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Arithmetical rank for some class of commutative rings

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Abstract. In this paper, we focus on the arithmetical rank of an ideal for some class of commutative rings issued from some constructions such as fiber product, amalgamation along an ideal and trivial ring extension.

Key Words: Fiber product, arithmetical rank, amalgamated duplication, trivial ring extension, Noetherian spectrum, radically principal.
2010 MSC: Primary 13A15 ; Secondary 13B02, 13B10.

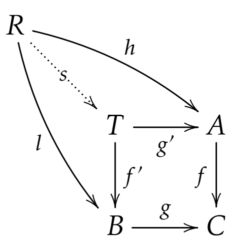
0. Introduction

All rings considered in this article are commutative with identity. Recall from [7] that, an ideal I of R is said radically of finite type if $\sqrt{I} = \sqrt{J}$ for some finitely generated subideal J of I. A geometric interpretation of this notion is the fact that I is radically of finite type if and only if $\operatorname{Spec}(R) \setminus V(I)$ is a quasi-compact open subset of $\operatorname{Spec}(R)$ with respect to the Zariski topology. If I is radically of finite type, the arithmetical rank of I is defined as the minimum number r of elements $a_1, \ldots, a_r \in R$ such that

$$\sqrt{(a_1, \ldots, a_r)} = \sqrt{I}.$$

We denote it by $\operatorname{ara}(I)$. If I is not radically of finite type we put $\operatorname{ara}(I) = \infty$. Since the zero ideal is generated by the empty set, by convention $\operatorname{ara}(I) = 0$ if and only if $\sqrt{I} = \sqrt{(0)}$. Recall that, a commutative ring R has Noetherian spectrum if $\operatorname{Spec}(R)$ is a Noetherian topological space with respect to the Zariski topology. In [7], the author has proven that R has Noetherian spectrum if and only if every ideal of R is radically of finite type if and only if every prime ideal is radically of finite type. A special case of commutative rings with Noetherian spectrum are radically principal rings that is commutative rings such that for every ideal I, $\operatorname{ara}(I) \leq 1$, for more details see [1].

Let A, B, C be commutative rings, and let $f : A \to C$, $g : B \to C$ be ring morphisms. Then there exist a commutative ring T and two morphisms of rings $g' : T \to A$ and $f' : T \to B$ such that $g \circ f' = f \circ g'$ with the following property: given any ring R and any two ring morphisms $h : R \to A$ and $l : R \to B$ such that $f \circ h = g \circ l$, there exists a unique morphism $s : R \to T$ such that $h = g' \circ s$ and $l = f' \circ s$.



T is called the fiber product of A and B over C, denoted $A \times_C B$. It is unique up to isomorphism. A simple description of $A \times_C B$ as a subring of the direct product $A \times B$ is given by $A \times_C B = \{(a,b) \in A \times B \mid f(a) = g(b)\}$. Let $C' = f(A) \cap g(B)$, $A' = f^{-1}(C')$, $B' = g^{-1}(C')$. Then A', B', C' are subrings of A, B, C respectively and there is a canonical isomorphism between $A \times_C B$ and $A' \times_{C'} B'$, for more details see [5]. As a particular case, if $f : A \to B$ is a morphism of rings and J is an ideal of B, the amalgamation of A with B along J with respect to f denoted $A \bowtie^f J$, is the fiber product $A \times_{\frac{B}{J}} B$. It is easy to see that $A \bowtie^f J = \{(a, f(a) + j) \mid a \in A, j \in J\}$.

In this paper, we study the arithmetical rank of prime ideals of the fiber product, in particular, we give a characterization for the fiber product to have Noetherian spectrum.

1 Prime ideals in the fiber product

Let $f : A \to C$, $g : B \to C$, and set $C' = f(A) \cap g(B)$, $A' = f^{-1}(C')$, $B' = g^{-1}(C')$. Then $A \times_C B = A' \times_{C'} B'$. If f and g are surjective then $A' = A$, $B' = B$ and $C' = C$. By T we mean the fiber product of A and B along C that is $T = A \times_C B$. For $P \in \operatorname{Spec}(A')$ denote $\overline{P} = \{(a,b) \in T | a \in P\}$, also for $Q \in \operatorname{Spec}(B')$ denote $\underline{Q} = \{(a,b) \in T \mid b \in Q\}$. Finally, denote $I_f = \{(a,0) \mid a \in \operatorname{Ker} f\}$ and $I_g = \{(0,b) \mid b \in \operatorname{Ker} g\}$.

Remark 1.1. 1. It is clear that I_f and I_g are ideals of T and $I_f I_g = 0$.

2. If $a \in A'$ then there exists $b \in B'$ such that $(a,b) \in T$.

3. If $b \in B'$ then there exists $a \in A'$ such that $(a,b) \in T$.

In the following result we determine all prime ideals of the fiber product.

Proposition 1.2. *With the above notations we have.*

1. *$\overline{P}$ and $\underline{Q}$ are prime ideals of T, where $P \in \operatorname{Spec}(A')$ and $Q \in \operatorname{Spec}(B')$.*

2. *$\operatorname{Spec}(T) = \{\overline{P} \mid P \in \operatorname{Spec}(A')\} \cup \{\underline{Q} \mid Q \in \operatorname{Spec}(B')\}$.*

Proof. 1. $\overline{P}$ is clearly an ideal of T. Let $(a,b), (a',b') \in T$ with $(a,b)(a',b') \in \overline{P}$, then $(aa', bb') \in \overline{P}$, so $aa' \in P$. Since $P \in \operatorname{Spec}(A')$, we have $a \in P$ or $a' \in P$, so that $(a,b) \in \overline{P}$ or $(a',b') \in \overline{P}$. Hence $\overline{P}$ is prime ideal of T. By a same argument $\underline{Q}$ is a prime ideal of T.

2. Let J be a prime ideal of T. Note that $I_f I_g = \{(0,0)\} \subseteq J$. Since J is a prime ideal of T, we have $I_f \subseteq J$ or $I_g \subseteq J$.
First case: If $I_g \subseteq J$. Set $P = \{a \in A' \mid (a,b) \in J \text{ for some } b \in B'\}$. P is clearly an ideal of A'. Let $aa' \in P$ with $a, a' \in A'$, then $(aa', c) \in J$ for some $c \in B'$. We have $(a,b), (a',b') \in T$ for some $b, b' \in B'$. Since $f(a) = g(b)$ and $f(a') = g(b')$ and $f(aa') = g(c)$, we have

$$\begin{aligned} g(bb' - c) &= g(bb') - g(c) = g(b)g(b') - g(c) = f(a)f(a') - g(c) \\ &= f(aa') - g(c) = 0. \end{aligned}$$

So $(0, bb' - c) \in I_g \subseteq J$. Therefore, $(aa', bb') = (aa', c) + (0, bb' - c) \in J$. Thus $(a,b)(a',b') \in J$. Since J is a prime ideal of T, we have $(a,b) \in J$ or $(a',b') \in J$, so that $a \in P$ or $a' \in P$. It follows that P is a prime ideal of A'. Next, we show that $J = \overline{P}$. It is clear that $J \subseteq \overline{P}$ because if $(a,b) \in J$, then $a \in P$, so $(a,b) \in \overline{P}$. Let $(a,b) \in \overline{P}$, that is $a \in P$. Then $(a,b') \in J$ for some $b' \in B'$. Now, $(a,b) = (a,b') + (0, b - b')$ and $(0, b - b') \in I_g$, so that $(a,b) \in J$. It follows that $\overline{P} = J$.
Second case : $I_f \subseteq Q$. Consider the ideal $Q = \{b \in B' \mid (a,b) \in J \text{ for some } a \in A'\}$. As in the previous case and by the same argument, Q is a prime ideal of B' and $\underline{Q} = J$.

□

2 Arithmetical rank in the fiber product

Definition 2.1. Let R be a commutative ring and I be an ideal of R.

1. The ideal I is radically finite it there exists a finitely generated subideal J of I such that $\sqrt{I} = \sqrt{J}$.
2. If I is radically of finite type the arithmetical rank of I is defined as the minimum number r of elements $a_1, \ldots, a_r \in R$ such that $\sqrt{(a_1, \ldots, a_r)} = \sqrt{I}$. We denote it by $\operatorname{ara}(I)$.

Remark 2.2. 1. Every finitely generated ideal is radically finite.

2. Let $R = \mathbb{Z}[X_k, k \in \mathbb{N}]/J$ where $J = (X_i X_k,\ \ i,k \in \mathbb{N})$ is the ideal generated by all monomials $X_i X_k$. Let $I = (2, x_k, k \in \mathbb{N})$ the ideal generated by 2 and all $x_k = \overline{X_k}$. Then I is not finitely generated but radically finite since $\sqrt{I} = \sqrt{(2)}$. Moreover, $\operatorname{ara}(I) = 1$.

Theorem 2.3. *1. If P is a prime ideal of A' then* $\operatorname{ara}(\overline{P}) \leq \operatorname{ara}(P) + \operatorname{ara}(\operatorname{Ker} g)$.

2. If Q is a prime ideal of B' then $\operatorname{ara}(\underline{Q}) \leq \operatorname{ara}(Q) + \operatorname{ara}(\operatorname{Ker} f)$.

Proof. 1. If one of the ideals P or $\operatorname{Ker} g$ is not radically finite then by definition $\operatorname{ara}(P) + \operatorname{ara}(\operatorname{Ker} g) = \infty$ and in this case the inequality holds.
Assume that $\operatorname{ara}(P) = n \in \mathbb{N}$ and $\operatorname{ara}(\operatorname{Ker} g) = m \in \mathbb{N}$. Let $p_1, \ldots, p_n \in P$, $b_1, \ldots, b_m \in \operatorname{Ker} g$ such that $P = \sqrt{(p_1, \ldots, p_n)}$ and $\sqrt{\operatorname{Ker} g} = \sqrt{(b_1, \ldots, b_m)}$. For each $p_k \in P \subseteq A'$, there exists $q_k \in B'$ such that $f(p_k) = g(q_k)$, that is $(p_k, q_k) \in T$. We show that $\overline{P} = \sqrt{((0, b_1), \ldots, (0, b_m), (p_1, q_1), \ldots, (p_n, q_n))}$. First, it is easy to see that

$$\sqrt{((0, b_1), \ldots, (0, b_m), (p_1, q_1), \ldots, (p_n, q_n))} \subseteq \sqrt{\overline{P}} = \overline{P}.$$

Let $z = (a, b) \in \overline{P}$, so that $a \in P$. Then $a^N = \sum_{k=1}^{n} \alpha_k p_k$ for some $N \in \mathbb{N}$ and $\alpha_k \in A'$. For $1 \leq k \leq n$, let $\beta_k \in B'$ such that $f(\alpha_k) = g(\beta_k)$ that is $(\alpha_k, \beta_k) \in T$. Then

$$z^N = (a^N, b^N) = (0, b^N - \sum_{k=1}^{n} \beta_k q_k) + \sum_{k=1}^{n} (\alpha_k, \beta_k)(p_k, q_k).$$

Note that

$$\begin{aligned} g(b^N - \sum_{k=1}^{n} \beta_k q_k) &= g(b^N) - \sum_{k=1}^{n} g(\beta_k) g(q_k) \\ &= f(a^N) - \sum_{k=1}^{n} f(\alpha_k) f(p_k) \\ &= f(a^N - \sum_{k=1}^{n} \alpha_k p_k) = 0. \end{aligned}$$

That is $b^N - \sum_{k=1}^{n} \beta_k q_k \in \operatorname{Ker} g \subseteq \sqrt{\operatorname{Ker} g}$. There exists $l \in \mathbb{N}$ such that $(b^N - \sum_{k=1}^{n} \beta_k q_k)^l = \sum_{i=1}^{m} c_i b_i$ with $c_i \in B'$. For each c_i, fix $\lambda_i \in A'$ such that $(\lambda_i, c_i) \in T$. Then

$$(0, (b^N - \sum_{k=1}^{n} \beta_k q_k))^l = (0, (b^N - \sum_{k=1}^{n} \beta_k q_k)^l) = \sum_{i=1}^{m} (\lambda_i, c_i)(0, b_i).$$

It follows that $(0, b^N - \sum_{k=1}^{n} \beta_k q_k) \in \sqrt{(0,b_1),\ldots,(0,b_m)}$, therefore

$$(0, b^N - \sum_{k=1}^{n} \beta_k q_k) \in \sqrt{(0,b_1),\ldots,(0,b_m),(p_1,q_1),\ldots,(p_n,q_n)}.$$

Since $\sum_{k=1}^{n} (\alpha_k, \beta_k)(p_k, q_k) \in \sqrt{(0,b_1),\ldots,(0,b_m),(p_1,q_1),\ldots,(p_n,q_n)}$, it follows that

$$z^N \in \sqrt{((0,b_1),\ldots,(0,b_m),(p_1,q_1),\ldots,(p_n,q_n))}.$$

Thus

$$z \in \sqrt{((0,b_1),\ldots,(0,b_m),(p_1,q_1),\ldots,(p_n,q_n))}.$$

Consequently, $\overline{P} = \sqrt{((0,b_1),\ldots,(0,b_m),(p_1,q_1),\ldots,(p_n,q_n))}$.

2. By the same argument.

□

Remark 2.4. If Ker$g$ is nilpotent, then ara$(\overline{P}) \leq$ ara(P). Similarly, ara$(\underline{Q}) \leq$ ara(Q) if Ker$f$ is nilpotent.

As a consequence, we have the following corollary which is a characterization for the fiber product to have Noetherian spectrum.

Corollary 2.5. *$A \times_C B$ has Noetherian spectrum if and only if so are A' and B'.*

Proof. Assume that T has Noetherian spectrum. Consider the rings morphism $\psi : T \to A'$ defined by $(a,b) \mapsto a$, we have Ker$\psi = I_g$. It is clear, from the definition, that ψ is a surjective map .Therefore, $\frac{T}{\text{Ker}\psi} = \frac{T}{I_g} \cong A'$. Since T has Noetherian spectrum, so is A'. Considering the rings morphism $T \to B'$ defined by $(a,b) \mapsto b$, its kernel is the ideal I_f. It follows that B' has Noetherian spectrum by the fact that $\frac{T}{I_f} \cong B'$.
Conversely, assume that A' and B' are rings with Noetherian spectrum. The ring T has Noetherian spectrum if every prime ideal of T is radically finite. Let J be a prime ideal of T, then $J = \overline{P}$ for some $P \in$ Spec(A') or $J = \underline{Q}$ for some $Q \in$ Spec(B'). Since A' and B' have Noetherian spectrum. In the first case by the previous theorem we have ara$(J) \leq$ ara(P) + ara(Ker$g$), so ara(J) is finite since ara(P) and ara(Ker$g$) are finite, and in the second case, we have ara$(J) \leq$ ara(Q) + ara(Ker$f$) is finite. Therefore T is a ring with Noetherian spectrum. □

Recall from [1], that a commutative ring R is radically principal if for every ideal I of R, $\sqrt{I} = \sqrt{J}$ for some principal ideal J of R if and only if for every prime ideal P of R, $P = \sqrt{J}$ for some principal subideal J of P.

Corollary 2.6. *1. If T is a radically principal ring then A' and B' are radically principal rings.*

2. If A' and B' are radically principal rings, then for every $J \in$ Spec(T),

$$\text{ara}(J) \leq 1 + \max(\text{ara}(\text{Ker}f), \text{ara}(\text{Ker}g)) \leq 2.$$

Proof. 1. This follows from [1, Proposition 2.6] and the fact that $\frac{T}{I_g} \cong A'$ and $\frac{T}{I_f} \cong B'$.

2. If A' and B' are radically principal rings then $\text{ara}(P) \leq 1$ and $\text{ara}(Q) \leq 1$, where $P \in \text{Spec}(A')$ and $Q \in \text{Spec}(B')$. Thus, for every $J \in \text{Spec}(T)$, we have $\text{ara}(J) \leq 1 + \text{ara}(\text{Ker} f)$ or $\text{ara}(J) \leq 1 + \text{ara}(\text{Ker} g)$. Therefore, $\text{ara}(J) \leq 1 + \max(\text{ara}(\text{Ker} f), \text{ara}(\text{Ker} g)) \leq 2$.

□

Corollary 2.7. *If A' and B' are radically principal rings and $\text{Ker} f$ and $\text{Ker} g$ are two nilpotent ideals then T is a radically principal ring.*

Proof. If $\text{Ker} f$ and $\text{Ker} g$ are two nilpotent ideals.Then $\sqrt{\text{Ker} f} = \sqrt{0}$ and $\sqrt{\text{Ker} g} = \sqrt{0}$, that is $\text{ara}(\text{Ker} f) = 0$ and $\text{ara}(\text{Ker} g) = 0$. By the previous corollary it follows that for every $J \in \text{Spec}(T)$ we have $\text{ara}(J) \leq 1$. Therefore T is a radically principal ring. □

Theorem 2.8. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then*

1. *If $A \bowtie^f J$ is a radically principal ring then A and $f(A) + J$ are radically principal rings.*

2. *Let J and $f^{-1}(J)$ be two nilpotent ideals. If A and $f(A) + J$ are radically principal rings then $A \bowtie^f J$ is a radically principal rings.*

Proof. In fact, $A \bowtie^f J$ is the fiber product $A \times_{\frac{B}{J}} B$, where $\pi : B \to \frac{B}{J}$ is the canonical projection and $\widehat{f} = \pi \circ f$:

$$\begin{array}{ccc} A \bowtie^f J := A \times_{\frac{B}{J}} B & \xrightarrow{p_A} & A \\ \downarrow{\scriptstyle p_B} & & \downarrow{\scriptstyle \widehat{f}} \\ B & \xrightarrow{\pi} & \frac{B}{J} \end{array}$$

we have $C' = \widehat{f}(A) \cap \pi(B) = \dfrac{f(A)+J}{J} \cap \dfrac{B}{J} = \dfrac{f(A)+J}{J}$ and $A' = \widehat{f}^{-1}(C') = \widehat{f}^{-1}(\dfrac{f(A)+J}{J}) = f^{-1}(f(A)+J)$
Clearly $\text{Ker}\pi = J$ and $\text{Ker}\widehat{f} = f^{-1}(J)$. We show that $A = A'$ and $B' = f(A) + J$.
By definition A' is a subring of A.
Now, let $x \in A$, then $f(x) \in f(A) + J$, so $x \in A'$. It follows that $A = A'$.
$B' = \pi^{-1}(C') = \pi^{-1}(\dfrac{f(A)+J}{J}) = f(A) + J$.

1. If $A \bowtie^f J = T$ is a radically principal ring. By Corollary 2.6 we have $A = A'$ and $f(A) + J = B'$ are radically principal rings.

2. If $A = A'$ and $f(A) + J = B'$ are radically principal rings, since $J = \text{Ker}\pi$ and $f^{-1}(J) = \text{Ker}\widehat{f}$ are two nilpotent ideals, then by Corollary 2.7, $A \bowtie^f J = T$ is a radically principal ring.

□

Theorem 2.9. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then $A \bowtie^f J$ has Noetherian spectrum if and only if A and $f(A) + J$ have Noetherian spectrum .*

Proof. The theorem follows directly from Corollary 2.5 where $A = A'$ and $f(A) + J = B'$ and $A \bowtie^f J = T$. □

3 Arithmetical rank in trivial ring extension

Let R be a commutative ring and M be an R module. The trivial ring extension of R by M is the ring $R \propto M$ whose underline group is $R \times M$ and multiplication given by $(a,m)(b,n) = (ab, an + bm)$.

Lemma 3.1. *Let R be a commutative ring and M be an R-module. Let $(a_1,m_1),\dots,(a_r,m_r) \in R \propto M$. Then*

1. $\sqrt{((a_1,m_1),\dots,(a_r,m_r))} = \sqrt{((a_1,0),\dots,(a_r,0))}$.
2. $\sqrt{((a_1,m_1),\dots,(a_r,m_r))} = \sqrt{(a_1,\dots,a_r)} \propto M$.

Proof. 1. For $1 \le k \le r$, we have $(a_k,m_k) = (a_k,0) + (0,m_k)$. Since $(0,m_k)^2 = 0$, $(0,m_k) \in \sqrt{((a_1,0),\dots,(a_r,0))}$, therefore $(a_k,m_k) \in \sqrt{((a_1,0),\dots,(a_r,0))}$. Thus

$$\sqrt{((a_1,m_1),\dots,(a_r,m_r))} \subseteq \sqrt{((a_1,0),\dots,(a_r,0))}.$$

Conversely, for $1 \le k \le r$, $(a_k,0) = (a_k,m_k) - (0,m_k)$, since $(0,m_k)^2 = 0$, we have $(0,m_k) \in \sqrt{((a_1,m_1),\dots,(a_r,m_r))}$. Therefore $(a_k,0) = (a_k,m_k) - (0,m_k) \in \sqrt{((a_1,m_1),\dots,(a_r,m_r))}$. It follows that

$$\sqrt{((a_1,m_1),\dots,(a_r,m_r))} = \sqrt{((a_1,0),\dots,(a_r,0))}.$$

2. Let $(a,m) \in \sqrt{((a_1,m_1),\dots,(a_r,m_r))} = \sqrt{((a_1,0),\dots,(a_r,0))}$. Then $(a,m)^N = \sum_{k=1}^{r}(\alpha_k,m'_k)(a_k,0) = \sum_{k=1}^{r}(\alpha_k a_k, a_k m'_k)$ for some $N \in \mathbb{N}$ and $(\alpha_k,m'_k) \in R \propto M$. It follows that $a^N = \sum_{k=1}^{r}\alpha_k a_k$, so that $a \in \sqrt{(a_1,\dots,a_r)}$. Thus $(a,m) \in \sqrt{(a_1,\dots,a_r)} \propto M$. Conversely, let $(a,m) \in \sqrt{(a_1,\dots,a_r)} \propto M$. We see that $a \in \sqrt{(a_1,\dots,a_r)}$, so $a^N = \sum_{k=1}^{r}\alpha_k a_k$ for some $N \in \mathbb{N}$ and $\alpha_k \in R$. Now, write

$$(a,m)^N = (a^N, Na^{N-1}m) = \sum_{k=1}^{r}(\alpha_k,0)(a_k,0) + (0, Na^{N-1}m).$$

Since $(0,Na^{N-1}m)^2 = 0$, it follows that $(0,Na^{N-1}m) \in \sqrt{((a_1,0),\dots,(a_r,0))}$. Thus $(a,m)^N \in \sqrt{((a_1,0),\dots,(a_r,0))}$. Therefore $(a,m) \in \sqrt{((a_1,0),\dots,(a_r,0))}$.

□

Theorem 3.2. *Let R be a commutative ring and M be an R-module. Let J be an ideal of $R \propto M$. Then J is radically finite if and only if so is J_0, where $J_0 = \{a \in R \mid (a,m) \in J$ for some $m \in M\}$. In this case* $\text{ara}(J) = \text{ara}(J_0)$.

Proof. Assume that J is radically finite and set $r = \text{ara}(J)$. From the definition, we have

$$\sqrt{J} = \sqrt{(a_1,m_1),\dots,(a_r,m_r)}$$

for some $(a_k,m_k) \in J$. By the previous Lemma, we have $\sqrt{J} = \sqrt{(a_1,0),\dots,(a_r,0)} = \sqrt{(a_1,\dots,a_r)} \propto M$. By definition, each $a_k \in J_0$, so $\sqrt{(a_1,\dots,a_r)} \subseteq \sqrt{J_0}$. Let $a \in J_0$, then $(a,m) \in J$ for some $m \in M$. Since $(a,m) \in \sqrt{J}$, $(a,m)^N = \sum_{k=1}^{r}(\alpha_k,m'_k)(a_k,0)$ for some $N \in \mathbb{N}$ and $(\alpha_k,m'_k) \in R \propto M$, it follows that $a^N = \sum_{k=1}^{r}\alpha_k a_k$, thus $a \in \sqrt{(a_1, \ldots, a_r)}$. As a consequence $\sqrt{J_0} = \sqrt{(a_1, \ldots, a_r)}$ and J_0 is radically finite. In particular $\operatorname{ara}(J_0) \le r = \operatorname{ara}(J)$.

Assume that J_0 is radically finite and set $s = \operatorname{ara}(J_0)$. From the definition $\sqrt{J_0} = \sqrt{(a_1, \ldots, a_s)}$ for some $a_k \in R$. For each $a_k \in J_0$, there exists $m_k \in M$ such that $(a_k, m_k) \in J$. By the previous Lemma, we have $\sqrt{(a_1, m_1), \ldots, (a_s, m_s)} = \sqrt{J_0} \propto M$. Clearly, $\sqrt{(a_1, m_1), \ldots, (a_s, m_s)} \subseteq \sqrt{J}$. If $(a, m) \in \sqrt{J}$ then it is easy to see that $a^N \in J_0$ for some $N \in \mathbb{N}$, so that $(a, m) \in \sqrt{J_0} \propto M$, therefore $\sqrt{J} \subseteq \sqrt{J_0} \propto M$. Thus $\sqrt{J} = \sqrt{J_0} \propto M = \sqrt{(a_1, m_1), \ldots, (a_s, m_s)}$. It follows that J is radically finite and $\operatorname{ara}(J) \le s = \operatorname{ara}(J_0)$. □

Corollary 3.3. *Let R be a commutative ring and M be an R−module .*

1. *$R \propto M$ has Noetherian spectrum if and only if R has Noetherian spectrum.*
2. *$R \propto M$ is a radically principal ring if and only if R is a radically principal ring.*

Proof.
1. If $R \propto M$ has Noetherian spectrum and I is an ideal of R, then the ideal $J = I \propto M$ is radically finite, so I is radically finite since $J_0 = I$. If R has Noetherian spectrum and J is an ideal of $R \propto M$ then J is radically finite since J_0 is radically finite.
2. If $R \propto M$ is radically principal then so is $R = \frac{R \propto M}{0 \propto M}$. If R is radically principal and J is any ideal of $R \propto M$ then $\operatorname{ara}(J) = \operatorname{ara}(J_0) \le 1$, therefore J is radically principal.

□

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References

[1] M. Aqalmoun and M. El Ouarrachi, *Radically principal rings*, Khayyam J. Math., 6 (2) (2020), 243-249.

[2] M. F. Atiyah and I.G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley Series in Math., Westview Press, (1969).

[3] M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. 6(3) (2007), 443-459.

[4] A. El Khalfi, H. Kim and N. Mahdou, *Amalgamation extension in commutative ring theory: a survey*, Moroccan J. Algebra, Geom. Appl., 1(1) (2022), 139-182.

[5] A. Facchini, *Fiber products and Morita duality for commutative rings*, Rend. Sem. Mat. Univ. Padova 67 (1982), 143-159.

[6] J. A. Huckaba, *Commutative rings with zero divisors*, Marcel Dekker, New York, Basel (1988).

[7] J. Ohm and R. Pendleton, *Ring with Noetherian spectrum*, Duke Math. J. 35 (1968) 631-639.