Moroccan Journal of Algebra and Geometry with Applications
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco
Volume 1, Issue 2 (2022), pp 418-427
ISSN: 2820-7114

Title :

s-quasi-modular closure of a finite purely inseparable extension

Author(s):

El Hassane Fliouet

Moroccan Journal of Algebra and Geometry with Applications
Vol. 1(2) (2022), 418–427
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco ISSN: 2820-7114

s-quasi-modular closure of a finite purely inseparable extension

El Hassane Fliouet
Regional Center for the Professions of Education and Training, Agadir, Morocco
e-mail: fliouet@yahoo.fr

Communicated by El Hassan El Kinani
(Received 10 August 2022, Revised 24 October 2022, Accepted 31 October 2022)

Abstract. Given a field k of characteristic $p \neq 0$. Let K/k be a finite purely inseparable field extension of j-th exponent e_j. Recall that K is modular over k if and only if for any $n \in \mathbb{N}$, K^{p^n} and k are linearly disjoint over $K^{p^n} \cap k$. This notion, which plays a central role in the development of the Galois theory relating to purely inseparable extensions, was used by M.E. Sweedler to characterize purely inseparable extensions of bounded exponent which were tensor products of simple extensions. Since then, many authors have studied various properties of modular field extensions, including the existence of modular closures. Similarly, K/k is said to be s-quasi-modular if for all $i \in \{1,\dots,e_s\}$, K^{p^i} and k are $K^{p^i} \cap k$ linearly disjoint. Motivated by R. Rasala's work, We characterize the notion of s-quasi modularity and we then a method which makes it possible to build the s-quasi modular closure of K/k. In particular, if $s = n$, we find the Rasala result.

Key Words: Purely inseparable extension, modular extension, s-quasi-modular.
2010 MSC: Primary 12F15.

0. Introduction

Given a finite purely inseparable extension K/k. The notion of modularity, defined by M.E. Sweedler in the 60s, is to the purely inseparable theory what normal is to the separable theory. In particular, Sweedler has shown in ([14], p. 4031) that K is modular over k if and only if K can be written as the tensor product of simple extensions of k. The author then showed that any purely inseparable field extension K/k has an unique minimal modular closure, and that the intersection of modular extensions is again modular (for more information see also W.C. Waterhouse [15]). In the same vein, R. Rasala provides in a simple way a method for building the minimal modular closure using the structure equations adopted by G. Pickert. This paper grew out of an attempt to find an analogue to this result for s-quasi-modular closures. So our first aim in this paper is to give, as in the case of modularity, necessary and sufficient conditions for K/k to be s-quasi modular (cf. Theorem 2.7 and Theorem 2.9).

Now let $B = \{\alpha_1,\dots,\alpha_n\}$ be a canonically ordered r-basis (Rasala uses in [11] the term normal generating sequence) of K/k and e_s the s-th exponent of K/k (however we will often use the usual notation $o_s(K/k)$ to designate the s-th exponent of K/k). Let i be a positive integer $< e_j$. By virtue of ([3], Proposition 9), $\{\alpha_1^{p^{e_j-i}},\dots,\alpha_j^{p^{e_j-i}}\}$ is a canonically ordered r-basis of $k(\alpha_1^{p^{e_j-i}},\dots,\alpha_j^{p^{e_j-i}})/k$, and its list of exponents is $(e_1-(e_j-i),\dots,e_j-(e_j-i))$, and so there exist unique constants $C_\varepsilon \in k$ checking $\alpha_j{}^{p^{e_j}} = (\alpha_j{}^{p^{e_j-i}})^{p^i} = \sum\limits_{\varepsilon\in\Lambda_j} C_{j,\varepsilon}(\alpha_1,\dots,\alpha_{j-1})^{p^i\varepsilon}$, where $\Lambda_j = \{(s_1,\dots,s_{j-1})$ such that $0 \leq s_1 < p^{e_1-e_j},\dots,0 \leq s_{j-1} < p^{e_{j-1}-e_j}\}$.

Definition 0.1. The p^i-root of $C_{j,\varepsilon}$ will be called the i-coefficients of α_j relative to $k(\alpha_1,\dots,\alpha_{j-1})$ and, if there is no confusion, we simply say the i-coefficients of α_j.

By convention, if $i \geq e_j$, the i-coefficients of α_j will have the meaning of e_j-coefficients of α_j. We also agree (if there is no confusion) to call the set of e_s-coefficients of α_i, for $i = 1,\ldots,n$, by the set of e_s-coefficients of K/k which we denote in the sequel by A_1. Now consider the extension $S_1(K/k)$ of K obtained by adjoining A_1 to K. More precisely, $S_1(K/k) = K(A_1) = k(\alpha_1,\ldots,\alpha_{s-1}, ((C_{j,\varepsilon})^{p^{-e_s}})_{\varepsilon\in(\bigcup\Lambda_j)_{j\leq s}}, ((C_{j,\varepsilon})^{p^{-e_j}})_{\varepsilon\in(\bigcup\Lambda_j)_{j>s}})$. Since $K \subseteq S_1(K/k)$ and the exponent of $k(A_1)$ over k does not exceed e_s, therefore $o_s(K/k) = e_s \leq o_s(S_1(K/k)/k) \leq o_1(k(A_1)/k) \leq e_s$, and consequently $o_s(S_1(K/k)/k) = e_s$. Let m_1 be the largest integer such that $o_{m_1}(S_1(K/k)/k) = e_s$. According to the r-basis completion algorithm ([3], Proposition 8), there exist $\alpha_s^1,\ldots,\alpha_{n_1}^1$ elements of A_1 such that $\{\alpha_1,\ldots,\alpha_{s-1},\alpha_s^1,\ldots,\alpha_{n_1}^1\}$ (which will be denoted by B_1) be a canonically ordered r-basis of $S_1(K/k)$. In particular, $k(\alpha_1,\ldots,\alpha_{s-1}, \alpha_s^1,\ldots,\alpha_{m_1}^1) \simeq k(\alpha_1,\ldots,\alpha_{s-1}) \otimes_k k(\alpha_s^1) \otimes_k \ldots \otimes_k k(\alpha_{m_1}^1)$. If $m_1 = n_1$, we take $S_2(K/k) = S_1(K/k)$, otherwise, let A_2 be the set of e_{m_1+1}-coefficients relative to B_1. Now consider the extension $S_2(K/k)$ of $S_1(K/k)$ obtained by adjoining A_2 to $S_1(K/k)$. Also, there exist $\alpha_{m_1+1}^2,\ldots,\alpha_{n_2}^2$ elements of A_2 such that $\{\alpha_1,\ldots,\alpha_{s-1},\alpha_s^1,\ldots,\alpha_{m_1}^1,\alpha_{m_1+1}^2,\ldots, \alpha_{n_2}^2\}$ (denoted by B_2) is a canonically ordered r-basis of $S_2(K/k)/k$ checking:

1. $o_{m_1+1}(S_2(K/k)/k) = o_{m_1+1}(S_1(K/k)/k)$.
2. $k(\alpha_1,\ldots,\alpha_{s-1},\alpha_s^1,\ldots,\alpha_{m_1}^1,\alpha_{m_1+1}^2,\ldots,\alpha_{m_2}^2) \simeq k(\alpha_1,\ldots,\alpha_{s-1})\otimes_k k(\alpha_s^1)\otimes_k\ldots\otimes_k k(\alpha_{m_1}^1) \otimes_k k(\alpha_{m_1+1}^2) \otimes_k \ldots\otimes_k k(\alpha_{m_2}^2)$, where m_2 is the largest integer such that $o_{m_2}(S_2(K/k)/k)$ equal to $o_{m_1+1}(S_2(K/k)/k)$.

Inductively, we obtain a tower of fields $S_r(K/k) = S_1(S_{r-1}/k)$ obtained by adjoining A_r (set of $(m_{r-1}+1)$-coefficients) to $S_{r-1}(K/k)$ if $m_{r-1} \neq n_{r-1}$ and, $S_r(K/k) = S_{r-1}(K/k)$ otherwise. One fact we obtain is that the tower stabilizes after a finite number of steps. Indeed, there also exists $\alpha_{m_{r-1}+1}^r,\ldots,\alpha_{n_r}^r$ elements of A_r such that $\{\alpha_1,\ldots,\alpha_{s-1},\alpha_s^1,\ldots,\alpha_{m_1}^1, \ldots, \alpha_{m_{r-1}+1}^r,\ldots,\alpha_{n_r}^r\}$ (denoted by B_r) is a canonically ordered r-basis of $S_r(K/k)/k$ checking:

1. $o_{m_{r-1}+1}(S_r(K/k)/k) = o_{m_{r-1}+1}(S_r(K/k)/k)$.
2. $k(\alpha_1,\ldots,\alpha_{s-1},\alpha_s^1,\ldots,\alpha_{m_1}^1,\alpha_{m_1+1}^2,\ldots,\alpha_{m_2}^2\ldots,\alpha_{m_{r-1}+1}^r,\ldots,\alpha_{m_r}^r) \simeq k(\alpha_1,\ldots,\alpha_{s-1})\otimes_k k(\alpha_s^1)\otimes_k\ldots\otimes_k k(\alpha_{m_1}^1)\otimes_k \ldots\otimes_k k(\alpha_{m_r}^r)$, where m_r is the largest integer such that $o_{m_r}(S_r(K/k)/k)$ equal to $o_{m_{r-1}+1}(S_{r-1}(K/k)/k)$.

Since $o_{m_1}(S_1(K/k)/k) \geq o_{m_2}(S_2(K/k)/k) \geq \ldots \geq o_{m_r}(S_r(K/k)/k)$, the sequence $(o_{m_i}(S_i(K/k)/k))$ will stationary starting at j, and therefore the sequence $(S_i(K/k))_i$ will also stationary at j, i.e, $S_j(K/k) = S_{j+1}(K/k)$. It is the purpose of this note to prove the following:

Theorem 0.2 (Main theorem)**.** The field $S_j(K/k)$ at which the tower stabilizes is the unique minimal extension of K such that $S_j(K/k)$ is s-quasi-modular over k.

The rest of this paper is organized as follows. In the next section, we review some basic terminologies and results concerning finite purely inseparable extensions, we begin to study the degree of irrationality and exponents related to a finite purely inseparable extension, then we shed light on modular extensions. In the third section, we introduce the notion of s-quasi modularity which is a natural generalization of modularity and we prove some results characterizing this notion. This section which is the heart of this work prepares the way to the proof of our main result. Examples are presented illustrating the application of the results obtained.

Throughout this paper k always designates a field of characteristic $p > 0$, Ω an algebraic closure of k, and all fields under consideration will be purely inseparable extensions of a common ground field k, they are to be viewed as contained in Ω. If K is such a field, K^{p^n} denotes the field of all p^n-th powers of elements from K.

1 Definitions and preliminary results

1.1 Irrationality degree

Definition 1.1. Let K/k be a purely inseparable extension. A minimal set of generators of K over k, which we prefer to call a minimal r-generator of K/k, is a subset G of K such that $K = k(G)$ and $G' \subset G$ implies $k(G') \subset k(G)$ where G denotes proper inclusion.

It is well known that if K/k is a finite purely inseparable extension of k, the minimum number of generators of K/k is r, the exponent determined by the degree $[K : k(K^p)] = p^r$ (cf [8], Theorem 6).

Definition 1.2. A relative p-basis (or simply r-basis) of K over k is a minimal set of generators of K over (K^p). A relatively p-independent (or r-independent) subset B of K/k is a subset B of K such that B is a minimal r-generator of $k(K^p)(B)/k(K^p)$, i.e, for all proper sub-sets B' of B, $k(K^p, B') \subset k(K^p, B)$.

These concepts are introduced in [12]. Recall that K is said to have an exponent (or, to be of bounded exponent) over k if there exists $e \in \mathbb{N}$ such that $K^{p^e} \subseteq k$, and the smallest integer that satisfies this relation will be called the exponent (or height) of K/k. Paul T. Rygg [13] showed that if K/k has a finite exponent, then there exist minimal r-generators of K over k and any two such sets have the same cardinal number. This result for the case of exponent $e = 1$ is given by MacLane ([8], Theorem 12).

Moreover, as in linear algebra, the r-dependence in $K/k(K^p)$ is a dependence relation (cf. [7], Lemma 6.1), and consequently, according to (cf. [7], Theorem 1.3) we have:

1. *Every extension K/k has an r-basis, and any two r-bases of K/k have the same cardinality.*
2. *Any relatively r-independent set of K/k can be extended to an r-basis of K/k.*
3. *Any generator G of $K/k(K^p)$ contains an r-basis of K/k.*

Pickert ([10], p. 881) has shown that if K is a finite inseparable extension of k and G is a minimal r-generators of K over k, then G is r-independent of K/k. More generally, it is shown by Paul T. Rygg [13] that if K/k has an exponent, a subset B of K is an r-basis of K/k if and only if B is a minimal r-generator of K/k. However, a minimal r-generator may not exist in the general case (cf. [9], Lemma 1.16, Proposition 1.23). In this note, unless otherwise stated, we assume that K is a finite purely inseparable extension of k.

Let us denote in the sequel by $di(K/k) = |B|$, where B is a minimal r-generator of K/k, the irrationality degree of K/k. Similarly, $di(k/k^p)$ will designate the imperfection degree of k, and it is denoted by $di(k)$.

We will often use the following theorem.

Theorem 1.3 ([5], Theorem 2.7)**.** For any family $k \subseteq L \subseteq L' \subseteq K$ of purely inseparable extensions, we have $di(L'/L) \leq di(K/k)$.

without losing any generality, a similar result for a more general field extension is given in ([5], Theorem 2.7). Now let K_1 and K_2 be intermediate fields of K/k, we immediately check that if B_1 and B_2 are two r-bases, respectively of K_1/k and K_2/k, then B_1 and $B_1 \cup B_2$ are two r-generators, respectively of $K_1(K_2)/K_2$ and $K_1(K_2)/k$. Furthermore, $di(K_1(K_2)/K_2) \leq di(K_1/k)$ and $di(K_1(K_2)/k) \leq di(K_1/k) + di(K_2/k)$. More precisely, we get the following result.

Proposition 1.4 ([2])**.** *If K_1/k and K_2/k are k-linearly disjoint, we then have:*

(i) *$B_1 \cup B_2$ is an r-basis of $K_1(K_2)/k$.*

(ii) *B_1 is an r-basis of $K_1(K_2)/K_2$.*

We will also need the following result which is a well-known consequence of the transitivity of linear disjointness.

Proposition 1.5 ([6], p. 35, Lemma 2.5.3)**.** *If K_1/k and K_2/k are k-linearly disjoint, then for every subfields L_1 and L_2 of K_1/k and K_2/k respectively, $L_2(K_1)$ and $L_1(K_2)$ are $k(L_1,L_2)$-linearly disjoint. In particular, $L_2(K_1)\cap L_1(K_2)=k(L_1,L_2)$.*

Corollary 1.6. *For any subset G of K_2 such that $K_1(K_2)=K_1(G)$, we have $K_2=k(G)$. In particular, if G is a linear basis of $K_1(K_2)$ over K_1, then G is also a linear basis of K_2 over k.*

Proposition 1.7. *Let L be a subfield of K over k and M a subset of k such that M is an r-basis of $k(K^p)/K^p$. Then $di(K/k)=di(k/k^p(M))$. In particular, $di(L/k)=di(K/k)$ if and only if $k(L^p)$ and K^p are L^p-linearly disjoint.*

Proof. Let G be a subset of k such that $M\cup G$ is a p-basis of k. By virtue of [9, p. 14, Proposition 1.22], for each positive integer n, $K(M^{p^{-n}})\simeq k(M^{p^{-n}})\otimes_k K$, so $di(K(M^{p^{-n}})/k(M^{p^{-n}}))=di(K/k)$. Furthermore, $k^{p^{-n}}\simeq k(M^{p^{-n}})\otimes_k k(G^{p^{-n}})$, it follows that $di(k^{p^{-n}}/k(M^{p^{-n}}))=di(k(G^{p^{-n}})/k)=|G|$. In particular,

$$|G|=di(k^{p^{-1}}/k(M^{p^{-1}}))\leq di(k^{p^{-1}}(K)/k(M^{p^{-1}})), \text{(cf. Theorem 0.2)}.$$

Since $K\subseteq k^{p^{-n_0}}$, where n_0 is the exponent of K/k, and $k^{p^{-1}}(K)=K(M^{p^{-1}})\subseteq K(M^{p^{-n_0}})$, so

$$di(k^{p^{-1}}(K)/k(M^{p^{-1}}))=di(K(M^{p^{-n_0}})/k(M^{p^{-n_0}}))\leq di(k^{p^{-n_0}}/k(M^{p^{-n_0}}))=|G|.$$

Therefore $di(K/k)=di(k/k^p(M))=|G|$. □

The previous result extends to finite type extensions. It links the invariant of inseparability to the degree of irrationality ([4]).

1.2 Exponents of a finite purely inseparable extension

In this paragraph, we will us some basic definitions and notations as it is mentioned in [3]. Consider an element x of K, recall that the least positive integer e such that $x^{p^e}\in k$ is called the exponent of x over k, and is denoted by $o(x/k)$. Clearly, the maximum of the set of exponents of elements of K is the exponent of K over k, that is, the smallest integer e such that $K^{p^e}\subseteq k$, which will be denoted throughout this paper by $o_1(K/k)$. An r-basis $B=\{a_1,a_2,\dots,a_n\}$ of K/k is said to be canonically ordered if for $j=1,2,\dots,n$, we have $o(a_j/k(a_1,a_2,\dots,a_{j-1}))=o_1(K/k(a_1,a_2,\dots,a_{j-1}))$.

Lemma 1.8 ([2], Lemma 1.3)**.** *The integer $o(a_j/k(a_1,\dots,a_{j-1}))$ thus defined satisfies*

$$o(a_j/k(a_1,\dots,a_{j-1}))=\inf\{m\in\mathbb{N}|\, di(k(K^{p^m})/k)\leq j-1\}.$$

We immediately deduce the structure theorem, one of the fundamental theorems which makes it possible to introduce, by means of exponents, the structure equations of a purely inseparable extension discovered for the first time by Picker [10] in 1949, and can be stated as follows: If K/k is finite purely inseparable of multiplicity (irrationality degree) m, then there is an ordering of the generators, namely $a_1,\dots,a_m$, that the following conditions hold for $i=1,\dots,m$:

1. ${a_i}^{q_i}\in k({a_1}^{q_i},\dots,{a_{i-1}}^{q_i})$, where $q_i=p^{e_i}$ and $e_i>0$.
2. ${a_i}^{p^{-1}}\notin k(a_1,\dots,a_{i-1})$.

3. $e_1 \geq e_2 \geq \ldots \geq e_m$.

Conversely, if K/k is generated by the m elements $a_1, \ldots, a_m$ satisfying the first two conditions above, then the exponents $e_1 \ldots, e_m$ are invariants of the extension. In particular, the structure equations have the following form: $a_i{}^{q_i} = \sum_{\alpha \in I_i} a_{i,\alpha} a^{q_i \alpha}$. Here I_i is a suitable multi-index set and $a_{i,\alpha} \in k$.

Definition 1.9. The invariant defined by $o_i(K/k) = o(a_i/k(a_1, \ldots, a_{i-1}))$ if $1 \leq i \leq n$ and $o_i(K/k) = 0$ if $i > n$ is called the i-th exponent of K/k.

Also as a consequence of Lemma 1.8 and Theorem 1.3, we get the following proposition which will be used repeatedly.

Proposition 1.10. *For any intermediate field L of K/k, $o_i(L/k) \leq o_i(K/k)$.*

In a more exact way, the structure theorem can be seen as follows.

Proposition 1.11. *([3], Proposition 9), Let $\{\alpha_1, \ldots, \alpha_n\}$ be a canonically ordered r-basis of K/k and e_j the j-th exponent of K/k. Then we have:*

- $k(K^{p^{e_j}}) = k(\alpha_1^{p^{e_j}}, \ldots, \alpha_{j-1}^{p^{e_j}})$.
- *Let $\Lambda_j = \{(i_1, \ldots, i_{j-1})$ such that $0 \leq i_1 < p^{e_1 - e_j}, \ldots, 0 \leq i_{j-1} < p^{e_{j-1} - e_j}\}$. Then $\{(\alpha_1, \ldots, \alpha_{j-1})^{p^{e_j}\xi}$ such that $\xi \in \Lambda_j\}$ is a linear basis of $k(K^{p^{e_j}})$ over k.*
- *Let s be a positive integer $\leq e_1$ and j the largest integer such that $e_j > s$. Then $\{\alpha_1^{p^s}, \ldots, \alpha_j^{p^s}\}$ is a canonically ordered r-basis of $k(K^{p^s})/k$, and its list of exponents is $(e_1 - s, \ldots, e_j - s)$.*

The next proposition is needed in the proof of the main result.

Proposition 1.12. *Given an intermediate field of K/k. Let e_s be the s-th exponent of K/k, $1 \leq s \leq n$. The following assertions are equivalent:*

1. *For each $i \in \{0, \ldots, e_s - 1\}$, $di(k(F^{p^i})/k) = di(k(K^{p^i})/k)$.*
2. *For each $i \in \{0, \ldots, e_s - 1\}$, $k(F^{p^{i+1}})$ and $k^p(K^{p^{i+1}})$ are $k^p(F^{p^{i+1}})$ linearly disjoint.*
3. *For each $i \in \{0, \ldots, e_s - 1\}$, $k(F^{p^{i+1}})$ and $K^{p^{i+1}}$ are $F^{p^{i+1}}$ linearly disjoint.*
4. *$k(F^{p^{e_s}})$ and $K^{p^{e_s}}$ are $F^{p^{e_s}}$ linearly disjoint.*

Proof. Taking into account Proposition 1.7 and Proposition 1.5, the assertions (1) and (2) are equivalent. It suffices to show that (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (2). Suppose that (2) holds. The proof is done by induction on i. It is clear that the result (3) is valid for $i = 0$ (cf. Proposition 1.5). If $k(F^{p^j})$ and K^{p^j} are F^{p^j} linearly disjoint for each $j \in \{0, \ldots, i\}$ where i is a positive integer $< e_s - 1$, which is equivalent to $k^p(F^{p^{j+1}})$ and $K^{p^{j+1}}$ are $F^{p^{j+1}}$ linearly disjoint. But as $k(F^{p^{j+1}})$ and $k^p(K^{p^{j+1}})$ are $k^p(F^{p^{j+1}})$ linearly disjoint (by (2)), therefore by transitivity of linear disjointness $k(F^{p^{j+1}})$ and $K^{p^{j+1}}$ are $F^{p^{j+1}}$ linearly disjoint, whence (3) follows by induction. The condition (4) results immediately from (3). Conversely, suppose that $k(F^{p^{e_s}})$ and $K^{p^{e_s}}$ are $F^{p^{e_s}}$ linearly disjoint. It follows that for each $i \in \{0, \ldots, e_s - 1\}$, $k^{p^{e_s - i}}(F^{p^{e_s}})$ and $K^{p^{e_s}}$ are $F^{p^{e_s}}$ linearly disjoint which is equivalent to $(k^{p^{e_s-i}}(F^{p^{e_s}}))^{p^{i-e_s}}$ and $(K^{p^{e_s}})^{p^{i-e_s}}$ are $(F^{p^{e_s}})^{p^{i-e_s}}$ linearly disjoint. This leads to $k(F^{p^i})$ and K^{p^i} are F^{p^i} linearly disjoint for every $i \in \{0, \ldots, e_s - 1\}$. Furthermore, if for each $i \in \{0, \ldots, e_s - 1\}$, $k(F^{p^{i+1}})$ and $K^{p^{i+1}}$ are $F^{p^{i+1}}$ linearly disjoint, so in particular $k(F^{p^{i+1}})$ and $k^p(K^{p^{i+1}})$ are $k^p(F^{p^{i+1}})$ linearly disjoint. $\square$

1.3 Modular extension

A subset B of K which we will prefer called a modular r-basis (M. Weisfeld used the term sub-basis see [16, p. 435]) of K over k if and only if it fulfills the following conditions: $B \cap k = \emptyset$, $K = k(B)$, and, for any finite subset $\{b_1, \dots, b_t\}$ of B, the canonical homomorphism of the tensor product $k(b_1)\otimes_k \dots \otimes_k k(b_t)$ into K is a monomorphism. This is equivalent, by [9, p. 14, Definition 1.21], to for every finite subset $\{b_1, \dots, b_t\}$ of B, $[k(b_1, \dots, b_t) : k] = \prod_{i=1}^{t}[k(b_i) : k]$, that is, $k(b_1, \dots, b_t)$ is a tensor product over k of the simple extensions $k(b_1), \dots, k(b_t)$. Sweedler showed in [14, p. 403, Theorem 1] that if K over k has a finite exponent, then K is modular over k, i.e. for all (positif) integer n, K^{p^n} and k are $k \cap K^{p^n}$-linearly disjoint, if and only if K can be written as the tensor product of simple extensions of k, that is, K/k has a modular r-basis.

Let e_j be the j-th exponent of a finite purely inseparable extension K/k and let $\{\alpha_1, \dots, \alpha_n\}$ be a canonically ordered r-basis of K/k, so for all $1 < j \leq n$, there are unique constants $C_\varepsilon \in k$ such as ${\alpha_j}^{p^{e_j}} = \sum_{\varepsilon\ in \Lambda_j} C_\varepsilon(\alpha_1, \dots, \alpha_{j-1})^{e_j\varepsilon}$, where $\Lambda_j = \{(i_1, \dots, i_{j-1}) | 0 \leq i_1 < p^{e_1 - e_j}, \dots, 0 \leq i_{j-1} < p^{e_{j-1}-e_j}\}$.

Proposition 1.13 (Modularity criterion)**.** *The following properties are equivalent:*

1. *K/k is modular.*
2. *For every canonically ordered r-basis $\{\alpha_1, \dots, \alpha_n\}$ of K/k, the $C_\varepsilon \in k \cap K^{p^{e_j}}$ for all $1 < j \leq n$.*
3. *There exists a canonically ordered r-basis $\{\alpha_1, \dots, \alpha_n\}$ of K/k such that the $C_\varepsilon \in k \cap K^{p^{e_j}}$ for all $1 < j \leq n$.*

Example 1.14. Let k_0 be a perfect field of characteristic $p \neq 0$, $k = Q(X, Y, Z)$ the field of rational fractions in indeterminates X, Y, Z, and $K = k(\alpha_1, \alpha_2)$ where $\alpha_1 = X^{p^{-2}}$ and $\alpha_2 = X^{p^{-2}}Y^{p^{-1}} + Z^{p^{-1}}$. We check immediately that:

- $o_1(K/k) = 2$ et $o_2(K/k) = 1$,
- $\alpha_2^p = Y\alpha_1^p + Z$.

If K/k is modular, according to the modularity criterion, we will have $Y \in k \cap K^p$ and $Z \in k \cap K^p$, and therefore $Y^{p^{-1}}$ and $Z^{p^{-1}} \in K$. Hence $k(X^{p^{-2}}, Y^{p^{-1}}, Z^{p^{-1}}) \subset K$, and consequently

$$di(k(X^{p^{-2}}, Y^{p^{-1}}, Z^{p^{-1}})/k) = 3 < di(K/k) = 2, \text{contradiction.}$$

2 s-quasi modular extension

Definition 2.1. Let K/k be a finite purely inseparable extension of irrationality degree n and exponents e_s. Let s belong to $\{1, \dots, n\}$, K/k is called s-quasi-modular if for each $i = 1, \dots, e_s$, K^{p^i} and k are $k \cap K^{p^i}$ linearly disjoint.

Let F be an intermediate field of K/k. taking into account ([9], Proposition 3.3), it is equivalent to say that:

- There exists an intermediate field J of K/k such that $K \simeq J \otimes_k F$, J/k is modular and in particular, K/F is modular.

- There exists a canonically ordered r-basis $\{\alpha_1,\ldots,\alpha_e\}$ of K/F such that $\alpha_i{}^{p^{m_i}} \in (K^{p^i}\cap k)(K_i{}^{p^i})$ where $m_i = o_i(K/F)$ and $K_i = F^{p^{m_i}}(\alpha_1{}^{p^{m_i}},\ldots,\alpha_{i-1}{}^{p^{m_i}})$, $i = 1,\ldots,m_1$.

Lemma 2.2. *If K/k is s-quasi-modular, there exists a canonically ordered r-basis $\{\alpha_1,\ldots,\alpha_n\}$ of K/k such that $K \simeq k(\alpha_1,\ldots,\alpha_{s-1})\otimes_k k(\alpha_s)\otimes_k \ldots \otimes_k k(\alpha_n)$*

Proof. Let $\{\alpha_1,\ldots,\alpha_n\}$ be a canonically ordered r-basis of K/k. If e_i designates the i-th exponent of K/k, then $k(K^{p^{e_i}}) = k(\alpha_1{}^{p^{e_i}},\ldots,\alpha_{i-1}{}^{p^{e_i}})$. Since $K^{p^{e_i}}$ and k are $k\cap K^{p^{e_i}}$ linearly disjoint for $i = 1,\ldots,e_s$, then $K^{p^{e_i}}$ and $k(\alpha_1{}^{p^{e_i}},\ldots,\alpha_{i-1}{}^{p^{e_i}})$ are $(k\cap K^{p^{e_i}})(\alpha_1{}^{p^{e_i}},\ldots,\alpha_{i-1}{}^{p^{e_i}})$ linearly disjoint, and therefore $K^{p^{e_i}} = K^{p^{e_i}}\cap k(K^{p^{e_i}}) = (k\cap K^{p^{e_i}})(\alpha_1{}^{p^{e_i}},\ldots,\alpha_{i-1}{}^{p^{e_i}})$, and consequently the result holds from ([9], Proposition 3.3). □

By construction, we have:

Lemma 2.3. *$S_1(K/k)$ is the unique minimal extension of K containing the e_s-coefficients of $\alpha_1,\ldots,\alpha_n$.*

Proof. Immediate. □

The following proposition characterizes the notion of s-quasi-modularity by relating it to the modularity, in which case we have the tools to test the modularity of an extension.

Proposition 2.4. *If $k^{p^{-e_s}}\cap K/k$ is modular and $o_i(k^{p^{-e_s}}\cap K/k) = o_i(K/k) = e_i$ for all $i\in\{s,\ldots,n\}$, then K/k is s-quasi-modular.*

Proof. To simplify the writing, we set $F = k^{p^{-e_s}}\cap K$. As $o_i(F/k) = o_i(K/k) = e_i$ for all $i\in\{s,\ldots,n\}$, according to Lemma 1.8, $di(k(F^{p^i})/k) = di(k(K^{p^i})/k)$ for all $i\in\{0,\ldots,e_s-1\}$, and consequently, for all $i\in\{0,\ldots,e_s\}$, $k(F^{p^i})$ and K^{p^i} are F^{p^i} linearly disjoint by virtue of Proposition 1.12. On the other hand, since F/k is modular then F^{p^i} and k are $k\cap F^{p^i}$ linearly disjoint for all (positive) integer i, so this property holds in particular for every $i\in\{0,\ldots,e_s\}$. By transitivity of linear disjointness, we have K^{p^i} and k are $k\cap F^{p^i}$ linearly disjoint for all $i\in\{0,\ldots,e_s\}$, notably K^{p^i} and k are $k\cap K^{p^i}$ linearly disjoint for all $i\in\{0,\ldots,e_s\}$, which implies that K/k est s-quasi-modular. □

Here is an example of an s-quasi-modular extension that is not modular.

Example 2.5. Let k_0 be a perfect field of characteristic $p\neq 0$ and X, Y, Z independent indeterminates over k_0. Let $k = k_0(X,Y,Z)$ and $K = k(m_1,m_2,m_3)$, where $m_1 = X^{p^{-3}}$, $m_2 = X^{p^{-3}}Y^{p^{-2}} + Z^{p^{-2}}$ and $m_3 = Y^{p^{-1}}$. We check that

$$\begin{aligned} k^{p-1}\cap K &= k(X^{p^{-1}},Y^{p^{-1}},Z^{p^{-1}}) \\ &\simeq k(X^{p^{-1}})\otimes_k k(Y^{p^{-1}})\otimes_k k(Z^{p^{-1}}). \end{aligned}$$

Clearly $k^{p-1}\cap K/k$ is modular, by the previous proposition K/k is 1-quasi-modular. But K/k is not modular, indeed if K/k is modular, from the structure equations we get:

$$\begin{aligned} m_2{}^{p^2} &= YX^{p^{-1}} + Z \\ &= Ym_1{}^{p^2} + Z \end{aligned}$$

with Y and Z belong to k. By virtue of the modularity criterion, we have $Y,Z\in k\cap K^{p^2}$, and thus $Y^{p^{-2}}, Z^{p^{-2}}\in K$. It follows that $k(X^{p^{-2}},Y^{p^{-2}},Z^{p^{-2}})\subseteq K$, according to Proposition 1.10,

$$2 = o_3(k(X^{p^{-2}},Y^{p^{-2}},Z^{p^{-2}})/k) \leq o_3(K/k) = 1,$$

it's a contradiction, and therefore K/k is not modular.

Remark 2.6. The converse of the previous proposition is also true (we can use the proof of the following theorem).

Now giving us as in the case of modularity, by means of the structural equations, a necessary and sufficient condition for K/k to be s-quasi-modular.

Theorem 2.7. K/k is s-quasi-modular if and only if K contains the e_s-coefficients relative to a canonically ordered r-basis of K/k.

Proof. Let $\{\alpha_1,\dots,\alpha_n\}$ be a canonically ordered r-basis of K/k and $e_i = o_i(K/k)$. By Proposition 1.11, there exist unique constants $C_{i,\varepsilon} \in k$ such that

$$\alpha_i{}^{p^{e_i}} = (\alpha_i{}^{p^{e_i-e_s}})^{p^{e_s}} = \sum_{\varepsilon\in\Lambda_i} C_{i,\varepsilon}(\alpha_1,\dots,\alpha_{i-1})^{p^{e_s}\varepsilon},$$

for $i \in \{1,\dots,s-1\}$ (respectively $\alpha_i{}^{p^{e_i}} = \sum_{\varepsilon\in\Lambda_i} C_{i,\varepsilon}(\alpha_1,\dots,\alpha_{i-1})^{p^{e_i}\varepsilon}$, for $i \in \{s,\dots,n\}$) where Λ_i is a suitable multi-index set. As $\{(\alpha_1,\dots,\alpha_{i-1})^{p^{e_s}\varepsilon}\}_{\varepsilon\in\Lambda_i}$ is linearly independent over k, in particular, it is remains also linearly independent over $k\cap K^{p^{e_s}}$. We complete this system to a linear basis G of $K^{p^{e_s}}$ over $k\cap K^{p^{e_s}}$. As $K^{p^{e_s}}$ and k are $k\cap K^{p^{e_s}}$ linearly disjoint, then G is also a linear basis of $k(K^{p^{e_s}})$ over k. By identification, the $C_{i,\varepsilon}$ belong to $k\cap K^{p^{e_s}}$, or again the $(C_{i,\varepsilon})^{p^{-e_s}}$ belong to $k^{p^{-e_s}}\cap K$. In the same way, we show that the e_i-coefficient of α_i, for $i = s,\dots,n$, belong to K.

Conversely, taking into account Proposition 2.4, it suffices to show that $k^{p^{-e_s}}\cap K/k$ is modular and for each $i \in \{s,\dots,n\}$, $o_i(k^{p^{-e_s}}\cap K/k) = e_i$. To do this, we will first construct by reverse induction a sequence $L_{s-1}\supseteq\dots\supseteq L_1$ checking:

1. L_j decomposes as follows: $L_j = k(\alpha_1,\dots,\alpha_j)\otimes_k k(\lambda_{j+1})\otimes_k\dots\otimes_k k(\lambda_n)$.
2. for each $r = j+1,\dots,s$, $o_r(L_j/k) = o_s(K/k)$ and for each $r = s+1,\dots,n$, $o_r(L_j/k) = o_r(K/k)$.

According to Lemma 1.8, there exists a canonically ordered r-basis $\{\alpha_1,\dots,\alpha_n\}$ of K/k such that $K \simeq k(\alpha_1,\dots,\alpha_{s-1})\otimes_k k(\alpha_s)\otimes_k\dots\otimes_k k(\alpha_n)$, therefore for $j = s-1$, $L_j = K$ is suitable, with$\alpha_i = \lambda_i$ for $i = s,\dots,n$. Suppose that L_i is constructed, so L_i has the form $L_i = k(\alpha_1,\dots,\alpha_i)\otimes_k k(\lambda_{i+1})\otimes_k \dots\otimes_k k(\lambda_n)$. Now let's put $L_{i-1} = k(\alpha_1,\dots,\alpha_{i-1},\alpha_i{}^{p^{e_i-e_s}})\otimes k(\lambda_{i+1})\otimes_k\dots\otimes_k k(\lambda_n)$. From this we obtain structure equations of the form $\alpha_i{}^{p^{e_i}} = (\alpha_i{}^{p^{e_i-e_s}})^{p^{e_s}} = \sum_{\varepsilon\in\Lambda} C_\varepsilon(\alpha_1,\dots,\alpha_{i-1})^{p^{e_s}\varepsilon}$, where Λ is a suitable multi-index set. As the e_s-coefficients belong to K, i.e, the $(C_\varepsilon)^{p^{-e_s}}$ belong to K, hence $\alpha_i{}^{p^{e_i-e_s}} \in k(\alpha_1,\dots,\alpha_{i-1},\lambda_{i+1},\dots\lambda_n,((C_\varepsilon)^{p^{-e_s}})_{\varepsilon\in\Lambda})$ which we denote by L. Since $L_{i-1}\subseteq L\subseteq K$, then

$$n = di(L_{i-1}/k) \le di(L/k) \le di(K/k) = n \text{ Theorem 1.3,}$$

and consequently $di(L/k) = n$. Also, since $\{\alpha_1,\dots,\alpha_n\}$ is an r-basis of K/k, then $\{\alpha_1,\dots,\alpha_{i-1},\alpha_s,\dots,\alpha_n\}$ which coincides with $\{\alpha_1,\dots,\alpha_{i-1},\lambda_s,\dots,\lambda_n\}$ is r-independent in L/k. However, if there exists $j\in\{i+1,\dots,s-1\}$ such that $\lambda_j \in k(L^p)(\alpha_1,\dots,\alpha_{i-1},\lambda_{i+1},\dots,\lambda_{j-1},\lambda_{j+1},\dots,\lambda_n)$, as

$$L_{i-1} = k(\alpha_1,\dots,\alpha_{i-1},\alpha_i{}^{p^{e_i-e_s}})\otimes k(\lambda_{i+1})\otimes_k\dots\otimes_k k(\lambda_n),$$

we would have $e_s = o(\lambda_j/k(\alpha_1,\dots,\alpha_{i-1},\lambda_{i+1},\dots,\lambda_{j-1},\lambda_{j+1},\dots,\lambda_n)) \le o_1(k(\alpha_1,\dots,\alpha_{i-1},\lambda_{i+1},\dots,\lambda_{j-1},\lambda_{j+1},\dots,\lambda_n)(((C_\varepsilon)^{p^{-e_s+1}})_{\varepsilon\in\Lambda})/k(\alpha_1,\dots,\alpha_{i-1},\lambda_{i+1},\dots,\lambda_{j-1},\lambda_{j+1},\dots,\lambda_n)) \le e_s-1$, contradiction. We deduce that $\{\alpha_1,\dots,\alpha_{i-1},\lambda_{i+1},\dots,\lambda_n\}$ is r-independent in L/k. So there exists $\varepsilon\in\Lambda$ such that $\{\alpha_1,\dots,\alpha_{i-1},\lambda_{i+1},\dots,\lambda_n,C_\varepsilon{}^{p^{-e_s}}\}$ is an r-basis of L/k. Let $\lambda_i = C_\varepsilon{}^{p^{-e_s}}$. As $L_{i-1}\subseteq L$, then $e_s = o_i(L_{i-1}/k) \le o_i(L/k) \le o_1(L/k(\alpha_1,\dots,\alpha_{i-1})) \le e_s$, and consequently $L_{i-1} = L = k(\alpha_1,\dots,\alpha_{i-1},\lambda_i,\dots,\lambda_n) \simeq k(\alpha_1,\dots,\alpha_{i-1})\otimes_k k(\lambda_i)\otimes_k\dots{}_k k(\lambda_n)$, since $o_j(L/k) = o_j(L_{i-1}/k)$ for every $j \in \{1,\dots,n\}$. Whence by induction, there exist $\lambda_2,\dots,\lambda_n$ such that $L_1 \simeq k(\alpha_1)\otimes_k k(\lambda_2)\otimes_k \dots \otimes_k k(\lambda_n)$. Since the first s exponents of $k^{p^{-e_s}} \cap K$ over k are $\geq e_s$ and the other exponents do not exceed those of K/k, it follows that $k^{p^{-e_s}} \cap K \simeq k({\alpha_1}^{p^{e_1-e_s}})\otimes_k k(\lambda_2)\otimes_k \dots \otimes_k k(\lambda_n)$, and hence K/k is s-quasi-modular. □

The previous theorem can be rephrased easily into the following property:

Theorem 2.8. The statements below are equivalent.

1. K/k is s-quasi-modular.
2. For every canonically ordered r-basis $\{\alpha_1,\dots,\alpha_n\}$ of K/k, for all $1 < j \leq n$,
$$ {\alpha_j}^{p^{e_j}} \in k \cap K^{p^{\varepsilon_j}}({\alpha_1}^{p^{e_j}},\dots,{\alpha_j}^{p^{e_{j-1}}}), \text{ where } \varepsilon_j = e_j \text{ if } s \leq j \text{ and } \varepsilon_j = e_s \text{ if } j \leq s. $$
3. There exists a canonically ordered r-basis $\{\alpha_1,\dots,\alpha_n\}$ of K/k, for all $1 < j \leq n$,
$$ {\alpha_j}^{p^{e_j}} \in k \cap K^{p^{\varepsilon_j}}({\alpha_1}^{p^{e_j}},\dots,{\alpha_j}^{p^{e_{j-1}}}), \text{ where } \varepsilon_j = e_j \text{ if } s \leq j \text{ and } \varepsilon_j = e_s \text{ if } j \leq s. $$

2.1 Main Result (s-quasi-modular closure Theorem)

Proof. (Proof of Main theorem) By construction the e_s-coefficients relative to $\{\alpha_1,\dots,\alpha_{s-1},\alpha_s^1,\dots,\alpha_{m_1}^1, \dots, \alpha_{m_{j-1}+1}^j,\dots,\alpha_{m_j}^j\}$ (namely $m_j = n_j$) belong to $S_j(K/k)$, hence $S_j(K/k)$ is s-quasi modular. If L is an extension of K such that L is s-quasi-modular over k, By application repeat from Lemma 2.2, we immediately verify that the $S_i(K/k)$ are included in L. □

As an immediate consequence, we have:

Theorem 2.9 (s-quasi-modular closure Theorem)**.** There exists an unique minimal field F, $F \supseteq K$, such that:

1. F/k is s-quasi-modular.
2. $o_i(K/k) = o_i(F/k)$ for each $i \in \{1,\dots,s\}$.
3. F/k is finite.

Proof. By construction, $S_j(K/k)$ satisfies all the conditions of the above Proposition. □

Definition 2.10. The unique minimal extension of K which is s-quasi-modular over k will be called the s-quasi modular closure of K/k.

It is clear that if F_s designates the s-quasi-modular closure of K/k, we obtain a tower of fields $F_1 \supseteq \dots \supseteq F_n$ with F_1 is the modular closure of K/k.

We now construct K/k such that the unique minimal extension of K which is s-quasi-modular over k is not elementary over k, i.e, the s-quasi-modular closure of K/k does not coincide with the modular closure of K/k.

Example 2.11. Let k_0 be a perfect field of characteristic $p \neq 0$ and let x, y, z, t_1, t_2, t_3 be algebraically independent over k_0. Let $k = k_0(x,y,z,t_1,t_2,t_3)$ and $K = k(m_1,m_2,m_3)$, where $m_1 = x^{p^{-4}}$, $m_2 = {t_1}^{p^{-1}}x^{p^{-4}} + y^{p^{-3}}$ and $m_3 = Z^{p^{-2}}x^{p^{-3}} + {t_2}^{p^{-1}}Z^{p^{-2}} + {t_3}^{p-1}$.

Clearly (m_1, m_2, m_3) is a canonically ordered r-basis of K/k such that $o_1(K/k) = o(m_1/k) = 4$, $o_2(K/k) = o(m_2/k(m_1)) = 3$ and $o_3(K/k) = o(m_3/k(m_1, m_2)) = 2$. We also check that

$$\begin{aligned} {m_3}^{p^2} &= Z{m_1}^{p^3} + {t_2}^p Z + {t_3}^p \\ ({m_2}^p)^{p^2} &= {t_1}^{p^2}({m_1}^p)^{p^2} + y, \end{aligned}$$

so, by construction, we have $S_1(K/k) = k(m_1, m_2, m_3, t_1, y^{p^{-2}}, Z^{p^{-2}}, {t_2}^{p^{-1}}Z^{p^{-2}} + {t_3}^{p-1})$. As $t_1, y^{p^{-2}} \in K$ (namely $y^{p^{-2}} = {m_2}^p - t_1{m_1}^p$), then $K = k(m_1, m_2, \beta, \gamma)$, where $\beta = Z^{p^{-2}}$ and $\gamma = {t_2}^{p^{-1}}Z^{p^{-2}} + {t_3}^{p^{-1}}$. Similarly we have $(m_1, m_2, \beta, \gamma)$ is an r-basis of $S_1(K/k)$ and $(4, 3, 2, 1)$ is the list of exponents of $S_1(K/k)$. As $\gamma^p = t_2\beta^p + t_3$, β^{p^2}, so $S_2(K/k) = k(m_1, m_2, \beta, {t_2}^{p^{-1}}, {t_3}^{p^{-1}})$. Using the structure equations again, we get $S_2(K/k) = S_3(K/k)$ and therefore $S_2(K/k)$ is the s-quasi modular closure of K/k, but $S_2(K/k)$ is not modular. In effect, it is clear that the system $(1, {m_1}^{p^3})$ is linearly independent over k, so in particular it is linearly independent over $k \cap S_2(K/k)^{p^3}$. We extend this system to a linear basis B of $S_2(K/k)^{p^3}$ over $k \cap S_2(K/k)^{p^3}$. If $S_2(K)/k$ is modular, $S_2(K)^{p^3}$ and k are $k \cap S_2(K)^{p^3}$-linearly disjoint, and consequently B is also a linear basis of $k(S_2(K)^{p^3})$ over $S_2(K)^{p^3}$. Since ${m_2}^{p^3} = {t_1}^{p^2}{m_1}^{p^3} + y$ with ${t_1}^{p^2}$, y belong to k and ${m_2}^{p^3}$ is written uniquely as a sum of elements of B, then by identification we will have ${t_1}^{p^2}, y \in k \cap S_2(K)^{p^3}$, and thus ${t_1}^{p^{-1}}$, $y^{p^{-3}} \in S_2(K)$. It follows that $L = k(X^{p^{-2}}, Y^{p^{-2}}, Z^{p^{-2}}, {t_1}^{p^{-1}}, {t_2}^{p^{-1}}, {t_3}^{p^{-1}}) \subseteq S_2(K)$, as a result $6 = di(L/k) \le di(S_2(K)/k) = 5$, it's a contradiction, and therefore $S_2(K)/k$ is not modular.

References

[1] M. F. Becker and S. MacLane, The minimum number of generators for inseparable algebraic extensions, Bull. Amer. Math. Soc. 46 (1940), 182–186.

[2] M. Chellali and E. Fliouet, Extensions purement inséparables d'exposant non borné, Arch. Math. 40 (2004), 129–159.

[3] M. Chellali and E. Fliouet, Sur les extensions purement inséparable, Arch. Math. 81 (2003), 369–382.

[4] E. Fliouet, On the invariants of inseparable field extensions, Arch. Math. 117(2) (2021), 155–164.

[5] E. Fliouet, Absolutely lq-finite extensions, Acta Math. Vietnam 44(3) (2019), 751–779.

[6] M. D. Fried and M. Jarden, Field arithmetic, Springer-Verlag Berlin Heidelberg (2008).

[7] G. Karpilovsky, Topics in field theory, North-Holland Mathematics Studies 155, Elsevier Science Publishers B.V., (1989).

[8] S. MacLane, A lattice formulation for transcendence degrees and p-bases, Duke Math. J. 4 (1938), 455–468.

[9] J. N. Mordeson and B. Vinograde, Structure of arbitrary purely inseparable extension fields, Springer-Verlag, Berlin, LNM 173, (1970).

[10] G. Pickert, Inseparable Körperweiterungen, Math. Z. 52 (1949), 81–135.

[11] R. Rasala, Inseparable splitting theory, Trans. Amer. Math. Soc. 162 (1971) 411–448.

[12] O. Teichmiiller, p-Algebren, Deutsche Math. 1 (1936), 362–388.

[13] P. T. Rygg, On minimal sets of generators of purely inseparable field extensions, Proc. Am. Math. Soc. 14(5) (1963), 742–745.

[14] M. E. Sweedler, Structure of inseparable extensions, Ann. Math. 87(2) (1968), 401–410.

[15] W. C. Waterhouse, The structure of inseparable field extensions, Trans. Am. Math. Soc. 211 (1975), 39–56.

[16] M. Weisfeld, Purely inseparable extensions and higher derivations, Trans. Amer. Math. Soc. 116 (1965), 435–469.