose Cartesian equations were given in the statement of (a).

Let us proceed in this paragraph by suitably adapting the proof of Theorem 2.5 (c). Let $\mathfrak{G}$ be the (bowed) geodesic passing though A and C. Let $\mathcal{L}$ be the (Euclidean) tangential half-line to $\mathfrak{G}$ at A that emanates from A and goes toward C. Observe the following three facts: the bound vector $\mathbf{u}$ from A to B points vertically upward; C is in the first quadrant, by (iv); and by (i), $\mathcal{L}$ makes an (acute) angle with $\mathbf{u}$ that has radian measure α. It follows that $\mathcal{L}$ points into Q_1 (rather than into the second quadrant) and that $\mathcal{L}$ (as a Euclidean ray of a Euclidean line) is neither vertical nor horizontal. Let m denote the slope of $\mathcal{L}$. We have that $m \neq 0$. In fact, $m > 0$ (since the angle between $\mathcal{L}$ and $\mathbf{u}$ is acute). Therefore, by the first assertion in Lemma 2.4,

$$\alpha = \pi/2 - \tan^{-1}(m),$$

and so $\tan^{-1}(m) = \pi/2 - \alpha$. Thus, $m = \tan(\tan^{-1}(m)) = \tan(\pi/2 - \alpha) = \cot(\alpha)$. Hence, the slope of the tangent line to $\mathfrak{G}$ at A is the same as the slope of the tangent line to $\mathcal{G}$ at A (namely, $\cot(\alpha)$). Therefore, by the uniqueness assertion in [4, Corollary 2.2], $\mathfrak{G} = \mathcal{G}$.

Let us proceed in this paragraph by suitably adapting the reasoning in the preceding paragraph. For the sake of completeness, we provide the details. Let $\mathfrak{H}$ be the (bowed) geodesic passing though B and C. Let L be the (Euclidean) tangential half-line to $\mathfrak{H}$ at B that emanates from B and goes toward C. Observe the following three facts: the bound vector $\mathbf{v}$ from B to A points vertically downward (in fact, $\mathbf{v} = -\mathbf{u}$); C is in the first quadrant, by (iv); and by (i), L makes an (acute) angle with $\mathbf{v}$ that has radian measure β. It follows that L points into Q_1 (rather than into the second quadrant) and that L (as a Euclidean ray of a Euclidean line) is neither vertical nor horizontal. Let m^* denote the slope of L. We have that $m^* \neq 0$. In fact, $m^* < 0$ (since the angle between L and $\mathbf{v}$ is acute). Therefore, by the second set of assertions in Lemma 2.4,

$$\beta = \pi/2 + \tan^{-1}(m),$$

and so $\tan^{-1}(m) = \beta - \pi/2$. Thus,

$$m = \tan(\tan^{-1}(m)) = \tan(\beta - \pi/2) = -\tan(\pi/2 - \beta) = -\cot(\beta).$$

Hence, the slope of the tangent line to $\mathfrak{H}$ at B is the same as the slope of the tangent line to $\mathcal{H}$ at B (namely, $-\cot(\beta)$). Therefore, by the uniqueness assertion in [4, Corollary 2.2], $\mathfrak{H} = \mathcal{H}$. Consequently, C, which is the intersection point of $\mathfrak{G}$ and $\mathfrak{H}$, is also the intersection point of $\mathcal{G}$ and $\mathcal{H}$. The proof is complete. □

Remark 2.7. (a) The "constructive" aspect of our proof of Corollary 2.6 (which was mentioned in the statements of each of the three parts of that result) depended, in part, on the following fact: for each $r \geq 1$, there exists a unique $d \geq 0$ such that $\cosh(d) = r$, with $d = 0$ if and only if $r = 1$. Indeed, the "construction" of c leading up to the proof of Theorem 2.5 depended on this fact. For the sake of completeness, we next establish an explicit formula for d in terms of r, namely,

$$d = \ln(r + \sqrt{r^2 - 1}).$$

It seems interesting that a proof of the just-displayed formula does not explicitly require calculus.

For a proof, note that one is solving for d such that $(e^d + e^{-d})/2 = r$ or, equivalently, such that $e^{2d} - 2re^d + 1 = 0$. Another equivalent, thanks to the quadratic formula, is that

$$e^d = r \pm \sqrt{r^2 - 1}.$$

It may appear at first glance that there are two viable candidates for d, namely,

$$d_1 := \ln(r - \sqrt{r^2 - 1}) \text{ and } d_2 := \ln(r + \sqrt{r^2 - 1}).$$

However, if $r > 1$, the candidate d_1 is extraneous. (In detail, if $r > 1$, then: $(r-1)^2 < r^2 - 1$, whence $r - 1 < \sqrt{r^2 - 1}$, whence $(0 <)\ r - \sqrt{r^2 - 1} < 1$, whence $d_1 < \ln(1) = 0$.) Thus, for all $r \geq 1$, the unique $d \geq 0$ such that $\cosh(d) = r$ is given by

$$d = (d_2 =) \ln(r + \sqrt{r^2 - 1}).$$

(b) The construction in Corollary 2.6 (a) admits a number of analogues, and the uniqueness result in Corollary 2.6 (c) can be generalized in several ways. Perhaps the most straightforward of these analogues or generalizations concerns the actual construction of Δ in Corollary 2.6 (a). This involves the condition that the vertex C is in the first quadrant. Granted, that conclusion was established in Theorem 2.5 (b). But if one changes the construction in Theorem 2.5 by requiring that the slope of the tangent line to $\mathcal{G}$ at A is $-\cot(\alpha)$ (< 0) and the slope of the tangent line to $\mathcal{H}$ at B is $\cot(\beta)$ (> 0), then *for this new construction*, its vertex called C is in the second quadrant. With the help of Lemma 2.4, one can show this and eventually conclude that the newly constructed hyperbolic triangle also satisfies the conclusion of Corollary 2.6 (a). As Corollary 2.6 (b) predicts, the two versions of Δ are congruent. For a class that has the appropriate background, this congruence is most easily established by appealing to [11, Theorem 4.4.1 (b)], as each of these versions of Δ can be obtained from the other version by (the rigid hyperbolic motion of) reflecting through the y-axis.

As for generalizations of Corollary 2.6 (c): at the cost of more cumbersome calculations in such assertions of uniqueness, the condition (ii) in Corollary 2.6 (c) could be generalized by taking A to be *any* point (x_0, y_0) in the upper half-plane (that is, with $x_0 \in \mathbb{R}$ and $y_0 > 0$), as the statements of [4, Theorem 2.1 and Proposition 2.3] are strong enough to handle such data; condition (iii) in Corollary 2.6 (c) could be generalized by taking B to be any point in the upper half-plane that is *below* $A(x_0, y_0)$ (on the line $x = x_0$), with essentially no additional effort, since the downward-pointing straight geodesic half-line emanating from (x_0, y_0) gives just as good a "ruler" as the upward-pointing counterpart did in Theorem 2.5 and Corollary 2.6 (and, as we have already seen, Lemma 2.4 can handle both upward- and downward-pointing straight geodesic half-lines); and, as we saw in the preceding paragraph, condition (iv) in Corollary 2.6 (c) could be generalized by taking C to be in the second quadrant of the Euclidean (half-)plane.

(c) In the Introduction, we promised that Section 2 would include "an accessible, natural, complete and relatively short proof of the hyperbolic realization theorem." The reader can decide whether the proof of Corollary 2.6 (a) and the path that we took to arrive there can fairly be described as accessible, complete and relatively short. (Perhaps, some of what we say in (d) will affect the reader's decision, as such matters may be viewed as being comparative in nature.) We will be content here to explain why we believe that our proof/path deserve(s) to be considered "natural." In particular, we would argue that the triangle Δ constructed and studied in Theorem 2.5 and Corollary 2.6 (a) is natural and the conclusion that it has the asserted properties seems to have a sense of inevitability. Indeed, once one has read parts (a), (c) and (d) of Theorem 2.5, that reader knows that *if* there exists a triangle $\Delta^* = \Delta ABC^*$ that satisfies the conclusion of the (hyperbolic) realizability theorem, then Δ *must* also work, the point being that Δ is congruent to Δ^* (so, the radian measures of corresponding angles agree) because ASA ("Angle-Side-Angle") is a criterion for congruence of triangles in any neutral geometry (cf. [11, Proposition 26, page 243]). Thus, once one has read parts (a), (c) and (d) of Theorem 2.5, the reader who is already aware that the realizability theorem is a known result can conclude that our triangle Δ definitely does have the properties asserted in Corollary 2.6 (a), even though the reader may not yet have finished reading the proof of parts (e), (f) and (g) of Theorem 2.5 or started reading the proof of Corollary 2.6 (a). As many students and researchers know, the first proof of a result is often the most difficult to obtain and may not be the clearest proof of the result, since the first-time prover has to overcome a psychological burden of doubt, whereas someone seeking a proof of a known result can relax to some extent, facing much less of a worrisome burden of doubt, in the same way that homework is often less stressful than actual research.

(d) To close the section, we next give the promised frank assessment of the proofs of the realizability theorem that appeared in [10] and in [11]. We will also address what we see as the appropriateness and the timeliness of providing a new proof of this result.

For a period of more than 20 years, my colleagues and I taught a senior-level undergraduate course on the classical geometries to mathematics majors (most of whom intended to become high school mathematics teachers). In working with the upper half-plane model, nearly all of those students (and their teachers) preferred to use clear, natural and accessible Euclidean methods instead of what they/we perceived as contrived formulas that were difficult to memorize. As a result, most of the instructors developed methods that were distinct from those of the textbook (though, of course, their methods gave answers that agreed with the answers resulting from use of the textbook's methods). This was particularly the case for formulas dealing with hyperbolic distance (especially, hyperbolic length along a bowed geodesic) and the measure of angles. In regard to the latter, I was moved to write the following in [4, Remark 2.16 (e)]: "We believe that Stahl's proof calculating the measures of [certain data from [4, Example 2.15]] is slightly less accessible to most undergraduate classes than the approach given above in Example 2.15." Stahl's methodology for measuring angles that was met with such dissatisfaction at the University of Tennessee had appeared in [10, Proposition 6.2] (and, as discussed in the next paragraph, was reproduced with no essentially no changes in [11]). That methodology was used frequently in Stahl's proof of the hyperbolic realization theorem in the first edition of his textbook [10, Theorem 6.7]. That fact may explain why many of my colleagues and students disliked that proof. Perhaps we were not outliers in having such an opinion, for (as discussed in the next paragraph) the second edition of Stahl's textbook featured a completely different proof of that theorem. With the passage of many years since I last looked at that first edition, I must admit that a recent re-reading of the proof of the realization theorem in the first edition indicated to me that the proof is likely correct. However, I still find that proof to be poorly organized, unmotivated and fraught with contrived methodology. As Stahl is no longer alive to defend his work, let me offer the following counterpoint to my criticism. A comparison of the proof of [10, Theorem 6.7] with the proofs of Theorem 2.5 and Corollary 2.6 (a) (as given above) shows that Stahl (in his first edition) and I (here) have each done *constructive* proofs, not merely *existence* proofs. However, significant differences remain between Stahl's approach/result and mine, including the uniqueness result in our Corollary 2.6 (c) and its generalizations in (b) above. In (c), I explained why I believe that the triangle constructed and studied in the above proofs of Theorem 2.5 and Corollary 2.6 (a) is natural and that the conclusion in Corollary 2.6 (a) has a sense of inevitability. By way of contrast, I also believe, on re-reading [10] recently, that Stahl's proof meanders with a poor sense of direction, but I may be too close to this matter to be as unbiased as I would like. If a reader comes to the conclusion that my proof in Section 2 is probably the result of a subconscious catalytic process that transpired over a period of many years beginning with my initial reading of [10], I would not object. I believe that the proof in Section 3 is much less susceptible to any such charge. In any case, we can hopefully agree that it is healthy to consider whether/when a later proof of a known result deserves to be called new or better.

Another proof of the hyperbolic realization theorem was given in the second edition of Stahl's textbook [11, Theorem 6.1.4]. Although this proof is superficially attractive, I have concluded that it is somewhat lacking in rigor, for the following reason. This proof uses what Stahl calls "a continuous function" [11, page 83] (namely, the function determined by the assignment $b \mapsto \gamma(b)$ on [11, pages 82-83]) and then invokes the Intermediate Value Theorem for continuous functions. A number of my colleagues and students share my opinion that the definition of this function should have been made more precise in [11] and that the technically demanding details establishing its continuity should not have been left to the reader in [11]. I would add that the proof of [11, Theorem 6.1.4] also makes frequent use of [11, Proposition 6.1.1], which is essentially the same as [10, Proposition 6.2] (whose features that my colleagues, students and I found to be inconvenient were discussed in the preceding paragraph).

It seems natural to ask why an esteemed and experienced author like Stahl would decide to replace a constructive proof in a first edition with an intuitively pleasant existence proof (whose details are reportedly unpleasant) in a second edition. Sadly, Stahl is no longer alive to answer this question, but one of the three comments that he made under the heading "New to the Second Edition" in the Preface to [11] was the following [11, page viii]: "The calculational proof of the determination of the triangle by its angles alone has been replaced by a synthetic argument." In my experience, the opposite of "synthetic" is usually called "analytic," and so I will suppose that by "calculational," Stahl meant "analytic." During my masters-level research on geometry in 1964-65 (and in my earlier studies of philosophy and logic), I gathered that mathematicians had formed a consensus on what the words "analytic" and "synthetic" meant in regard to methods of proof in geometry. Part of that research involved reading an earlier edition of Moise's text [9], as well as [2]. Roughly speaking, I had concluded that reasoning from axioms with minimal calculation could be considered "synthetic," while proofs involving calculations with coordinates were typically considered "analytic." (The situation is, admittedly, a bit blurred in regard to the use of real numbers. Surely, they are implicit in Euclid's *Elements*, thanks to the contributions of Eudoxus. Moreover, in the spirit of Birkhoff's work in the early 20^{th} century, American authors such as Moise could be viewed as doing a "synthetic" approach to the foundations of geometry (as a modernization of Hilbert's *Grunglagen der Geometrie*), while still using "ruler axioms" with supporting calculations. At the same time, far fewer calculations or uses of real numbers appeared in contemporaneous European "synthetic" modernizations of the *Grundlagen* such as [2].) However, I formed the distinct impression that *any* study that worked within models, rather than directly with the axioms of the underlying geometric theory, was definitely to be considered "analytic." With that interpretation, *both* of Stahl's proofs of the realizability theorem would be considered "analytic" (as would our proof of Corollary 2.6). My understanding of the analytic/synthetic dichotomy was reinforced in 1966-67 during my doctoral studies by my work grading homework for a class on the classical geometries. It seems clear from the above quotation from [11] that Stahl was using the words "calculational" and "synthetic" with meanings that differ significantly from what I had discerned from my masters-level research on geometry and my subsequent grading activities as a doctoral student. While my masters and doctoral studies occurred at two different universities and my understanding of the meaning of "analytic" and "synthetic" has not changed significantly as a result of my subsequent employment at, and visits to, several other universities, it is certain that none of us could hope (or want?) to visit "most" centers of mathematical activity. So, I cannot conclude with any plausibility that my understanding of those terms is "right" and Stahl's was "wrong". But I can, and do, conclude that I did not find the above quotation from [11] to be enlightening. In particular, that quotation has not helped me to find an answer to the question that was raised at the beginning of this paragraph. And I would maintain that one cannot convert a "calculational" model-based proof into a "synthetic" proof just by omitting the details of some necessary calculations! Closing with a more conciliatory tone, I do hope that this paragraph can serve as a plea that when one is explaining choices that were made as to the writing of certain mathematics, one should guard against using unexplained, emotive, interpretive words whose meaning may be lost on some earnest readers whose mathematical background is different from the author's.

3 An identity in neutral geometry

As explained in Remark 2.7 (c), once we knew that the hyperbolic realization theorem was a known result, we could somewhat relax after reading the proof of parts (a)-(d) of Theorem 2.5, for the following reason. The triangle Δ that was being studied in Section 2 was certain to satisfy the desired conclusions in parts (e)-(g) of Theorem 2.5 and in Corollary 2.6 (a) because Δ was congruent to a triangle that was known to satisfy those conclusions. Indeed, that moment of "relaxation" occurred when one noticed that the data from both triangles matched up according to the "ASA" congruence criterion. This section investigates whether yet another proof of the hyperbolic realization theorem is possible if one studies another triangle (which will also be denoted by Δ) whose data match up with the above-mentioned triangle according to the "SAS" congruence criterion. In fact, the first part of Section 3 will recount the initial results of this project, as we began the work on it with the "SAS" criterion in mind, in view of the basic role that this congruence criterion has played in many approaches to the axiomatic foundations of plane hyperbolic geometry (and, more generally, of neutral geometry). It will turn out that this section's Δ also has the expected radian measures for its interior angles, but that conclusion will not be reached as easily as the corresponding conclusion transpired in the proofs of Theorem 2.5 and Corollary 2.6 (a) in Section 2. In fact, to reach the desired conclusions in this section, I will need to use Corollary 2.6(a)! This unexpected turn of events will produce an unexpected dividend, namely, what is apparently a new identity for neutral geometry, in Corollary 3.8. Before presenting this (to the author, surprising) chain of developments, we will set up some riding notation and hypotheses and give an easy Euclidean observation.

It will be convenient to fix the following **riding hypotheses and notation** from this point on until the end of the proof of Corollary 3.2. Let $0 < \alpha \leq \beta \leq \gamma$ in $\mathbb{R}$ such that $\alpha + \beta + \gamma \leq \pi$. Put

$$u := \frac{\cos(\alpha)\cos(\gamma) + \cos(\beta)}{\sin(\alpha)\sin(\gamma)}, \; w := \sqrt{u^2 - 1},$$

$$v := \frac{\cos(\alpha)\cos(\beta)+\cos(\gamma)}{\sin(\alpha)\sin(\beta)},\ z := \sqrt{v^2-1},$$

$$N := uz - wv\cos(\alpha) \text{ and } D := \sqrt{(uv - wz\cos(\alpha))^2 - 1}.$$

Of course, each of u, w, v, z, N and D is a function of (at least some of) α, β and γ. Although the above notation has suppressed these functional relationships in the interest of simplifying notation, we trust that the reader will find such suppressions to be harmless. Also, let the equation $N = D\cos(\beta)$ be denoted by $(*)$. When $(*)$ holds, we will say that α, β and γ *satisfy the equation* $(*)$. Our reasons for introducing the above notation will become apparent as one reads the rest of Section 3.

Proposition 3.1. *Given the above riding hypotheses and notation, suppose, in addition, that* $\alpha+\beta+\gamma=\pi$. *Then* $u=1$, $w=0$, $v=1$, $z=0$, $N=0$ *and* $D=0$. *Hence,* $N = D\cos(\beta)$; *that is,* α, β *and* γ *satisfy the equation* $(*)$.

Proof. It will suffice to prove that $u = 1$ and $v = 1$, as the remaining assertions will then follow easily. Therefore, as $\alpha+\beta+\gamma=\pi$, it will suffice to prove the following: if θ, φ and ψ are positive real numbers such that $\theta+\varphi+\psi=\pi$, then

$$\cos(\theta)\cos(\varphi)+\cos(\psi) = \sin(\theta)\sin(\varphi) \neq 0$$

To that end, using the facts that $\cos(\pi) = -1$ and $\sin(\pi) = 0$, along with the standard expansions for the cosine of a difference or a sum (as in [5, page 332]), we have

$$\cos(\psi) = \cos(\pi-(\theta+\varphi)) = \cos(\pi)\cos(\theta+\varphi)+\sin(\pi)\sin(\theta+\varphi) =$$

$$-(\cos(\theta+\varphi)) = -(\cos(\theta)\cos(\varphi)-\sin(\theta)\sin(\varphi)) =$$

$\sin(\theta)\sin(\varphi)-\cos(\theta)\cos(\varphi)$. The desired equation follows at once. Finally, since θ and φ are each strictly between 0 and π, both $\sin(\theta)$ and $\sin(\varphi)$ are nonzero (in fact, positive) real numbers, so $\sin(\theta)\sin(\varphi)\neq 0$. The proof is complete. □

Note that another way to prove that $u = 1$ and $v = 1$ in the context of Proposition 3.1 is to apply the final conclusion of Proposition 2.1.

A classic result in Euclidean geometry states that the sum of the radian measures of the three interior angles of any triangle (in Euclidean geometry) is π. Therefore, Proposition 3.1 has the following application to Euclidean geometry.

Corollary 3.2. *Let* Δ *be any triangle in Euclidean geometry and let* $\alpha\le\beta\le\gamma$ *be the radian measures of the three interior angles of* Δ. *Then* $N = D\cos(\beta)$; *in other words,* α, β *and* γ *satisfy the equation* $(*)$.

To obtain the (much harder) analogue of Corollary 3.2 for plane hyperbolic geometry, we will need to strengthen the riding hypotheses that were in effect earlier in this section. From this point on until the end of the proof of Corollary 3.7, we fix the following **riding hypotheses and notation**: let $0<\alpha\le\beta\le\gamma$ in $\mathbb{R}$ such that $\alpha+\beta+\gamma<\pi$; let the symbols u, w, v, z, N, D and the equation $(*)$ be defined in terms of α, β and γ exactly as they were prior to the statement of Proposition 3.1; and, as in Section 2 , let Q_1 denote the first quadrant.

In order to study the properties of a suitable hyperbolic triangle, we now proceed to **add the information in this paragraph to the riding hypotheses and notation that will be in effect until the end of the proof of Corollary 3.7**. Using Corollary 2.2, there exist uniquely determined positive real numbers b and c such that

$$\cosh(b) = \frac{\cos(\gamma)\cos(\alpha)+\cos(\beta)}{\sin(\gamma)\sin(\alpha)} \text{ and } \cosh(c) = \frac{\cos(\alpha)\cos(\beta)+\cos(\gamma)}{\sin(\alpha)\sin(\beta)}.$$

Then by using Remark 2.7 (a), one can easily obtain formulas for b and c in terms of α, β and γ. Consider the points $A(0,1)$ and $B(0,e^c)$; observe that $A \neq B$ since $c \neq 0$. Using [4, Corollary 2.2], construct a (uniquely determined) bowed geodesic $\mathcal{G}$ which passes through A such that the slope of the tangent line to $\mathcal{G}$ at A is $m_1 := \cot(\alpha)$ (> 0). Let C denote the point on $\mathcal{G}$ such that C lies in Q_1 and the hyperbolic distance from A to C (that is, the hyperbolic length along the bowed geodesic $\mathcal{G}$ from A to C) is b. (Note that C is uniquely determined because of the "ruler" property of the hyperbolic half-line that is induced by $\mathcal{G}$, emanates from A and points into Q_1.) Let Δ denote the hyperbolic triangle ΔABC. By Theorem 2.5 (c), the radian measure of the interior angle of Δ with vertex A is α. Let β^* (resp., γ^*) be the radian measure of the interior angle of Δ with vertex B (resp., vertex C); and let a^* be the hyperbolic length of the side BC of Δ, that is, of the "hyperbolic line segment" of the bowed geodesic connecting B to C. (WARNING: You may not assume at this time that this Δ is the same as the hyperbolic triangle that was denoted by Δ in Section 2; these two hyperbolic triangles *are* indeed the same, but that is a fact which can only be proved, as a consequence of AAA being a congruence criterion in plane hyperbolic geometry, after we have proved Theorem 3.6 (a). Moreover, while it is clear that the symbols A, B, m_1 and $\mathcal{G}$ which were defined earlier in this paragraph have the same meanings as their counterparts did in Section 2 (in regard to the triangle that was denoted by Δ in Section 2), you may not assume at this time that this "same meaning" conclusion applies to any of the other relevant data, such as C.) The main technical facts about the above data will be obtained in the results 3.3-3.5.

We can now begin to explain why some of the above notation was introduced. It is clear that $u = \cosh(b)$ and $v = \cosh(c)$. Thus, in view of the identity $\sinh(t) = \sqrt{\cosh^2(t) - 1}$ for all $t \geq 0$, we also have that $w = \sinh(b)$ and $z = \sinh(c)$. Also, by Remark 2.7 (a),

$$b = \ln(u + \sqrt{u^2 - 1}) = \ln(u + w) \text{ and } c = \ln(v + \sqrt{v^2 - 1}) = \ln(v + z).$$

These observations suggest that it will be pragmatic to make the following definition.

It will be convenient to say that we *know* a certain real number r (resp., a certain point P in $\mathbb{R}^2$) that is of interest if the value of r (resp., if the x- and y-coordinates of P) could be obtained via a formula (resp., via formulas) for r (resp., for those coordinates) stated in terms of the assumed values of the riding data α, β and γ. Thus, we already know each of u, w, v, z, N, D, b and c. The next result tells us some more in regard to what we know about the ambient hyperbolic triangle Δ.

Proposition 3.3. *Given the above riding hypotheses and notation, then:*

(a) We know (in the sense of the above definition) each of the following: b, c, A, B, a^*, β^* and γ^*.

(b) $\beta = \beta^*$ if and only if $\gamma = \gamma^*$.

Proof. (a) By the last two paragraphs that preceded the statement of this result, we know b and c. Thus, by definition, we know $A(0,1)$ and $B(0,e^c)$. We turn next to a^*, β^* and γ^*.

By the hyperbolic law of cosines (HLC) in Lemma 2.3 (a), we have

$$\cos(\alpha) = \frac{\cosh(b)\cosh(c) - \cosh(a^*)}{\sinh(b)\sinh(c)}.$$

By viewing this display as a linear equation in the "unknown" $\cosh(a^*)$ and then solving that equation, it follows (bearing in mind that we already know b and c) that we know $\cosh(a^*)$. So, it follows from Remark 2.7 (a) that we know a^*. It also follows that we know $\sqrt{\cosh^2(a^*) - 1} = \sinh(a^*)$, and so we also know $\sin(\alpha)/\sinh(a^*)$. Therefore, by the hyperbolic law of sines (HLS) in Lemma 2.3 (b), we also know $\sin(\beta^*)/\sinh(b)$ and $\sin(\gamma^*)/\sinh(c)$. As we also know b and c, it follows that we know $\sin(\beta^*)$ and $\sin(\gamma^*)$. Therefore, we know the right-hand side in the next display (which is provided by the HLC):

$$\cos(\beta^*) = \frac{\cosh(a^*)\cosh(c) - \cosh(b)}{\sinh(a^*)\sinh(c)}.$$

It follows that we know $\cos(\beta^*)$ and, hence, we know $\cos^{-1}(\cos(\beta^*)) = \beta^*$ (where the last step held because $0 \le \beta^* \le \pi$). It remains only to prove that we know γ^*. That, in turn, can be shown by tweaking the preceding reasoning, as follows. Use the HLC to get

$$\cos(\gamma^*) = \frac{\cosh(a^*)\cosh(b) - \cosh(c)}{\sinh(a^*)\sinh(b)},$$

observe that we know the right-hand side of this display, whence we know $\cos(\gamma^*)$, and conclude that we know $\cos^{-1}(\cos(\gamma^*)) = \gamma^*$ (where the last step held because $0 \le \gamma^* \le \pi$).

(b) We will show that if $\beta = \beta^*$, then $\gamma = \gamma^*$, leaving the (similar) proof of the converse to the reader. Using the defining property of $\cosh(c)$, the law of hyperbolic trigonometry in Lemma 2.3 (c) and the assumption that $\beta = \beta^*$, we get

$$\frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)} = \cosh(c) =$$

$$\frac{\cos(\alpha)\cos(\beta^*) + \cos(\gamma^*)}{\sin(\alpha)\sin(\beta^*)} = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma^*)}{\sin(\alpha)\sin(\beta)}.$$

It follows that $\cos(\alpha)\cos(\beta) + \cos(\gamma) = \cos(\alpha)\cos(\beta) + \cos(\gamma^*)$, whence $\cos(\gamma) = \cos(\gamma^*)$, whence $\gamma = \gamma^*$ (since $\cos|_{[0,\pi]}$ is a one-to-one function). The proof is complete. □

Part (b) of Proposition 3.3 will be put to significant use below. On the other hand, the proof of Proposition 3.3 (a) gave us renewed contact with the laws of hyperbolic trigonometry from Lemma 2.3. Reformulations of one of those laws, the HLC, will see repeated use in the proof of Theorem 3.4, which will establish what is perhaps the most important equation in Section 3.

Recall that N and D are certain real numbers that are defined in terms of the riding hypotheses and notation.

Theorem 3.4. Given the above riding hypotheses and notation, we have $N/D = \cos(\beta^*)$, and hence, $N = D\cos(\beta^*)$.

Proof. By the HLC, we get

$$\text{(1)} \qquad \cos(\beta^*) = \frac{\cosh(a^*)\cosh(c) - \cosh(b)}{\sinh(a^*)\sinh(c)} \qquad \text{and}$$

$$\text{(2)} \qquad \cos(\alpha) = \frac{\cosh(b)\cosh(c) - \cosh(a^*)}{\sinh(b)\sinh(c)}.$$

Solving (2) for $\cosh(a^*)$ and substituting the result into (1), we get that $\cos(\beta^*) =$

$$\text{(3)} \qquad \frac{(\cosh(b)\cosh(c) - \cos(\alpha)\sinh(b)\sinh(c))\cosh(c) - \cosh(b)}{\sinh(a^*)\sinh(c)}.$$

Using the identity $\sinh^2(t) = \cosh^2(t) - 1$, we can rewrite (3) as

$$\frac{\cosh(b)(\cosh^2(c) - 1) - \cos(\alpha)\sinh(b)\sinh(c))\cosh(c)}{\sinh(a^*)\sinh(c)} =$$

$$\text{(4)} \qquad \frac{\cosh(b)\sinh^2(c) - \cos(\alpha)\sinh(b)\sinh(c)\cosh(c)}{\sinh(a^*)\sinh(c)}.$$

Since $c > 0$, we have $\cosh(c) > 1$, and so $\sinh(c) \neq 0$. Hence, we can factor $\sinh(c)/\sinh(c)$ out of the last display. This gives

$$(5) \qquad \cos(\beta^*) = \frac{\cosh(b)\sinh(c) - \cos(\alpha)\sinh(b)\cosh(c)}{\sinh(a^*)}.$$

As noted prior to Proposition 3.3, $u = \cosh(b)$, $v = \cosh(c)$, $w = \sinh(b)$ and $z = \sinh(c)$. Hence, (5) gives

$$(6) \qquad \cos(\beta^*) = \frac{uz - \cos(\alpha)wv}{\sinh(a^*)} = \frac{N}{\sinh(a^*)}.$$

Therefore, it will suffice to prove that $\sinh(a^*) = D$. As $\sinh(a^*) > 0$, an equivalent task is to prove that $\sinh^2(a^*) = D^2$; equivalently, that $\cosh^2(a^*) = D^2 + 1$. Thus, as $\cosh(a^*) \geq 1$, it will suffice to prove that $\cosh(a^*) = \sqrt{D^2+1}$, that is, that

$$(7) \qquad \cosh(a^*) = |uv - \cos(\alpha)wz| \quad \text{or, equivalently,}$$

$$(8) \qquad \cosh(a^*) = |\cosh(b)\cosh(c) - \cos(\alpha)\sinh(b)\sinh(c)|.$$

By (2), $\cosh(a^*) = \cosh(b)\cosh(c) - \cos(\alpha)\sinh(b)\sinh(c)$. Therefore, since $\cosh(a^*) > 0$, the equation in (8) follows. The proof is complete. □

Recall that $(*)$ denotes the equation $N = D\cos(\beta)$. The next result characterizes when this equation holds.

Corollary 3.5. *Given the above riding hypotheses and notation (involving α, β, γ, N, D, $\Delta = \Delta ABC$ and β^*), then the following conditions are equivalent:*

(1) The (radian) measure of the (interior) angle of Δ that has vertex A (resp., vertex B; resp., vertex C) is α (resp., β; resp., γ);

(2) $\beta^* = \beta$;

(3) $\cos(\beta) = N/D$;

(4) $N = D\cos(\beta)$;

(5) α, β and γ satisfy the equation $(*)$.

Proof. By Theorem 3.4, $\cos(\beta^*) = N/D$, and so $\beta^* = \cos^{-1}(N/D)$ (since $0 \leq \beta^* \leq \pi$). It follows that $D \neq 0$. Therefore, (3) ⇔ (4). On the other hand, (3) is equivalent to $\cos^{-1}(N/D) = \beta$ (since $0 \leq \beta \leq \pi$). Thus, (3) ⇔ (2). Also, the equivalence (4) ⇔ (5) is immediate from the definition underlying (5). Thus, as (1) ⇒ (2) trivially, it remains only to prove that (2) ⇒ (1).

By Theorem 2.5 (c), the radian measure of the interior angle of Δ that has vertex A is α; and, by Proposition 3.3 (b), $\beta^* = \beta$ if and only if $\gamma^* = \gamma$. Consequently, (2) ⇒ (1). The proof is complete. □

The next result shows that, as expected, the interior angles of the ambient triangle Δ have the desired radian measures. However, one should not view Theorem 3.6 (a) as this paper's second proof of the hyperbolic realizability theorem, since the proof of Theorem 3.6 (a) will use Corollary 2.6.

Theorem 3.6. Let $\alpha \leq \beta \leq \gamma$ be three (not necessarily distinct) positive real numbers such that $\alpha + \beta + \gamma < \pi$. Then:

(a) Take b and c to be the uniquely determined positive real numbers such that

$$\cosh(b) = \frac{\cos(\gamma)\cos(\alpha) + \cos(\beta)}{\sin(\gamma)\sin(\alpha)} \text{ and } \cosh(c) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)}.$$

Take A to be the point $(0,1)$; take B to be the point $(0, e^c)$; put $m_1 := \cot(\alpha)$; let $\mathcal{G}$ denote the (bowed) geodesic that has the Cartesian equation $x^2 - 2m_1 x + y^2 = 1$; and let C be the point on $\mathcal{G}$ such that

C lies in the first quadrant and the hyperbolic distance from A to C is b. Consider the hyperbolic triangle $\Delta = \Delta ABC$ (in the upper half-plane model). Then the (radian) measure of the (interior) angle of Δ with vertex A (resp., with vertex B; resp., with vertex C) is α (resp., β; resp., γ).

(b) α, β and γ satisfy the equation $(*)$.

Proof. (a) The hyperbolic realizability theorem is a known result. (We have already noted published proofs of it in [10, Theorem 6.7] and [11, Theorem 6.1.4], and a new proof of it was given in Corollary 2.6 (a).) Let $\Delta_* = \Delta A_*B_*C_*$ denote the hyperbolic triangle that was studied in Corollary 2.6 (a). Recall from that result that the (radian) measure of the (interior) angle of Δ_* with vertex A_* (resp., vertex B_*; resp., vertex C_*) is α (resp., β; resp., γ). Next, note that $A = A_*$ and $B = B_*$. Also, by [4, Corollary 2.2], the sides AC and A_*C_* (of Δ and Δ_*, respectively) are subsets of the same bowed geodesic $\mathcal{G}$. Moreover, C is in the first quadrant, by the above construction; and C_* is also in the first quadrant, by Theorem 2.5 (b). Therefore, with the understanding that an angle is defined as the union of two half-lines that each emanate from the same point (namely, the vertex of the angle), we can conclude that the angles $\angle BAC$ and $\angle B_*A_*C_*$ (of Δ and Δ_*, respectively) are the same angle, hence are congruent to one another and necessarily have the same (radian) measure, which must be α. In addition, since the sides AB and A_*B_* (of Δ and Δ_*, respectively) are the same side, they must be congruent and also have the same hyperbolic length. That common hyperbolic distance (from A to B; equivalently, from A_* to B_*) is c, by Theorem 2.5 (a).

We will use the SAS congruence criterion to show that the ambient triangle Δ is congruent to Δ_*, with vertex A (resp., vertex B; resp., vertex C) corresponding to vertex A_* (resp., vertex B_*; resp., vertex C_*). In view of what we have already shown, the desired congruence of hyperbolic triangles will hold if we show that the sides AC and A_*C_* (of Δ and Δ_*, respectively) are congruent or, equivalently, that the hyperbolic distance from A to C is the same as the hyperbolic distance from A_* to C_*. By the above construction, the hyperbolic distance from A to C is b. Let b_* denote the hyperbolic distance from A_* to C_*. By applying the law of hyperbolic trigonometry in Lemma 2.3 (c) to Δ_*, we have

$$\cosh(b_*) = \frac{\cos(\gamma)\cos(\alpha) + \cos(\beta)}{\sin(\gamma)\sin(\alpha)}.$$

Thus $\cosh(b) = \cosh(b_*)$. Hence, $b = b_*$ since, as we saw earlier, $\cosh|_{[0,\infty)}$ is a one-to-one function. It now follows from the SAS congruence criterion that the hyperbolic triangles Δ and Δ_* are congruent. Hence, the angles $\angle ABC$ and $\angle A_*B_*C_*$ are congruent, since they are corresponding angles in this congruence. Of course, congruent angles must have the same radian measure. Denoting the radian measure of an angle $\angle$ by $\mathrm{m}(\angle)$, we therefore have that

$$\beta = \mathrm{m}(\angle A_*B_*C_*) = \mathrm{m}(\angle ABC) = \beta^*.$$

Consequently, since $\beta = \beta^*$, it follows from Proposition 3.3 (b) that the radian measure of the interior angle $\angle BCA$ of Δ is γ. This completes the proof of (a).

(b) As the proof of (a) established that $\beta = \beta^*$, an application of the implication (2) $\Rightarrow$ (5) from Corollary 3.5 completes the proof. □

A classic result in plane hyperbolic geometry states that the sum of the radian measures of the three interior angles of any triangle (in plane hyperbolic geometry) is less than π (cf. [11, Theorem 7.2.1], but also see [6, Theorems 6.1 and 10.1]). Therefore, the riding hypotheses on α, β and γ apply to these radian measures, and so Theorem 3.6 (b) has the following application to plane hyperbolic geometry.

Corollary 3.7. *Let $\alpha \le \beta \le \gamma$ be the radian measures of the three interior angles of some hyperbolic triangle in (the upper half-plane model of) plane hyperbolic geometry.Then, using the riding notation, we have $\cos(\beta) = N/D$, and so α, β and γ satisfy the equation $(*)$; that is $N = D\cos(\beta)$.*

We can now present, as our final result, the titular identity in neutral geometry.

Corollary 3.8. *Let $\alpha \leq \beta \leq \gamma$ be the radian measures of the three interior angles of some triangle in some neutral geometry. Then, using the above notation for N and D that was introduced above (although we are* **not** *assuming that $\alpha+\beta+\gamma < \pi$ here), we have that α, β and γ satisfy the equation $(*)$; that is $N = D\cos(\beta)$. This identity can be rewritten more explicitly by expressing N and D as*

$$N := uz - wv\cos(\alpha) \text{ and } D := \sqrt{(uv - wz\cos(\alpha))^2 - 1},$$

where

$$u := \frac{\cos(\alpha)\cos(\gamma) + \cos(\beta)}{\sin(\alpha)\sin(\gamma)}, \; w := \sqrt{u^2 - 1},$$

$$v := \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)} \text{ and } z := \sqrt{v^2 - 1}.$$

Proof. The given triangle is in either plane Euclidean geometry or (up to isomorphism, the upper half-plane model of) plane hyperbolic geometry. In the former (resp., latter) context, the conclusion was established in Corollary 3.2 (resp., Corollary 3.7). □

We close by collecting some open questions and some partial results.

Remark 3.9. (a) The first paragraph of this section mentioned that the methodology for this project's initial attempt to find a new proof of the hyperbolic realization theorem had used the SAS congruence criterion. In fact, we only considered using the ASA congruence criterion (instead of SAS) for that purpose, a decision that led to the development of what is here called Section 2, after obtaining (what is here called) Corollary 3.5 but failing to find a direct proof of Theorem 3.6. That failure was described above as being part of what we view as a "surprising chain of developments." That surprise was due, in part, to what I perceive to be a preference for the use of SAS instead of ASA in various axiomatic presentations of plane hyperbolic or neutral geometry. In this regard, I would like to raise the following questions. If one replaces SAS with ASA in a list of axioms for plane hyperbolic (resp., neutral) geometry, must every model of the resulting set of axioms be isomorphic to plane hyperbolic (resp., neutral) geometry? In other words, does ASA imply SAS in the presence of other reasonable axioms? A fruitful way to start working on these questions may be to consider their analogue for plane Euclidean geometry. I must admit that, although I find these questions interesting, time constraints have not permitted me to pursue them, but I do hope that some reader(s) will find them interesting.

(b) In (a), I mentioned that this project's initial attempt reached an impasse after I had proved Corollary 3.5, as I subsequently failed to find a direct proof of Theorem 3.6 ("direct" in the sense that it could use Corollary 3.5 if necessary but it would not use the hyperbolic realizability theorem). So, in June 2021, I reached out to a small list of contacts and asked them to see if they could find a proof of (what we are here calling) the equation $\cos(\beta) = N/D$ without the use of non-Euclidean geometry (where N and D are as defined above, in terms of positive real numbers $\alpha \leq \beta \leq \gamma$ such that $\alpha+\beta+\gamma < \pi$). As none of those contacts had responded after a period of 12 months, I decided to write and submit this paper in its present form. At this time, I would like to raise the same question for this paper's readers. It is evident that an affirmative answer to this question could be combined with Corollary 3.5 to produce yet another proof of the hyperbolic realizability theorem.

Since June 2021, I have made two kinds of observations (it may not be appropriate to call them "progress") about the above question. The details of these observations are not being given here for reasons of space, but I hope that the following summary of these observations will be of help to any interested readers. For the first observation, it is helpful to let $x_1 := \sin(\alpha)$, $x_2 := \sin(\beta)$, $x_3 := \sin(\gamma)$,

$y_1 := \cos(\alpha)$, $y_2 := \cos(\beta)$ and $y_3 := \cos(\gamma)$. I have shown that under the above assumptions, N can be written as an explicit quotient whose numerator is an explicit function of y_1, y_2 and y_3 and whose denominator is $x_1^2 x_2 x_3$; the same conclusion holds for D in place of N; and hence that N/D is an explicit function of y_1, y_2 and y_3. (Given that the goal is to find a direct proof that $N/D = y_2$, this may be a step in the right direction.) Here is the second observation: I can replace the equation $\cos(\beta) = N/D$ (under the above assumptions) with the equation $N^2 = \cos^2(\beta)D^2$. This observation may be of help for the following two reasons: the expression for D^2 should be easier to manipulate than the expression for D because the former does not feature the symbol "$\sqrt{\ldots}$"; and similar reasoning may reduce the problem to showing that a certain polynomial identity holds for the six variables $\sin(\alpha)$, $\cos(\alpha)$, $\sin(\beta)$, $\cos(\beta)$, $\sin(\gamma)$ and $\cos(\gamma)$, in which case some computer software may be of help in resolving the question.

(c) In closing, we wish to raise the question of finding companions for Theorem 3.6 and Corollaries 3.7-3.8. In particular, can one devise an essentially different way to find *other* apparently nontrivial identities that hold for the radian measures of all three interior angles of all triangles in all neutral geometries?

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