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Relative prime (resp., semiprime) ideals with applications in $C(X)$

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Abstract. Let I and J be two ideals in a commutative ring R. The ideal I is called J-prime (resp., J-semiprime) if $a, b \in J$ (resp., $a \in J$) and $ab \in I$ (resp., $a^2 \in I$) imply $a \in I$ or $b \in I$ (resp., $a \in I$). Whenever $J \not\subseteq I$ and I is a J-prime (resp., J-semiprime) ideal, the ideal I is said to be a relative prime (resp., semiprime) ideal, and moreover, the ideal J is a p (resp., sp)-factor of I. The class of relative semiprime ideals includes relative z-ideals in any commutative ring and all non-essential ideals in reduced commutative rings. In this article, first we characterize some properties of these two classes of ideals in any commutative ring. Next, we apply the theory of relative prime (resp., semiprime)-ideals in the ring of continuous functions.

Key Words: Relative z-ideal, prime ideal, F-space.
2010 MSC: Primary 13A15; Secondary 54C40.

0. Introduction

Throughout this paper, R denotes a commutative ring with identity, and the ideals may be improper. A ring R is reduced if it has no non-zero nilpotent element. A ring R is regular if for each $a \in R$ there exists $b \in R$ such that $a = a^2 b$. We denote by $\mathrm{Min}(I)$ the set of all prime ideals minimal over the ideal I. It is well-known that if P is a prime ideal in a ring R, then $P \in \mathrm{Min}(I)$ if and only if for each $a \in P$, there exists $c \notin P$ and $n \in \mathbb{N}$ such that $(ac)^n \in I$. An ideal I of a commutative ring R is semiprime if I equals to the intersection of all prime ideals over it, i.e., $I = \bigcap_{P \in \mathrm{Min}(I)} P$. It follows that $I = \sqrt{I} = \{x \in R : x^n \in I, \text{ for some } n \in \mathbb{N}\}$, i.e., I is a radical ideal. The radical of the zero-ideal, i.e., the intersection of all prime ideals of the ring R is called the nilradical of R. For each element a in a ring R, the intersection of all maximal ideals in R containing a is denoted by M_a, and an ideal I in R is called a z-ideal if $M_a \subseteq I$, for all $a \in I$. It is easy to see that an ideal I is a z-ideal if and only if whenever $b \in R$, $a \in I$, and $M_b \subseteq M_a$, then $b \in I$. Intersection of all z-ideals containing I is the smallest z-ideal containing I and is denoted by I_z , see [12, 13, 15, 16, 19, 20]. A generalization of the class of z-ideals is the class of relative z-ideals. Let I and J be two ideals of a ring R. The ideal I is called a z_J-ideal if $M_a \cap J \subseteq I$, for each $a \in I$. Whenever $I \not\supseteq J$ and I is a z_J-ideal, then I is called a relative z-ideal. An ideal I is a z_J-ideal if and only if $I_z \cap J \subseteq I$. We refer the reader to [1] and [7] for more details about relative z-ideals.

In this paper, $C(X)$ ($C^*(X)$) is the ring of all (bounded) real-valued continuous functions on a completely regular Hausdorfff space X. The set $f^{-1}\{0\}$ called the zero-set of f and denoted by $Z(f)$. Any set that is a zero-set of some function in $C(X)$ is called a zero-set in X. The space βX is known as the *Stone-Čech compactification* of X. It is characterized as that compactification of X in which X is C^*-embedded as a dense subspace. The space υX is the *real compactification* of X, if X is C-embedded in this space as a dense subspace. For a completely regular Hausdorff space X, we have $X \subseteq \upsilon X \subseteq \beta X$. For any $p \in \beta X$, O^p (resp., M^p) is the set of all $f \in C(X)$ for which $p \in \mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X} Z(f)$

(resp., $p \in \mathrm{cl}_{\beta X} Z(f)$). Also, for $A \subseteq \beta X$, O^A (resp., M^A) is the intersection of all O^p(resp., M^p) where $p \in A$, and whenever $A \subseteq X$, we denote it by O_A (resp., M_A).

The paper consists of three sections. The first section is about an investigation of relative prime (resp., semiprime)-ideals. The characterization of relative z-ideals and the fact that every z-ideal is a semiprime ideal motivate us to introduce relative prime (resp., semiprime) ideals. If we consider an ideal J, then we observe that every ideal containing J is a J-prime (resp., semiprime) ideal. So they are trivial J-prime (resp., semiprime) ideals. For an ideal $I \not\supseteq J$, we apply the definition of prime and semiprime ideal, respectively, restricted to an ideal J. As we mentioned in the abstract, if $I \not\supseteq J$ and it is J-prime (semiprime), then we say I is a relative prime (semiprime). We observe that every relative z-ideal is a relative semiprime ideal. We also have seen that every non-essential ideal in a reduced ring is a relative semiprime ideal. For an ideal J of R with $J \not\subseteq J(R)$, it is proved that every proper semiprime ideal in J is a semiprime ideal in R if and only if J is a semiprime ideal in R. A semiprime ideal I of R need not be relative prime. We show that a semiprime ideal I is relative prime if and only if there is an irredundant ideal with respect to I. We observe that for any ring R, every ideal is relative semiprime if and only if R is a regular ring. The largest p (resp., sp)-factor of an ideal I of a ring R also is characterized.

In the second section we use the topological properties of Zariski topology on the sets $\mathrm{Max}(R)$ (resp., $\mathrm{Min}(R)$). We have seen that the zero-ideal in any ring R is relative prime if and only if the space $\mathrm{Min}(R)$ has an isolated point if and only if there is a non-zero element $a \in R$ such that $\mathrm{Ann}(a)$ is a prime ideal. For a subset A of $\mathrm{Min}(R)$, it is proved that $O_A = \bigcap_{P \in A} P$ is a relative prime ideal if and only if A is not a perfect subset of $\mathrm{Min}(R)$.

Section 3 applies these concepts in the ring of continuous functions. For a Tychonoff space X, we prove that for every ideal J of $C(X)$, the sum of every two J-prime ideals is a J-prime ideal if and only if the sum of every two z-ideals in J is a z-ideal in J if and only if X is an F-space. For a subset A of βX, we have shown that the ideal M^A is relative prime if and only if A is not a perfect subset of βX if and only if every prime ideal of $C(X)/M^A$ is essential. We conclude this section by showing that the ideal $C_\psi(X)$ is not a relative prime ideal in $C(X)$.

1 Relative semiprime (resp., prime) ideals

We begin with the following definitions.

Definition 1.1. Let I and J be two ideals of R. The ideal I is a J*-semiprime ideal*, if $a \in J$ and $a^2 \in I$ imply $a \in I$. Whenever $I \not\supseteq J$ and I is a J-semiprime ideal, we say that I is a *relative semiprime* ideal and J is called a sp-factor of I.

Definition 1.2. Let I and J be two ideals of R. The ideal I is called a J*-prime* ideal, if $a, b \in J$ and $ab \in I$ imply $a \in I$ or $b \in I$. Whenever $I \not\supseteq J$ and I is a J-prime, then we say that I is a *relative prime* ideal and J is called a p-factor of I.

The next proposition introduces two classes of relative semiprime ideals in any ring and in reduced rings, respectively. We recall that a nonzero ideal I in a ring R is called *essential* if it intersects every nonzero ideal nontrivially. It is well known that an ideal I in a commutative reduced ring R is an essential ideal if and only if $Ann(I) = 0$, see McConnel and Robson [18] for some properties of essential ideals in general rings, and see also Azarpanah [3, 4], Ghirati and Taherifar [10] and Taherifar [21, 22] for topological characterization of these ideals in $C(X)$ and in reduced rings.

Proposition 1.3. *The following statements hold.*

1. *Every relative z-ideal is a relative semiprime ideal.*

2. *Let I be a non-essential ideal in a reduced ring R. Then I is a relative semiprime ideal.*

Proof. (1) Let I be a relative z-ideal. There exists an ideal J of R such that I is a z_J-ideal and $I \nsupseteq J$. Suppose that $a \in J$ and $a^2 \in I$. Then we have $a \in M_a \cap J = M_{a^2} \cap J \subseteq I$. Thus I is a relative semiprime ideal.

(2) Put $J = Ann(I)$. Now let $a \in J$ and $a^2 \in I$. Then $a^3 = 0$. Therefore $a = 0 \in I$. This implies that I is a J-semiprime ideal. On the other hand $J \nsubseteq I$. Thus I is a relative semiprime ideal □

Clearly every semiprime (resp., prime) ideal in J is a J-semiprime (resp., J-prime) ideal in R. However there are many J-semiprime (resp., J-prime) ideals in R which are not semiprime (resp., prime) ideal in J. We also note that if P is a prime non z-ideal, then for any ideal J, where $P \nsupseteq J$, P is a J-semiprime (resp., J-prime) ideal and hence is a relative semiprime (resp., prime) ideal. But P is not a relative z-ideal, by [1, Lemma 2.1].

Part (3) of the following lemma shows that whenever I is a relative semiprime (resp., relative prime) ideal, then there exists an ideal K containing I properly such that I is a K-semiprime (resp., K-prime) ideal.

Lemma 1.4. *Let I, J and K be ideals in a ring R. The following statements hold.*

1. *If $I \subseteq J \subseteq K$, I is a J-semiprime ideal and J is a K-semiprime ideal, then I is a K-semiprime ideal.*
2. *If I is a J-semiprime (J-prime) ideal and $K \subseteq J$, then I is a K-semiprime (K-prime) ideal.*
3. *The ideal I is a J-semiprime (resp., J-prime) ideal if and only if I is $I+J$-semiprime (resp., $I+J$-prime) ideal.*
4. *The ideal I is a J-semiprime (resp., J-prime) ideal if and only if $I \cap J$ is a J-semiprime (resp., J-prime) ideal.*
5. *If J is a semiprime ideal, then I is a J-semiprime ideal if and only if $I \cap J$ is a semiprime ideal.*
6. *The ideal $I \cap J$ is both J-semiprime (resp., J-prime) ideal and I-semiprime (resp., I-prime) ideal if and only if I is J-semiprime (resp., J-prime) ideal and J is I-semiprime (resp., I-prime)ideal.*
7. *If M is a maximal ideal, then $I \cap M$ is a semiprime ideal if and only if I is a semiprime ideal.*

Proof. The proof the statements (1), (2), (4) and (6) are trivial and we left them to the reader. For part (3), suppose that $a_1, a_2 \in I$, $b_1, b_2 \in J$ and $(a_1 + b_1)(a_2 + b_2) \in I$. Then $b_1 b_2 \in I$. By hypothesis, this implies that $b_1 \in I$ or $b_2 \in I$ and hence $(a_1 + b_1) \in I$ or $(a_2 + b_2) \in I$. The converse follows from statement (2). Next, we show the statement (5). Let I be a J-semiprime ideal. Then by (4), $I \cap J \subseteq J$ is a J-semiprime ideal, and so by (1), it is a semiprime ideal. The converse is evident. Now for the statement (7), if $I \subseteq M$, then $I = I \cap M$ is a semiprime ideal. Otherwise, $I \nsubseteq M$, and so by (5), I is a M-semiprime ideal and hence the statement (3) implies that I is a $I + M = R$-semiprime ideal. The converse of the statement holds trivially. □

The following proposition which characterizes relative semiprime (resp., prime) ideals element wise, also helps us to construct a relative semiprime (resp., prime) ideal.

Proposition 1.5. *An ideal I of a ring R is relative semiprime (resp., prime) ideal if and only if there exists an $a \in R \setminus I$ such that I is an Ra-semiprime (resp., prime) ideal.*

Proof. If I is a relative semiprime (resp., prime) ideal, then by Lemma 1.4, I is a J-semiprime (resp., prime) ideal for some ideal J with $I \subset J$. Now it is enough to take $a \in J \setminus I$. For the reverse implication, assume that there is an $a \in R \setminus I$ such that I is a Ra-semiprime (resp., prime) ideal. Consider the ideal $J = Ra$. Clearly I is a J-semiprime (resp., prime) ideal and hence a relative semiprime (resp., prime) ideal, for $J \nsubseteq I$. □

The subsequent lemmas which are easy to prove are needed in the upcomming results.

Lemma 1.6. *Let R be a ring and let I and J be two ideals of R. Moreover, let $(I_\alpha : \alpha \in \mathcal{S})$ and $(J_\alpha : \alpha \in \mathcal{S})$ be two families of ideals of R.*

1. *If I_α is a J_α-semiprime ideal of R, for each $\alpha \in \mathcal{S}$, then the ideal $\bigcap_{\alpha\in S} I_\alpha$ is $\bigcap_{\alpha\in S} J_\alpha$-semiprime ideal.*
2. *If I is a J_α-semiprime (J_α-prime) ideal of R, for each $\alpha \in S$, then I is a $\bigcap_{\alpha\in S} J_\alpha$-semiprime (resp., prime) ideal.*
3. *If I_α is a J-semiprime ideal, for each $\alpha \in S$, then the ideal $\bigcap_{\alpha\in S} I_\alpha$ is a J-semiprime ideal.*
4. *If $I \subseteq J$ is a J-semiprime ideal of R and J is a semiprime ideal of R, then I is a semiprime ideal of R.*

By the following result, we observe that in any reduced ring, the zero ideal (0) is a relative semiprime ideal.

Lemma 1.7. *Let R be a ring and let I and J be two ideals of R. The following statements are equivalent.*

1. *There exists a semiprime ideal K containing I such that $K \cap J \subseteq I$.*
2. *The ideal I is a J-semiprime ideal.*
3. *$\sqrt{I} \cap J \subseteq I$ ($\sqrt{I} \cap J = I \cap J$).*

Corollary 1.8. *Let R be a ring and Let J be an ideal of R. Then an ideal K of J is semiprime in J if and only if $K = P \cap J$, for some semiprime ideal P of R*

Proof. If K is semiprime in J, then K is a J-semiprime ideal of R and hence by Lemma 1.7, $K = \sqrt{K} \cap J$, so we are done. The converse of the statement holds trivially. □

The following result shows that a relative semiprime ideal need not be a semiprime ideal. More precisely, whenever the ideal J is a non-semiprime ideal which is not included in the Jacobson radical of R, then there exists a proper J-semiprime ideal I in J such that I is not a semiprime ideal of R.

Proposition 1.9. *Let R be a ring and let J be an ideal of R such that $J \not\subseteq J(R)$. Every proper semiprime ideal in J is a semiprime ideal in R if and only if J is a semiprime ideal in R.*

Proof. Suppose that every proper semiprime ideal in J is a semiprime ideal in R and J is not semiprime in R. Since $J \not\subseteq J(R)$, there exists a maximal ideal $M \not\supseteq J$ in R. This implies the ideal $I = M \cap J$ is a J-semiprime ideal in J. Note that since J is not semiprime, then it is not M-semiprime (if J is M-semiprime, then it is a $J + M = R$-semiprime, a contradiction). Therefore we have an $f \in M$ such that $f^2 \in J$ but $f \notin J$. Thus $f^2 \in I$ but $f \notin I$, i.e., I is not a semiprime ideal of R. So we obtain a semiprime ideal of J which is not semiprime in R, a contradiction. Conversely, let I be a proper semiprime ideal in J. Corollary 1.8 implies that there is a semiprime ideal P of R such that $I = P \cap J$, and hence by our hypothesis, I is a semiprime ideal of R. □

A relative prime ideal need not be a prime ideal. For, suppose that P and Q are two distant prime ideals which are not comparable, then $P \cap Q$ is a P-prime ideal and hence a relative prime ideal, but is not a prime ideal. Also, a relative semiprime ideal need not be a semiprime ideal. To see this, consider a maximal ideal M and a non-semiprime ideal $Q \not\subseteq M$. Then $I = Q \cap M$ is a Q-semiprime ideal and so a relative semiprime ideal which is not semiprime. Since Q is not M-semiprime (if Q is M-semiprime, then it is $Q + M = R$-semiprime, a contradiction), so there exists $f \in M$ such that $f^2 \in Q$ but $f \notin Q$. This shows that $f^2 \in I$ and $f \notin I$.

Lemma 1.10. *Let R be a ring, and let I and J be two ideals of R.*

1. *If I is a (non-trivial) J-prime ideal, then there is a (unique) $P \in Min(I)$ such that $P \cap J \subseteq I$.*
2. *If $|Min(I)| > 1$ and there is a (unique) $P \in Min(I)$ such that $P \cap J \subseteq I$, then I is a (non-trivial) J-prime ideal.*

Proof. (1) Suppose that I is a non-trivial J-prime ideal. Note that $J \nsubseteq I$, $I \cap (J \setminus I) = \emptyset$ and $J \setminus I$ is a multiplicative closed subset. So there is a prime ideal P containing I such that $P \cap (J \setminus I) = \emptyset$. Thus $P \cap J \subseteq I$. Now let $Q \in Min(I)$ and $Q \cap J \subseteq I$. Then we have $P \cap J \subseteq Q$. But $J \nsubseteq Q$ (if $J \subseteq Q$, then $J = J \cap Q \subseteq I$, a contradiction). This implies that $P = Q$, by minimality of P.

(2) First we have $J \nsubseteq I$. For $J \subseteq I$ implies that $J \cap Q \subseteq I$ for each $Q \in Min(I)$ (this minimal prime exists, by hypothesis), which is a contradiction, by uniqueness of P . Now assume that $a, b \in J$ and $ab \in I$. Then $ab \in P$, so $a \in P \cap J \subseteq I$ or $b \in P \cap J \subseteq I$. Thus I is a non-trivial J-prime ideal. □

If R is a Noetherian ring, then every semiprime ideal is the intersection of a finite number of prime ideals and so by Lemma 1.10, it is a relative prime ideal. We need the following corollary which is also a consequence of [17, Lemma 2] and [13, 5.1].

Corollary 1.11. *[20, Lemma 1.3] Let J be an ideal of a ring R. Then an ideal K of J is a (proper) prime ideal in J if and only if $K = P \cap J$ for some (unique) prime ideal P of R containing K.*

Proof. If K is a (proper) prime ideal in J, then K is a (non-trivial) J-prime ideal and hence by Lemma 1.10, there is a (unique) prime ideal P containing K such that $K = P \cap J$. Conversely, it is obvious. □

Remark 1.12. A counterpart for part (1) of Lemma 1.4 does not hold for relative prime ideals. In other words, we may have $I \subset J$ is J-prime and $J \subset K$ is K-prime, but I is not K-prime. For let M_1, M_2 and M_3 be three distant maximal ideals in R, and $I = M_1 \cap M_2 \cap M_3$. Using Corollary 1.11, the ideal I is a J-prime ideal, where $J = M_2 \cap M_3$ and J is a K-prime ideal, where $K = M_3$. However, evidently, the ideal I is not a K-prime ideal.

We remind the reader that if I is a semiprime ideal of a ring R, and $P \in \mathrm{Min}(I)$ is an ideal such that $I \neq \bigcap_{Q \neq P \in \mathrm{Min}(I)} Q$, then the ideal P is called *irredundant* with respect to I (see [2]).

Proposition 1.13. *Let R be a ring and let I be a semiprime ideal of R. Then I is relative prime if and only if there is an irredundant ideal with respect to I.*

Proof. Let I be a relative prime ideal. Then there is a non-zero ideal $J \nsubseteq I$ such that I is J-prime. By Lemma 1.10, there exists a unique $P \in \mathrm{Min}(I)$ such that $P \cap J \subseteq I$. This implies that for each $Q \neq P \in \mathrm{Min}(I)$, we have $J \subseteq Q$ and hence $J \subseteq \bigcap_{Q \neq P \in \mathrm{Min}(I)} Q$. Thus $I \neq \bigcap_{Q \neq P \in \mathrm{Min}(I)} Q$. So we are thorough. Conversely, suppose that P is an irredundant ideal with respect to I. It is enough to take $J = \bigcap_{Q \neq P \in \mathrm{Min}(I)} Q$. Then $J \nsubseteq I$ and $P \cap J = I$. So by Lemma 1.10, I is a J-prime ideal and hence a relative prime ideal. □

Lemma 1.14. *The zero ideal in any ring R is relative prime if and only if R has a non-essential minimal prime ideal.*

Proof. By hypothesis, there is a non-zero ideal J such that (0) is J-prime. By Lemma 1.10, there is a unique $P \in \mathrm{Min}((0))$ and so $P \in \mathrm{Min}(R)$ such that $P \cap J = 0$. This implies that P is a non-essential minimal prime ideal. The converse implication holds trivially. □

It is well known that every ideal of R is a *rez*-ideal if and only if R is a regular ring if and only if every ideal of R is a z-ideal, see [1, Proposition 3.7] and [16, Theorem 1.2]. This motivates us to consider the following question: When is every ideal of R a relative semiprime ideal? We will answer this question in the next result (i.e., Proposition 1.16). To achieve this goal, we need the following lemma.

Lemma 1.15. *Let R be a ring and let I be an ideal of R. Moreover, let the set of sp (resp., p)-factors of the ideal I be nonempty. Then the set of sp (resp., p)-factors of the ideal I has a maximal element containing I.*

Proof. Let $\mathcal{C}$ be a chain of sp (resp., p)-factors of the ideal I. Clearly, we can observe that $K = \bigcup_{J\in\mathcal{C}} J$ is a sp (resp., p)-factor for I. The ideal K must be a sp (resp., p)-factor for I. Applying Zorn's Lemma, the set of sp (resp., p)-factors of the ideal I has a maximal element. □

Proposition 1.16. *For any ring R, the following statements are equivalent.*

1. *Every ideal of R is a relative semiprime ideal.*
2. *Every ideal of R is a semiprime ideal.*
3. *The ring R is regular.*
4. *Every ideal of R is a relative z-ideal.*

Proof. First we show that (1) implies (2). Assume that I is an ideal of R. Then, by our hypothesis, it is a relative semiprime ideal and therefore a maximal sp-factor J of I exists, by Lemma 1.15. If J is a proper ideal, then by part (3) of Lemma 1.4, there exists an ideal $K \supset J$ such that J is a K-semiprime ideal. Therefore, I is a K-semiprime ideal, by Lemma 1.6, which contradicts the maximality of J. Thus $J = R$, and so (2) is proved. Next suppose that (2) holds. We prove that the statement (3) holds. Consider $a \in R$. The ideal (a^2) is a semiprime ideal and $a^2 \in (a^2)$. Thus $a \in (a^2)$. This implies that $a = ka^2$, for some $k \in R$, and so R is a regular ring. The statement (3) implies the statement (4), by [1, Proposition 3.7]. Suppose that the statement (4) holds. Then every ideal of R is a relative semiprime ideal, and hence the statement (1) holds. □

Let R be a ring, I be an ideal of R and $a \in R$. We recall that the ideal $[I : a]$ is defined as follows:

$$[I : a] = \{x \in R : xa \in I\}.$$

Proposition 1.17. *Let R be a ring and let I be an ideal of R. The following statements hold.*

1. *If the largest p-factor of I exists, then it is equal to the set*

$$L_p = \{a \in R : [I : a] \ \textit{is prime}\}.$$

2. *If the largest sp-factor of I exists, then it is equal to the set*

$$L_{sp} = \{a \in R : [I : a^2] \ \textit{is semiprime}\}.$$

Proof. (1) Let K be the largest p-factor of I. First we show that the set $L_p = \{a \in R : [I : a] \text{ is prime}\}$ is an ideal. It is easily seen that, for $a \in R$, $[I : a]$ is prime if and only if I is Ra-prime. Now let $a, b \in L_p$. Then I is Ra and Rb-prime, respectively. This implies $R(a+b) \subseteq Ra + Rb \subseteq K$. By part (1) of Lemma 1.4, the ideal I is $R(a+b)$-prime, i.e., $[I : (a+b)]$ is prime, so $a+b \in L_p$. If $r \in R$ and $x \in L_p$, then $R(rx) \subseteq Rx$ implies I is $R(rx)$-prime, and hence $rx \in L_p$. Next suppose that $a, b \in L_p$ and $ab \in I$. We have $a, b \in Ra + Rb \subseteq K$. Thus I is $(Ra + Rb)$-prime. This fact together with the assumption $ab \in I$ imply $a \in I$ or $b \in I$. Hence I is L_p-prime. Finally, assume that $x \in K$. Then I is Rx-prime, by part (1) of Lemma 1.4. Thus $[I : x]$ is prime, i.e., $x \in L_p$. Therefore $K \subseteq L_p$, and hence $K = L_p$.

(2) It is easy to see that I is Ra-semiprime if and only if $[I : a^2]$ is semiprime. Now similar to the proof of part (1), we can see that L_{sp} is the largest sp-factor of I. □

Proposition 1.18. *Let I be a semiprime ideal of R.*

1. *The largest p-factor of I exists if and only if there is a unique irredundant ideal with respect to I.*
2. *If the largest p-factor of I exists, then it is a semiprime ideal.*

Proof. (1) Suppose that L_p is the largest p-factor of I. By Proposition 1.13, there is an irredundant $P \in \mathrm{Min}(I)$. For each $Q \neq P \in \mathrm{Min}(I)$ we have $L_p \subseteq Q$. But $\bigcap_{Q \neq P \in \mathrm{Min}(I)} Q$ is a p-factor for I, by Lemma 1.10. Therefore $L_p = \bigcap_{Q \neq P \in \mathrm{Min}(I)} Q$. Now let $P_1 \in \mathrm{Min}(I)$ be another irredundant ideal. Then by similar reason $L_p = \bigcap_{Q \neq P_1 \in \mathrm{Min}(I)} Q$. This implies that $\bigcap_{Q \neq P \in \mathrm{Min}(I)} Q \subseteq P$, a contradiction. Conversely, let $P \in \mathrm{Min}(I)$ be a unique irredundant. Then it is easy to see that $J = \bigcap_{Q \neq P \in \mathrm{Min}(I)} Q$ is the largest p-factor for I.

(2) If L_p exists, then by (1), there is a unique irredundant prime ideal P with respect to I. Now it is easy to see that $L_p = \bigcap_{Q \neq P \in \mathrm{Min}(I)} Q$, and so it is a semiprime ideal. □

The following result shows that for a maximal ideal M, M-prime ideals are exactly prime ideals in R and prime ideals in M.

Proposition 1.19. *Let I be an ideal of a ring R and M be a maximal ideal of it. Then the following statements are equivalent.*

1. *The ideal I is M-prime*
2. *The ideal I is prime in R or $I = P \cap M$, for some prime ideal P of R.*

Proof. Clearly (2) implies (1). Now let (1) holds. If $I \subseteq M$, then $I = P \cap M$ for some prime ideal P of R, by Lemma 1.11. Otherwise, I is $I + M = R$-prime, i.e., I is prime, by Lemma 1.4. Thus (2) is proved. □

It is well known that if I is an ideal of an ideal J in a ring R and I is semiprime, then I is an ideal of R. By using this fact we have the following result. The reader is refereed to [14], to see rings for which the sum of two semiprime ideals is a semiprime ideal.

Proposition 1.20. *Let R be a ring with the sum of every two semiprime ideals in it is a semiprime ideal (e.g., $C(X)$) and J be an ideal of R. Then the following statements are equivalent.*

1. *The sum of every two J-semiprime ideals is a J-semiprime ideal.*
2. *The sum of every two semiprime ideals of J is a semiprim ideal in J.*
3. *For any two semiprime ideals P and Q of R, $(P+Q) \cap J = P \cap J + Q \cap J$.*

Proof. Every semiprime ideal of J is a J-semiprime ideal in R, by the comment before to this proposition. Thus (1) implies (2). Now let (2) holds. We prove the statement (3). Consider two semiprime ideals P, Q in R. By Corollary 1.8, $P \cap J$ and $Q \cap J$ are two semiprime ideals in J and by hypothesis, $P \cap J + Q \cap J$ is a semiprime ideal in J. Suppose that $a = a_1 + a_2 \in (P+Q) \cap J$. Then $a^2 = aa_1 + aa_2 \in P \cap J + Q \cap J$. Thus $a \in P \cap J + Q \cap J$, i.e., $(P+Q) \cap J \subseteq P \cap J + Q \cap J$. On the other hand, $P \cap J + Q \cap J \subseteq (P+Q) \cap J$. Hence (3) is proved. Now we show that (3) implies (1). Let I and K be two J-semiprime ideals. By Lemma 1.7, there are semiprime ideals P and Q such that $P \cap J \subseteq I$ and $Q \cap J \subseteq K$. So, by hypothesis, $P \cap J + Q \cap J = (P+Q) \cap J \subseteq I + K$, where $P+Q$ is a semiprime ideal containing $I+K$. Now by Lemma 1.7, $I + K$ is a J-semiprime ideal. □

Proposition 1.21. *Let R be a ring with the sum of every two prime ideals in it is a prime ideal (e.g., $C(X)$) and J be an ideal of R. Then the following statements are equivalent.*

1. *The sum of every two J-prime ideals is a J-prime ideal.*

2. *The sum of every two prime ideals of J is a prime ideal in J.*

3. *For any two prime ideals P and Q of R, $(P+Q)\cap J = P\cap J + Q\cap J$.*

Proof. The proof is similar to that of Proposition 1.20. □

A ring R is said to be arithmetical if for every three ideals I, J and K in R, we have $I\cap(J+K) = I\cap J + I\cap K$. Now using Proposition 1.20 and Proposition 1.21, whenever R is an arithmetical ring in which the sum of every two prime ideals is a prime ideal, then for each ideal J of R, the sum of every two J-semiprime (resp., prime)-ideals is a J-semiprime (resp., prime)-ideal.

2 Relative prime ideals and Zariski topology

Let Min(R) be the set of minimal prime ideals of R and $V(a) = \{P \in \text{Min}(R) : a \in P\}$. Then $B = \{\text{Min}(R)\setminus V(a) : a \in R\}$ forms a base for open sets in Min(R). Equipped with this topology, the space Min(R) is zero-dimensional and Hausdorff. This topology is known as the *Zariski topology*. For $A \subseteq R$, we use $V(A)$ to denote the set of all $P \in \text{Min}(R)$ where $A \subseteq P$. For a subset H of Min(R) we denote by cl$H$ (resp., int$H$) the closure points of H (resp., the interior points of H) in Min(R).

Proposition 2.1. *For any ring R, the following statements are equivalent.*

1. *The ideal* (0) *is relative prime.*

2. *The space Min(R) has an isolated point.*

3. *There is an $a \neq 0 \in R$ such that* Ann(a) *is a prime ideal.*

Proof. Suppose the statement (1) holds. We show the statement (2). By Lemma 1.14, there is a non-essential minimal prime ideal $P \in \text{Min}(R)$. By [21, Lemma 3.1], int$V(P) = \text{int}\{P\} \neq \emptyset$. Therefore $\{P\}$ is an isolated point in Min(R). Now let (2) holds. So there is an isolated point $\{P\}$ in Min(R). By [21, Lemma 3.1], the ideal P is non-essential and so there is a non-zero ideal J such that $P\cap J = 0$. Thus (1) is proved. Finally we prove the equivalency of (1) and (3). Let the ideal (0) be relative prime. By Proposition 1.5, there exists $a \neq 0 \in R$ such that (0) is Ra-prime. We show $Ann(a)$ is a prime ideal. Let $xy \in Ann(a)$. Then $xaya = 0$. Thus $xa = 0$ or $ya = 0$. This shows that $x \in Ann(a)$ or $y \in Ann(a)$, i.e., $Ann(a)$ is a prime ideal. Now let (3) holds, $x = ra, y = sa \in Ra$ and $xy = 0$. Then $rsa^2 = 0$ and hence $rsa = 0$. This implies that $rs \in Ann(a)$. By hypothesis, $r \in Ann(a)$ or $s \in Ann(a)$, i.e., $x = 0$ or $y = 0$. Therefore (0) is Ra-prime. Hence (0) is a relative prime ideal. □

We use Max(R) as the set of all maximal ideals of R. For $a \in R$, let $M(a) = \{M \in \text{Max}(R) :\ a \in M\}$. It is easy to see that for any ring R, the set $\{D(a) : a \in R\}$, (where $D(a) = \text{Max}(R)\setminus M(a)$) forms a basis of open sets in Max(R). This topology is called the Zariski topology. For $A \subseteq R$, we use $\mathcal{M}(A)$ to denote the set of all $N \in \text{Max}(R)$ where $A \subseteq N$. For a subset H of Max(R) we denote by cl$H$ (resp., int$H$) the closure points of H (resp., the interior points of H) in Max(R), see [9] and [11].

Let A be a subset of Max(R), then $M_A = \{a \in R : A \subseteq M(a)\}$ is an ideal of R (see [10]). It is clear that $M_A = \bigcap_{M\in A} M$ and $\mathcal{M}(M_A) = \text{cl}A$. It is easy to see that $M_A \subseteq M_B$ if and only if cl$B \subseteq$ cl$A$, where A and B are subsets of Max(R). We also note that $J(R/M_A) = \bigcap_{M\in \text{cl}A} M/M_A = (0)$. Recall from [9] that a ring R is a *pm*-ring if every prime ideal is contained in a unique maximal ideal.

It is well known that if X is a dense subspace of a T_1 space Y (i.e., for every $y \in Y$, $\{y\}$ is closed), then $I(X) = I(Y)$ (i.e., the set of isolated points of X and T are the same).

Theorem 2.2. Let R be a *pm*-ring and $A \subseteq \text{Max}(R)$. The following are equivalent.

1. The set A is perfect with the subspace topology.

2. Every prime ideal of R/M_A is essential, i.,e., $\mathrm{Soc(R/M_A)} = (0)$.

3. Every maximal ideal of R/M_A is essential, i.,e., $\mathrm{Soc_m(R/M_A)} = (0)$.

4. The ideal M_A is not a relative prime ideal.

Proof. Using Theorem 4.1 in [23], the statements (1), (2) and (3) are equivalent. By Lemma 1.14, the staetments (2) and (4) are equivalent. □

For a subset A of $\mathrm{Min}(R)$, suppose that $O_A = \{a \in R : A \subseteq V(a)\}$. Then it is easy to see that O_A is an ideal of R, $V(O_A) = \mathrm{cl}A$ and $O_A = \bigcap_{P \in A} P$. We also can see that $O_A \subseteq O_B$ if and only if $\mathrm{cl}B \subseteq \mathrm{cl}A$, whenever $A, B \subseteq \mathrm{Min}(R)$.

Theorem 2.3. Let R be a ring and let A be a subset of $\mathrm{Min}(R)$. The following are equivalent.

1. The set A is a perfect subset of $\mathrm{Min}(R)$.

2. Every prime ideal of R/O_A is essential.

3. The ideal O_A is not a relative prime ideal.

Proof. Let (1) holds we show that (2) holds. Suppose that P/O_A is a non-essential prime ideal in R/O_A. Then there is a non zero ideal J/O_A such that $P/O_A \cap J/O_A = (0)$. This shows that $P \cap J = O_A$. Thus $V(P) \cup V(J) = V(O_A)$. If $V(P) \subseteq V(J)$, then $V(J) = \mathrm{cl}A$, and hence $J \subseteq O_A$, a contradiction. Thus $V(P) = \{P\} \not\subseteq V(J)$ and hence $\{P\} \cup V(J) = \mathrm{cl}A$. This say that $\{P\}$ is an isolated point in $\mathrm{cl}A$ and so is an isolated point in A, it contradicts that A is perfect. By Lemma 1.14, (3) and (2) are equivalent. The proof of (2) ⇒ (1) is similar to that of (3) ⇒ (1) in Theorem 2.2. □

3 Relative semiprime (resp., prime) ideals in $C(X)$

In this section, we examine the results obtained in the previous sections in rings of continuous functions and in some special ideals of it. We are going to get some topological connections by studying this special ring.

Proposition 3.1. *Let X be a Tychonoff space and let $f \in C(X)$. The following statements hold.*

1. *The principal ideal (f) in $C(X)$ is relative prime if and only if the zero-set $Z(f)$ contains an isolated point.*

2. *Every principal ideal in $C(X)$ is relative prime if and only if X is an almost P-space with a dense subset of isolated points.*

3. *The principal ideal (f) in $C(X)$ is relative semiprime if and only if $\mathrm{int}_X Z(f)$ is non-empty.*

4. *Every principal ideal in $C(X)$ is relative semiprime if and only if the space X is an almost P-space.*

Proof. (1) If the ideal (f) is relative prime, then by Lemma 1.14, the factor ring $C(X)/(f)$ has a non-essential minimal prime ideal. Now by [5, Theorem 4.2], $Z(f)$ contains an isolated point. The reverse implication is a consequence of [5, Theorem 4.2] and Lemma 1.14.

(2) From Lemma 1.14, every principal ideal in $C(X)$ is relative prime if and only if every factor ring of $C(X)$ modulo a principal ideal contains a nonessential minimal prime ideal. Now using [5, Corollary 4.7], the implication holds.

(3) Let the principal ideal (f) is relative semiprime. Then there exists an ideal $J \supset (f)$ such that $\sqrt{(f)} \cap J \subseteq (f)$. On the other hand $\sqrt{(f)} = P_f$. Thus (f) is a relative z°-ideal and hence a relative z-ideal, so by [7, Example 2.3], $\mathrm{int}Z(f) \neq \emptyset$. Conversely, suppose that $\mathrm{int}Z(f) \neq \emptyset$. Then $\mathrm{Ann}(f) \neq 0$, i.e., (f) is a non-essential ideal. Now by Proposition 1.3, (f) is a relative semiprime ideal.

The statement (4) follows from the statement (3). □

Proposition 3.2. *Let X be a Tychonoff space. The following statements are equivalent.*

1. *The zero ideal (0) in $C(X)$ is a relative prime ideal.*
2. *The space X has an isolated point.*

Proof. (1) implies (2). For, if the ideal (0) is relative prime in $C(X)$, then there is a non-essential prime ideal P in $C(X)$, by Proposition 2.1. Now by [3, Corollary 3.3], there is an isolated point $x \in X$ such that $P = M_x$. Suppose that (2) holds. If $x \in X$ is an isolated point, then by [3, Corollary 3.3], the ideal $M_x = O_x$ is a non-essential minimal prime ideal in $C(X)$, and so by Proposition 2.1, the statement (1) is deduced. □

Previous corollary together with Proposition 1.18, imply that the largest p-factor of the ideal (0) in $C(X)$ exists if and only if X has a unique isolated point. It is well known that the ideal O^p is a z-ideal and hence is an intersection of minimal prime ideals over O^p and hence an intersection of minimal prime ideals in $C(X)$. Now by Proposition 1.13, we obtain the following result.

Proposition 3.3. *Let X be a Tychonoff space and let $p \in \beta X$. The following are equivalent.*

1. *The ideal O^p is relative prime.*
2. *There is an irredundant ideal $P \in Min(C(X))$ with respect to O^p.*

Proposition 3.4. *Let X be a Tychonoff space and let $I \neq 0$ be a semiprime ideal in $C(X)$. The following statements are equivalent.*

1. *The ideal I is a prime ideal.*
2. *The ideal I is M^p-prime for each $p \in \beta X$.*
3. *The ideal I is P-prime for each prime ideal P.*
4. *The ideal I is P-prime for each minimal prime ideal P.*
5. *The ideal I is O^p-prime for each $p \in \beta X$.*

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ are obvious. Now we suppose that (5) holds, I is not prime and $p \in \beta X$. Then, there is $Q \in \mathrm{Min}(I)$ such that $Q \cap O^p \subseteq I$. But I is not a prime ideal, so there exists an ideal $Q' \neq Q \in \mathrm{Min}(I)$. Therefore $O^p \subseteq Q'$. This implies that $Q' \subseteq M^p$, i.e., $I \subseteq M^p$. Thus $I \subseteq \bigcap_{p \in \beta X} M^p = 0$, a contradiction. □

To find the proof of the following result see Birkhouff [8].

Lemma 3.5. *Let R be a lattice ordered ring and let I, J and K be absolutely convex ideals of R. Then $I \cap (J + K) = I \cap J + I \cap K$.*

It is well known that the sum of every two semiprime ideals of $C(X)$ is a semiprime ideal, see [19, Lemma 5.1]. Now, if J is an absolutely convex ideal in $C(X)$, then by Lemma 3.5 and Proposition 1.20 (resp., Proposition 1.21), the sum of two semiprime (resp., prime)-ideals in J is a semiprime (resp., prime)-ideal. However, the next result shows that if X is not an F-space, then there exists an ideal J of $C(X)$ such that the sum of two J-semiprime ideals is not a J-semiprime ideal. Recall that a topological space X is an F-space if every ideal of $C(X)$ is an absolutely convex ideal, see [12, Theorem 14.25].

Let $f \in J$ and let J be an ideal of $C(X)$. We denote by M_f^J the intersection of all maximal ideals in J containing f.

Lemma 3.6. *Let X be a Tychonoff space and let J be an ideal of $C(X)$.*

1. *For each $f \in J$, $M_f^J = M_f \cap J$.*
2. *If I is a z-ideal of J, then $I = I_z \cap J$.*

Proof. (1) By [20, Corollary 3.6], every maximal ideal in J is of the form $M \cap J$, where $M \not\supseteq J$. Thus $M_f^J = \bigcap_{f \in M \not\supseteq J} M \cap J = (\bigcap_{f \in M} M) \cap J = M_f \cap J$.

(2) I is a z-ideal in J, so I is semiprime in J and hence is an ideal of $C(X)$. By [6, Proposition 1.2], we have

$$I = \sum_{f \in I} M_f^J = \sum_{f \in I} (M_f \cap J) = (\sum_{f \in I} M_f) \cap J = I_z \cap J.$$

□

Corollary 3.7. *Let X be a Tychonoff space. The following statements are equivalent.*

1. *For every ideal J of $C(X)$, the sum of every two J-semiprime ideals is a J-semiprime ideal.*
2. *For every ideal J of $C(X)$, the sum of every two z-ideals in J is a z-ideal in J.*
3. *The space X is an F-space.*

Proof. (1) ⇒ (2) Let J be an ideal of $C(X)$ and I, K be two z-ideals in J. By Lemma 3.6, $I = I_z \cap J$ and $K = K_z \cap J$. By Proposition 1.20, $I + K = (I_z + K_z) \cap J = (I + K)_z \cap J$. This implies $I + K$ as a z-ideal in J, by Lemma 3.6.

(2) ⇒ (3) Let J be an ideal of $C(X)$ and I, K be two z-ideals in $C(X)$. Then $I \cap J + K \cap J$ is a z-ideal in J. Now let $f \in (I + K) \cap J$. Then $f = f_1 + f_2$, where $f_1 \in I$, $f_2 \in K$ and $f_1, f_2 \in J$. Therefore $f^2 = ff_1 + ff_2 \in I \cap J + K \cap J$. So $f \in M_f^J = M_{f^2}^J \subseteq I \cap J + K \cap J$. On the other hand, $I \cap J + K \cap J \subseteq (I + K) \cap J$. Hence by [7, Proposition 3.1], the sum of every two z_J-ideals is a z_J-deal, for each ideal J of $C(X)$. Thus by [7, Theorem 3.4], X is an F-space.

(3) ⇒ (1) If X is an F-space, then every ideal J of $C(X)$ is an absolutely convex ideal, and hence by Lemma 3.5 and Theorem 1.20, the sum of two J-semiprime ideals is a J-semiprime ideal. □

Recall that a space X is a P-space if every zero set (G_δ-set) in X is open or if every prime ideal in $C(X)$ is a z-ideal, see [12, 4J]. In Proposition 1.3, we have seen that every relative z-ideal is a relative semiprime ideal. However, the next result shows that whenever X is not a P-space, there exists a relative semiprime ideal in $C(X)$ which is not a relative z-ideal.

Proposition 3.8. *Let X be a Tychonoff space. The following statements are equivalent.*

1. *Every ideal of $C(X)$ is a relative semiprime ideal.*
2. *The space X is a P-space.*

3. *Every relative semiprime ideal of $C(X)$ is a relative z-ideal.*

4. *Every relative prime ideal of $C(X)$ is a relative z-ideal.*

5. *The sum of every two relative semiprime ideals is a relative z-ideal.*

6. *The sum of every two relative prime ideals is a relative z-ideal.*

Proof. By Proposition 1.16 and this fact that $C(X)$ is a regular ring if and only if X is a P-space (see [12, 4.J]), (1) implies (2). Clearly (2) implies (3) and (3) implies (4). Now suppose that (4) holds and P is a prime ideal of $C(X)$. Then P is a relative z-ideal and hence is a z-ideal, by [1, Lemma 2.1]. Therefore X is a P-space, so (1) is done. If X is a P-space, then every ideal of $C(X)$ is a relative z-ideal, so (2) implies (5) and trivially (5) implies (6). Finally, assume that (6) holds and X is not a P-space. Then there exists a prime ideal P which is not a z-ideal, and hence by [7, Example 2.3] (i.e., every non-essential ideal in $C(X)$ is a relative z-ideal), it is an essential ideal. Let M and M' be two maximal ideals not containing P such that $M+M'=C(X)$. Then $P\cap M$ and $P\cap M'$ are two non-zero ideals. Moreover they are proper subsets of P. This implies that they are relative prime ideals. Now by Lemma 3.5, we have $P\cap M+P\cap M'=P\cap(M+M')=P$. So by hypothesis, P is a relative z-ideal and hence it must be a z-ideal, a contradiction. □

The intersection of all essential maximal ideals in a ring R is denoted by $\mathrm{Soc}_{max}(R)$, see [23].

Theorem 3.9. Let X be a Tychonoff space and let $A\subseteq\beta X$. The following are equivalent.

1. The set A is perfect with respect to the subspace topology.

2. Every maximal ideal of $C(X)/M^A$ is essential, i.,e., $\mathrm{Soc}_{max}\left(C(X)/M^A\right)$ equals to the zero ideal.

3. Every prime ideal of $C(X)/M^A$ is essential, i.,e., $\mathrm{Soc}_{max}\left(C(X)/M^A\right)$ equals to the zero ideal.

4. The ideal M^A is not a relative prime ideal.

Proof. We prove that (1) and (3) are equivalent. First, let (1) holds and P/M^A is a non-essential ideal. Then $P\subseteq M^x$ for some $x\in cl_{\beta X}A$ and there is a non-zero ideal J/M^A such that $P/M^A\cap J/M^A=0$. Hence $P\cap J=M^A$, therefore we have, $\{x\}\cup\theta(J)=\theta(P)\cup\theta(J)=\theta(P\cap J)=\theta(M^A)=cl_{\beta X}A$. If $x\in\theta(J)$, then $\theta(J)=cl_{\beta X}A$, and thus $J\subseteq M^A$, this is a contradiction. Therefore $\{x\}\cup\theta(J)=cl_{\beta X}A$, where $x\notin\theta(J)$. This shows that x is an isolated point in the space $cl_{\beta X}A$ and so is an isolated point in A (for, A is dense in $cl_{\beta X}A$), a contradiction. Now let (3) holds and x is an isolated point in the space A. Then x is an isolated point in $cl_{\beta X}A$ and $cl_{\beta X}A=\{x\}\cup B$, where B is a closed subset of $cl_{\beta X}A$ and hence of βX. Consider the maximal ideal M^x/M^A and the ideal M^B/M^A. It is clear that $M^x/M^A\cap M^B/M^A=0$. On the other hand M^B/M^A is a non-zero ideal (if $M^B/M^A=0$, then $B=cl_{\beta X}A$, a contradiction). This shows that M^x/M^A is a non-essential prime ideal, this is a contradiction. □

For a topological space Y, let $I(Y)$ denotes the set of its isolated points, and let: $Y^0=Y$, $Y'=Y\setminus I(Y)$. For any ordinal η, let $Y^{\eta+1}=(Y^\eta)'$ and if η is a limit ordinal, let

$$Y^\eta=\bigcap_{\alpha<\eta}Y^\alpha.$$

The spaces Y^η are called *Cantor-Bendixson derivatives* of Y. Let $CB(Y)$ denotes the smallest ordinal α for which $Y^\alpha=Y^{\alpha+1}$. This is the CB-index of a space Y. Now the above theorem implies next result.

Proposition 3.10. *Let X be a Tychonoff space. The following are equivalent.*

1. *$CB(\beta X) \leq 1$.*
2. *$\mathrm{Soc}_{max}(C(X))$ is not a relative prime ideal.*
3. *Every maximal ideal in the factor ring $C(X)/\mathrm{Soc}_{max}(C(X))$ is essential.*
4. *Every prime ideal in the factor ring $C(X)/\mathrm{Soc}_{max}(C(X))$ is essential.*

The following result is Corollary 9.6 in [12].

Lemma 3.11. *No point in $\beta X \setminus X$ is a G_δ-point of βX.*

The following lemma needs for the next result. This is obvious, see [12].

Lemma 3.12. *For every point $p \in \beta X \setminus \upsilon X$, there exists a zero-set $Z \in Z[\beta X]$ such that $p \in Z$ and $Z \cap \upsilon X = \emptyset$.*

We recall that $C_\psi(X) = \{f \in C(X) : \mathrm{cl}_X(X \setminus Z(f))$ is pseudocompact $\}$ is an ideal of $C(X)$. It is well known that $C_\psi(X) = M^{\beta X \setminus \upsilon X}$ and hence it is a semiprime ideal. However, the next result shows that this ideal never is a relative prime ideal.

Proposition 3.13. *The ideal $C_\psi(X)$ is not a relative prime ideal.*

Proof. Suppose that $C_\psi(X) = M^{\beta X \setminus \upsilon X}$ is a relative prime ideal. Then by Theorem 3.9, $\beta X \setminus \upsilon X$ is not a perfect subset of βX. So there exists an isolated point p in the subspace $\beta X \setminus \upsilon X$ of βX. Thus there exists an open set $V_p \subseteq \beta X$ such that $p \in V_p$ and $V_p \cap (\beta X \setminus \upsilon X) = \{p\}$. By Lemma 3.12, there exists a zero-set $Z \in Z[\beta X]$ such that $p \in Z \subseteq \beta X \setminus \upsilon X$. Therefore, $V_p \cap Z = \{p\}$. This shows that $\{p\}$ is a G_δ point in βX, a contradiction, by Lemma 3.11. □

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