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Graded (m,n)-closed and graded weakly (m,n)-closed ideals

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Abstract. Let G be a group with identity e and R a G-graded commutative ring with $1 \neq 0$. In this paper we introduce the concept of the graded (m,n)-closed ideals and graded weakly (m,n)-closed ideals. A graded proper ideal I of R is called a graded (m,n)-closed (resp. graded weakly (m,n)-closed) ideal if whenever $a^m \in I$ (resp. $0 \neq a^m \in I$) for $a \in h(R)$, then $a^n \in I$. Many results are given, in particular we investigate the graded (weakly) (m,n)-closed ideals in the direct product $R_1 \times R_2$ of G-graded rings R_1, R_2 and in the trivial extension $R(+)M$ of a G-graded ring R by a graded R-module M.

Key Words: G-graded rings; Trivial extensions, (m,n)-closed ideals; Graded (m,n)-closed ideals; weakly (m,n)-closed ideals; Graded weakly (m,n)-closed ideals.
2010 MSC: Primary 13A02, Secondary 13A15, 47B47.

Dedicated to the memory of Muhammad Zafrullah

1 Introduction

We recall some basic properties of graded rings and modules used in the sequel. Let G be a multiplicative group with identity e. A ring R is called to be G-graded ring (or graded ring) if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. If the inclusion is equality, then the ring R is called strongly graded. The elements of R_g are called homogeneous of degree g and R_e is a subring of R and $1 \in R_e$. For $x \in R$, x can be written uniquely as $x = \sum_{g\in G} x_g$ where x_g is the component of x in R_g. Also, we write $h(R) = \cup_{g\in G} R_g$. If $r \in R_g$ is unit, then $r^{-1} \in R_{g^{-1}}$. A G-graded ring $R = \bigoplus_{g\in G} R_g$ is called a crossed product if R_g contains a unit for every $g \in G$. Note that a G-crossed product $R = \bigoplus_{g\in G} R_g$ is a strongly graded ring. Let R be a G-graded ring and I an ideal of R. Then I is called G-graded ideal if $I = \bigoplus_{g\in G}(I \cap R_g) = \bigoplus_{g\in G} I_g$, that is, if $x \in I$ and $x = \sum_{g\in G} x_g$, then $x_g \in I$ for all $g \in G$. If $R = \bigoplus_{g\in G} R_g$ and $R' = \bigoplus_{g\in G} R'_g$ are two G-graded rings, then a ring homomorphism $f : R \to R'$ with $f(1_R) = 1_{R'}$ is called a gr-homomorphism if $f(R_g) \subseteq R'_g$ for all $g \in G$. Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring and I a G-graded ideal of R. Then the quotient ring R/I is also G-graded ring. Indeed, $R/I = \bigoplus_{g\in G}(R/I)_g$ where $(R/I)_g = \{x + I \ : \ x \in R_g\}$.

Let R be a G-graded ring and $S \subseteq h(R)$ a multiplicatively closed subset of R. The the ring of fractions R_S is a G-graded ring which is called the gr-ring of fractions. Indeed, $R_S = \bigoplus_{g\in G}(R_S)_g$ where

$$(R_S)_g = \{\frac{r}{s} \ : \ r \in R, \ s \in S \quad \text{and} \quad g = (degs)^{-1}(degr)\}.$$

Consider the ring gr-homomorphism $f: R \to R_S$ defined by $f(r) = \frac{r}{1}$. For any graded ideal I of R, the ideal of R_S generated by $f(I)$ is denoted by I_S. Similar to non-graded case one can prove that

$$I_S = \{\frac{r}{s} \in R_S \ : \ r \in I, \ s \in S\}.$$

A proper graded ideal I of R is said to be graded prime if whenever $a, b \in h(R)$ such that $ab \in I$, then either $a \in I$ or $b \in I$.

A graded R-module is an R-module M such that $M = \bigoplus_{g\in G} M_g$ where M_g is an additive subgroup of M and for every $g, h \in G$ we have $R_g M_h \subseteq M_{gh}$. Since $R_e M_h \subseteq M_h$ we see that M_h is an R_e-submodule of M. The elements of $h(M) = \cup_{g\in G} M_g$ are called the homogeneous elements of M. A nonzero element $m \in M_g$ is said to be a homogeneous element of degree g. Every $m \in M$ can be uniquely represented as a sum $m = \sum_{g\in G} m_g$ with a finitely many nonzero $m_g \in M_g$. The nonzero elements m_g in this sum are called the homogeneous components of m. An R-submodule N of M is said to be a graded submodule if for every $n \in N$ all its homogeneous components are also in N, that is, $N = \bigoplus_{g\in G}(N \cap M_g)$. If $I = \bigoplus_{g\in G} I_g$ is a graded ideal of R, then I_g is an R_e-module for every $g \in G$. Let R be a G-graded ring. The *graded radical* of a graded ideal I, denoted by $\mathrm{Gr}(I)$ is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with ${x_g}^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in \mathrm{Gr}(I)$ if and only if $r^n \in I$ for some positive integer n. In particular we denote $N(R) = \mathrm{Gr}\{0\}$.

In this article, we define and study graded (weakly) (m,n)-closed ideals of a graded ring for positive integers m and n.

A proper graded ideal I of a graded ring R is said to be a graded (m,n)-closed ideal (resp. graded weakly (m,n)-closed) ideal of R if whenever $a \in R$ with $a^m \in I$ (resp. $0 \neq a^m \in I$), then $a^n \in I$. Besides other useful results we also investigate graded (weakly) (m,n)-closed ideals in the direct product $(R_1 \times R_2)$ of graded rings R_1, R_2 (Theorems 3.12, 4.10, 4.12) and in the trivial extension $(R(+)M)$ of a graded ring R by a graded R-module M (Theorems 3.14, 3.16, 4.13).

We assume throughout this article that all rings are commutative with $1 \neq 0$, all R-modules are unitary. For such a ring R, let $\mathrm{Nil}(R)$ be its ideal of nilpotent, $U(R)$ its set of units and $h(R)$ its set of homogeneous elements. Note that every proper graded ideal is graded (weakly) (m,n)-closed for $m \leq n$, so throughout we also assume that $m > n$.

2 Generalized purity of modules

3 Graded (m,n)-closed ideals

In this section, we present few properties of graded (m,n)-closed ideals and investigate graded (m,n)-closed ideals in direct product $(R_1 \times R_2)$ of G-graded rings R_1, R_2 and in the trivial extension $(R(+)M)$ of G-graded ring R by graded R-module M. For the sake of completeness, we begin with the definitions of (m,n)-closed and graded (m,n)-closed ideals.

Definition 3.1. [4] A proper ideal I of a ring R is said to be (m,n)-closed if whenever $a^m \in I$ for $a \in R$, then $a^n \in I$.

Definition 3.2. A proper graded ideal I of a G-graded ring R is said to be graded (m,n)-closed if whenever $a^m \in I$ for $a \in h(R)$, then $a^n \in I$.

Example 3.3. Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is a G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Let $I = 2R$. Then I is not $(2,1)$-closed ideal of R because $(1+i)^2 = 2i \in I$ and $(1+i) \notin I$. Similarly, $J = 4R$ is not $(4,3)$-closed ideal since $(1+i)^4 = -4 \in J$, but $(1+i)^3 = 2i - 2 \notin J$. However it is easy to check that I and J are graded $(2,1)$-closed and graded $(4,3)$-closed ideals of R, respectively.

Proposition 3.4. *Let R be a G-graded ring. If I is a graded (m,n)-closed ideal of R, then I_e is an (m,n)-closed ideal of R_e.*

Proof. Let $a \in R_e$ with $a^m \in I_e$. We know that $R_e \subset h(R)$ and $I_e \subset I$. Therefore $a \in h(R)$ and $a^m \in I$. Since I is a graded (m,n)-closed ideal of R, we conclude that $a^n \in I$. Thus $a^n \in I \cap R_e = I_e$. Hence I_e is an (m,n)-closed ideal of R_e, as we desired.

□

Let R be a graded ring and I a graded ideal of R. Then the following lemma says that for every $a \in h(R)$ the quotient $(I : a) := \{x \in R : ax \in I\}$ is a graded ideal of R.

Lemma 3.5. *Let R be a G-graded ring, I a graded ideal of R and $a \in h(R)$. Then $(I : a)$ is graded ideal of R.*

Proof. Let $r \in (I : a)$ for some $r \in R$, then $ra \in I$. Since R is graded ring,

$$r = \sum_{g\in G} r_g \quad \text{where} \quad r_g \in R_g$$

and therefore

$$\sum_{g\in G} r_g a = \left(\sum_{g\in G} r_g\right) a = ra \in I$$

.

As I is a graded ideal, $r_g a \in I$ for all $g \in G$. Hence $r_g \in (I : a)$ for all $g \in G$. Thus $(I : a)$ is a graded ideal of R, as we desired. □

Recall that an element a in a ring R is called idempotent if $a^2 = a$. The following is one of the main results of this section.

Theorem 3.6. *Let R be a G-graded ring, I a graded (m,n)-closed ideal of R and $a \in h(R)$. If a is idempotent and $a \notin I$, then $(I : a)$ is a graded (m,n)-closed ideal of R.*

Proof. From the lemma 3.5, $(I : a)$ is a graded ideal of R. Now, suppose that $r \in h(R)$ such that $r^m \in (I : a)$. Since a is idempotent, we have $(ra)^m = r^m a^m = r^m a \in I$. The fact that $a, r \in h(R)$ implies that there exists g and h in G such that $r \in R_g$ and $a \in R_h$. Therefore, $ra \in R_g R_h \subset R_{gh} \subset h(R)$ and $(ra)^m \in I$. Since I is a graded (m,n)-closed ideal of R, we conclude that $r^n a^n = r^n a \in I$. Thus $r^n \in (I : a)$ and hence $(I : a)$ is a graded (m,n)-closed ideal of R, as we desired. □

The next theorem is a graded analog of (m,n)-closed ideals ([4, Theorem 2.6]).

Theorem 3.7. *Let R be a G-graded ring, I a graded $(m,2)$-closed ideal of R and J a graded ideal of R.*

(1) If $J^m \subseteq I$, then for every $g \in G$, $2{J_g}^2 \subseteq I$.

(2) Suppose that $2 \in U(R)$. If $J^m \subseteq I$, then for every $g \in G$, ${J_g}^2 \subseteq I$.

Proof. (1) Let $g \in G$ and $x_g, y_g \in J_g$. Then $x_g^m, y_g^m, (x_g + y_g)^m \in I$. Since I is a graded $(m,2)$-closed ideal of R, it follows that $x_g^2, y_g^2, (x_g + y_g)^2 \in I$. Hence $2x_g y_g \in I$ and thus $2J_g^2 \subset I$, as we desired.

(2) follows directly from (1). □

The next theorem is the graded (m,n)-closed analog for well-known localization results about prime, radical, n-absorbing ([2, Theorem 4.1]) and (m,n)-closed ideals ([4, Theorem 2.8]).

Theorem 3.8. *Let R be a G-graded ring, I a graded (m,n)-closed ideal of R and $S \subseteq h(R)$ a multiplicatively closed subset of R such that $S \cap I = \emptyset$. Then,*

(1) I_S is a graded (m,n)-closed ideal of R_S.

(2) If $n = 2$, $2 \in S$ and $J^m \subseteq I_S$ for a graded ideal J of R_S, then for every $g \in G$, ${J_g}^2 \subseteq I_S$.

Proof. (1) Let $(r/s)^m \in I_S$ for some $r/s \in h(R_S)$. Then $r^m/s^m = b/t$ for some $b \in I \cap h(R)$ and $t \in S$. Hence there exists $s' \in S$ such that $s'tr^m = s'bs^m \in I$, and thus $(s'tr)^m \in I$. Since I is a graded (m,n) closed ideal and $s'tr \in R_{deg(s')deg(t)deg(r)} \subseteq h(R)$, we conclude that $(s'tr)^n \in I$ and thus $(r/s)^n = s'^n t^n r^n / s'^n t^n s^n \in I_S$. Hence I_S is a graded (m,n)-closed ideal of R_S.

(2) Suppose that $J^m \subseteq I_S$ for some graded ideal J of R_S. Since $2 \in S$, then $2 \in U(R_S)$ and thus, by Theorem 3.7(2), for every $g \in G$, ${J_g}^2 \subseteq I_S$.

□

Corollary 3.9. *Let R be a G-graded ring and I a proper graded ideal of R. Then I is a graded (m,n)-closed ideal of R if and only if $I_{S(P)}$ is a graded (m,n)-closed ideal of $R_{S(P)}$ where $S(P) = h(R) \cap R \setminus P$ for every prime (or maximal) ideal of R containing I.*

Proof. ($\Rightarrow$) This follows from Theorem 3.8(1). ($\Leftarrow$) Let P be a prime ideal of R with $I \subseteq P$ and denote $h(R) \cap R \setminus P$, a multiplicatively closed subset of R, by S. Let us suppose that $x^m \in I$ for some $x \in h(R)$ and consider $J = (I : x^n) = \{r \in R : rx^n \in I\}$. Then $(\frac{x}{1})^m \in I_S$, therefore $(\frac{x}{1})^n \in I_S$, since I_S is graded (m,n)-closed ideal of R_S. Thus $sx^n \in I$ for some $s \in S$ and henceforth $J \nsubseteq P$. Also, note that $J \nsubseteq Q$ for every prime ideal Q of R with $I \nsubseteq Q$. Hence $J = R$ and consequently $x^n \in I$. Thus I is a graded (m,n)-closed ideal, as we desired. □

The next theorem is a graded analog for [4, Theorem 2.10].

Theorem 3.10. *Let R and T be two G-graded rings and $f : R \to T$ a homogeneous homomorphism.*

(1) If J is a graded (m,n)-closed ideal of T, then $f^{-1}(J)$ is a graded (m,n)-closed ideal of R.

(2) If $f(R_g) = T_g$ for all $g \in G$, and I is a graded (m,n)-closed ideal of R containing Ker$f$*, then $f(I)$ is a graded (m,n)-closed ideal of T.*

Proof. (1) Firstly, note that $f^{-1}(J)$ is a graded ideal of R. Indeed, we know that $f^{-1}(J)$ is an ideal of R. Now, let $x = \sum_{g \in G} x_g \in f^{-1}(J)$ where $x_g \in R_g$ for all $g \in G$. Then $f(x) = \sum_{g \in G} f(x_g) \in J$ where $f(x_g) \in T_g$ because f is a homogeneous homomorphism. Since J is a graded ideal of T, therefore $f(x_g) \in J$ and hence $x_g \in f^{-1}(J)$, as asserted. Now, let us suppose that $x^m \in f^{-1}(J)$ for some $x \in h(R)$, then $f(x) \in h(T)$, as f is homogeneous, and $(f(x))^m \in J$. Since J is a graded (m,n)-closed ideal of T, therefore $(f(x))^n \in J$. Thus $x^n \in f^{-1}(J)$ and hence $f^{-1}(J)$ is a graded (m,n)-closed ideal of R, as we desired.

(2) It is clear that f is surjective and $f(I)$ is a graded ideal of T. Now, let $y^m \in f(I)$ for some $y \in h(T)$, then there exists $g \in G$ with $y \in T_g = f(R_g)$. So, there exists $x \in R_g$ such that $y^m = (f(x))^m \in f(I)$. As Ker$(f) \subseteq I$ we have $x^m \in I$. Since I is graded (m,n)-closed ideal, therefore $x^n \in I$. Thus $y^n \in f(I)$ and hence $f(I)$ is a graded (m,n)-closed ideal of T, as we desired.

□

Corollary 3.11. *The following assertions are equivalent.*

(1) Let $R \subseteq T$ be an extension of G-graded rings. If J is a graded (m,n)-closed ideal of T, then $R \cap J$ is a graded (m,n)-closed ideal of R.

(2) Let $I \subseteq J$ be proper graded ideals of R. Then J/I is a graded (m,n)-closed ideal of R/I if and only if J is a graded (m,n)-closed ideal of R.

If R_1 and R_2 are two G-graded rings, then $R_1 \times R_2$ is a G-graded ring by $(R_1 \times R_2)_g = (R_1)_g \times (R_2)_g$. Recall that an ideal of $R_1 \times R_2$ has the form $I_1 \times I_2$ for ideals I_1 of R_1 and I_2 of R_2. The following theorem determines when an ideal of $R_1 \times R_2$ is graded (m,n)-closed.

Theorem 3.12. *Let $R = R_1 \times R_2$, where R_1 and R_2 are G-graded rings, and J a proper graded ideal of R. Then the following statements are equivalent.*

(1) J is a graded (m,n)-closed ideal of R.

(2) $J = I_1 \times R_2$, $R_1 \times I_2$ or $I_1 \times I_2$ for graded (m,n)-closed ideals I_1 of R_1 and I_2 of R_2.

Proof. $(1) \Rightarrow (2)$. Assume that $J = I_1 \times I_2$ is a graded (m,n)-closed ideal of R. Due to symmetry, it suffices to prove that if $I_1 \neq R_1$, then I_1 is graded (m,n)-closed of R_1. For this purpose, assume that $I_1 \neq R_1$, then for every $a \in h(R_1)$ with $a^m \in I_1$ we have $(a,0)^m \in J$. Since $(a,0) \in h(R)$ and J is a graded (m,n)-closed ideal of R, we have $(a,0)^n \in J$. Thus $a^n \in I_1$ and hence I_1 is a graded (m,n)-closed ideal of R_1, as we desired.

$(2) \Rightarrow (1)$. Assume that $J = I_1 \times I_2$ for graded (m,n)-closed ideals I_1 of R_1 and I_2 of R_2. Let $(a,b) \in h(R)$ such that $(a,b)^m = (a^m, b^m) \in J$. Then $a^m \in I_1$ and $b^m \in I_2$. Since $a \in h(R_1)$ and $b \in h(R_2)$, therefore $a^n \in I_1$ and $b^n \in I_2$. Thus $(a,b)^n \in J$ and hence J is a graded (m,n)-closed ideal of R. The proofs of other two cases are similar.

□

Remark 3.13. *The above Theorem 3.12 is also a consequence of the Theorem 3.10. Indeed, $(1) \Rightarrow (2)$ is follows by Theorem 3.10(1) and $(2) \Rightarrow (1)$ follows by Theorem 3.10(2).*

Let R be a ring and M be an R-module. Then the ring $R(+)M$ with coordinate-wise addition and multiplication given by $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$ is a ring with unity $(1,0)$ (even R-algebra) called idealization of M or the trivial extension of R by M. Note that R naturally embeds into $R(+)M$ by $r \mapsto (r,0)$. If N is a submodule of M, then $0(+)N$ is an ideal of $R(+)M$ and $0(+)M$ is a nilpotent ideal of $R(+)M$ of index 2. It is well known that $I(+)N$ is an ideal of $R(+)M$ if and only if I is an ideal of R and N is a submodule of M such that $IM \subseteq N$, cf. [1, Theorem, 3.1].

Let G be an Abelian group. Suppose that $R = \bigoplus_{g \in G} R_g$ be a G-graded ring and $M = \bigoplus_{g \in G} M_g$ a G-graded R-module. Then $R(+)M$ is a G-graded ring with $(R(+)M)_g = R_g \bigoplus M_g$ for every $g \in G$ cf. [7, Proposition 3.1] and [5, Proposition 2]. Consequently, $h(R(+)M) = \{(a,x); a \in h(R), x \in h(M)\}$.

Theorem 3.14. *Let R be a G-graded ring, I a proper graded ideal of R, M a G- graded R-module and N a graded submodule of M such that $IM \subseteq N$.*

(1) If I is a graded (m,n)-closed ideal of R, then $J := I(+)N$ is a graded $(m,n+1)$-closed ideal of $R(+)M$.

(2) I is a graded (m,n)-closed ideal of R if and only if $I(+)M$ is a graded (m,n)-closed ideal of $R(+)M$.

Proof. (1) Since I is a graded ideal, by [7, Proposition 3.1] and [5, Proposition 2], J is a graded ideal of $R(+)M$. Now suppose that I is a graded (m,n)-closed ideal of R. Let $x = (a,c) \in h(R(+)M)$ such that $x^m = (a^m, ma^{m-1}c) \in J$. Since I is a graded (m,n)-closed ideal of R, $a \in h(R)$ and $IM \subseteq N$, we conclude that $(a^{n+1}, (n+1)a^n c) = x^{n+1} \in J$. Thus J is a graded $(m,n+1)$-closed ideal of $R(+)M$.

(2) Since I is a graded ideal, by [7, Proposition 3.1] and [5, Proposition 2], $I(+)M$ is a graded ideal of $R(+)M$. Now assume that I is a graded (m,n)-closed ideal of R. Let $(a,x)^m = (a^m, ma^{m-1}x) \in I(+)M$ for some $(a,x) \in h(R(+)M)$. Then $a \in h(R)$ and $a^m \in I$. Since I is a graded (m,n)-closed ideal of R, we have $a^n \in I$. Hence $(a,x)^n = (a^n, na^{n-1}x) \in I(+)M$. Thus $I(+)M$ is a graded (m,n)-closed ideal of $R(+)M$. Conversely assume that $I(+)M$ is a graded (m,n)-closed ideal of $R(+)M$. Let $a^m \in I$ for some $a \in h(R)$, then $(a,0) \in h(R(+)M)$ and $(a,0)^m \in I(+)M$. Since $I(+)M$ is a graded (m,n)-closed ideal of $R(+)M$, therefore $(a^n,0) = (a,0)^n \in I(+)M$. Thus $a^n \in I$ and hence I is a graded (m,n) closed ideal of R. □

Lemma 3.15. *Let R be a G-graded ring and M a G-graded R-module. Suppose that I is a graded (m,n)-closed ideal of R and N a graded submodule of M such that $IM \subseteq N$. Let $x = (a,c) \in h(R(+)M)$ for some $a \in h(R)$ and $c \in h(M)$. Then $x^m \in I(+)N$ if and only if $a^m \in I$.*

Proof. From the proof above of the Theorem 3.14(1), $I(+)N$ is a graded ideal of $R(+)M$. Suppose that $x = (a,c) \in h(R(+)M)$ with $x^m \in I(+)N$, then clearly $a^m \in I$. Conversely assume that $a^m \in I$. Since I is a graded (m,n)-closed ideal of R and $a \in h(R)$, therefore $a^n \in I$ and hence $a^{m-1} \in I$ (as $n < m$). Then $ma^{m-1}c \in IM \subseteq N$. Consequently, $x^m \in I(+)N$, as desired. □

The following theorem characterizes the graded (m,n)-closed ideals of trivial ring extension $R(+)M$.

Theorem 3.16. *Let R be a G-graded ring and M a G-graded R-module. Suppose that I is a graded ideal of R and N a graded submodule of M such that $IM \subseteq N$. Then the following assertions are equivalent.*

(1) $I(+)N$ is a graded (m,n)-closed ideal of $R(+)M$.

(2) I is a graded (m,n)-closed ideal of R and whenever $a^m \in I$ for some $a \in h(R)$ implies $na^{n-1}M_g \subseteq N$ for some $g \in G$.

Proof. $(1) \Rightarrow (2)$. Suppose that $I(+)N$ is a graded (m,n)-closed ideal of $R(+)M$. Then it is clear that I is a graded (m,n)-closed ideal of R. Assume that $a^m \in I$ for some $a \in R_g$ and some $g \in G$. Let $x = (a,c)$ for some $c \in M_g$. It is clear that $x \in h(R(+)M)$. As I is graded (m,n)-closed ideal of R and $a^m \in I$, therefore by Lemma 3.15, we have $x^m = (a^m, ma^{m-1}c) \in I(+)N$. Also, since $I(+)N$ is a graded (m,n)-closed ideal of $R(+)M$, it follows that $x^n = (a^n, na^{n-1}c) \in I(+)N$. Hence $na^{n-1}c \in N$ for every $c \in M_g$. Thus $na^{n-1}M_g \subseteq N$, as we desired.

$(2) \Rightarrow (1)$. Let $x^m = (a,c)^m = (a^m, ma^{m-1}c) \in I(+)N$ for some $x = (a,c) \in h(R(+)M)$. Since $a^m \in I$, $a \in h(R)$ and I is a graded (m,n)-closed ideal of R, we conclude that $a^n \in I$. On the other hand there exists $g \in G$ such that $c \in M_g$ and by assumption $na^{n-1}M_g \subseteq N$. Thus $x^n = (a^n, na^{n-1}c) \in I(+)N$. Hence $I(+)N$ is a graded (m,n)-closed ideal of $R(+)M$, as we desired. □

4 Graded weakly (m,n)-closed ideal

In this section, we give some basic properties of graded weakly (m,n)-closed ideals and investigate graded weakly (m,n)-closed ideals in direct product $R_1 \times R_2$ of G-graded rings R_1, R_2 and in trivial extension $R(+)M$ of a G-graded ring R by a G-graded R-module M. For the sake of completeness, we begin with the definitions of weakly (m,n)-closed and graded weakly (m,n)-closed ideals.

Definition 4.1. A proper ideal I of a ring R is said to be weakly (m,n)-closed if whenever $0 \neq a^m \in I$ for $a \in R$, then $a^n \in I$.

Definition 4.2. A proper graded ideal I of a G-graded ring R is said to be graded weakly (m,n)-closed if whenever $0 \neq a^m \in I$ for $a \in h(R)$, then $a^n \in I$.

Note that a graded (m,n)-closed ideal is always graded weakly (m,n)-closed ideal, the converse need not hold. The following example illustrates this fact.

Example 4.3. Consider $R = M_2(K)$ (the ring of all 2×2 matrices with entries from a field K and $G = \mathbb{Z}_4$). Then R is G-graded by $R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$, $R_1 = R_3 = 0$.

Consider $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$, then I is graded weakly $(2,1)$-closed ideal of R. However I is not a graded $(2,1)$-closed ideal, since $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R_2 \subseteq h(R)$ with $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in I$ but $A \notin I$.

Proposition 4.4. *If I is a graded weakly (m,n)-closed ideal of a G-graded ring R, then I_e is a weakly (m,n)-closed ideal of R_e.*

Proof. Let $a \in R_e$ such that $0 \neq a^m \in I_e$. Since $R_e \subseteq h(R)$, $I_e \subseteq I$ and I is a graded weakly (m,n)-closed ideal of R, it follows that $a^n \in I$ and then $a^n \in I \cap R_e = I_e$. Hence I_e is a weakly (m,n)-closed ideal of R_e, as we desired. □

An (m,n)-unbreakable-zero element was defined in [3] for weakly (m,n)-closed ideals, here we define it again in graded setup. It will be helpful for studying graded weakly (m,n)-closed ideals that are not graded (m,n)-closed.

Definition 4.5. Let R be a G-graded ring and I a graded weakly (m,n)-closed ideal of R. Then $a \in h(R)$ is an (m,n)-unbreakable-zero element of I if $a^m = 0$ and $a^n \notin I$.

Thus A graded weakly (m,n)-closed ideal I has an (m,n)-unbreakable-zero element if and only if I is not graded (m,n)-closed.

The following theorem is a graded analog of weakly (m,n)-closed ([3, Theorem 2.5]) and weakly semiprime ideals ([6, Theorem 2.3]).

Theorem 4.6. *Let R be a G-graded ring and I a graded weakly (m,n)-closed ideal of R. If $a \in h(R)$ is an (m,n)-unbreakable-zero element of I, then there exists $g \in G$ such that for every $y \in I_g$, $(a+y)^m = 0$.*

Proof. Assume that $a \in h(R)$ is an (m,n)-unbreakable-zero element of I. Then there exists $g \in G$ such that $a \in R_g$. Now, let $y \in I_g$. Then $a + y \in R_g$ and

$$(a+y)^m = a^m + \sum_{k=1}^{m} \begin{pmatrix} m \\ k \end{pmatrix} a^{m-k} y^k = 0 + \sum_{k=1}^{m} \begin{pmatrix} m \\ k \end{pmatrix} a^{m-k} y^k \in I_g \subset I.$$

However $(a+y)^n \notin I$, because $a^n \notin I$. Thus $(a+y)^m = 0$, since I is a graded weakly (m,n)-closed ideal of R, as we desired.

□

The next theorem is a graded analogue of [3, Theorem 2.6], it also extends [6, Theorem, 2.4].

Theorem 4.7. *Let R be a G-crossed product and I a graded weakly (m,n)-closed ideal of R. Then either I is a graded (m,n)-closed ideal of R or $I_e \subseteq Nil(R_e)$.*

Proof. Assume that I is not a graded (m,n)-closed ideal of R. Then I has an (m,n)-unbreakable-zero element, that is, there exists $a \in h(R)$ such that $a^m = 0$ and $a^n \notin I$. Let $g \in G$ such that $a \in R_g$ and let $x \in I_e$. As R is a crossed product, there exists u a unit element in $R_{g^{-1}}$ such that $b = au \in R_e$ and $b^m = 0$. Thus,

$$(b+x)^m = b^m + \sum_{k=1}^{m} \binom{m}{k} b^{m-k}x^k = 0 + \sum_{k=1}^{m} \binom{m}{k} b^{m-k}x^k \in I_e \subseteq I.$$

If $(b+x)^m \neq 0$, then, since I is a graded weakly (m,n)-closed ideal of R and $b+x \in R_e$, we obtain $(x+b)^n \in I_e \subseteq I$. Consequently, $b^n = a^n u^n \in I$ and (as u is unit) $a^n \in I$, a contradiction. Hence $(x+b)^m = 0$, that is, $x+b \in \mathrm{Nil}(R_e)$. Thus $x = (x+b) - b \in \mathrm{Nil}(R_e)$ and henceforth $I_e \subseteq \mathrm{Nil}(R_e)$, as we desired.

□

The next two theorems are the analogue of the results for graded (m,n)-closed ideals in Theorem 3.8 and Theorem 3.10, respectively. Their proofs are similar, and thus will be omitted.

Theorem 4.8. *Let R be a G-graded ring, I a graded weakly (m,n)-closed ideal of R and $S \subseteq h(R)$ a multiplicatively closed subset of R such that $S \cap I = \emptyset$. Then I_S is a graded weakly (m,n)-closed ideal of R_S.*

Theorem 4.9. *Let R and T be two G-graded rings and $f : R \to T$ a homogeneous homomorphism.*

(1) If f is injective and J is a graded weakly (m,n)-closed ideal of T, then $f^{-1}(J)$ is a graded weakly (m,n)-closed ideal of R. In particular, if R is a graded subring of T and J a graded weakly (m,n)-closed ideal of T, then $R \cap J$ is a graded weakly (m,n)-closed ideal of R.

(2) If $f(R_g) = T_g$ for all $g \in G$ and J is a graded weakly (m,n)-closed ideal of R containing $\mathrm{Ker} f$, *then $f(J)$ is a graded weakly (m,n)-closed ideal of T. In particular, if I is a graded weakly (m,n)-closed ideal of R such that $I \subseteq J$, then J/I is a graded weakly (m,n)-closed ideal of R/J if and only if J is a graded weakly (m,n)-closed ideal of R.*

In the next two theorems, we determine when an ideal of $R_1 \times R_2$ is graded weakly (m,n)-closed but not graded (m,n)-closed.

Theorem 4.10. *Let R_1 and R_2 be G-graded rings such that R_2 is G-crossed product and I_1 a graded ideal of R_1. Then the following statements are equivalent.*

(1) $I_1 \times R_2$ is a graded weakly (m,n)-closed ideal of $R_1 \times R_2$.

(2) I_1 is a graded (m,n)-closed ideal of R_1.

(3) $I_1 \times R_2$ is a graded (m,n)-closed ideal of $R_1 \times R_2$.

A similar result holds for $R_1 \times I_2$ when I_2 is a graded ideal of R_2.

Proof. (1) $\Rightarrow$ (2) Let $a \in (R_1)_g$. Since R_2 is G-crossed product, choose a unit $u \in (R_2)_g$. Then note that $(a,u) \in (R_1)_g \times (R_2)_g \subseteq h(R_1 \times R_2)$ and $0 \neq (a,u)^m = (a^m, u^m) \in I_1 \times R_2$. Since $I_1 \times R_2$ is a graded weakly (m,n)-closed ideal of R, we have $(a,u)^n = (a^n, u^n) \in I_1 \times R_2$. Hence $a^n \in I_1$, and I_1 is a graded (m,n)-closed ideal of R_1, as we desired.

(2) $\Rightarrow$ (3) follows from Theorem 3.12.

(3) $\Rightarrow$ (1) is clear by definition. □

Remark 4.11. The analog of (1) $\Rightarrow$ (2) of Theorem 3.12 is clearly holds for graded weakly (m,n)-closed ideals by Theorem 4.9(2), but the above theorem shows that the analog of (2) $\Rightarrow$ (1) does not hold for weakly (m,n)-closed ideals. For instance, if we take I_1 is a graded weakly (m,n)-closed ideal but not graded (m,n)-closed, then by above theorem $I_1 \times R_2$ is not a graded weakly (m,n)-closed ideal of $R_1 \times R_2$.

Theorem 4.12. *Let $R = R_1 \times R_2$, where R_1 and R_2 are G-crossed products and J be a proper graded ideal of R. Then the following statements are equivalent.*

(1) J is a graded weakly (m,n)-closed ideal of R that is not graded (m,n)-closed.

(2) $J = I_1 \times I_2$ for proper graded ideals I_1 of R_1 and I_2 of R_2 such that either

(a) I_1 is a graded weakly (m,n)-closed ideal of R_1 that is not graded (m,n)-closed, $y^m = 0$ whenever $y^m \in I_2$ for $y \in h(R_2)$ and if $0 \neq x^m \in I_1$ for some $x \in h(R_1)$, then I_2 is a graded (m,n)-closed ideal of R_2, or

(b) I_2 is a graded weakly (m,n)-closed ideal of R_2 that is not graded (m,n)-closed, $y^m = 0$ whenever $y^m \in I_1$ for $y \in h(R_1)$ and if $0 \neq x^m \in I_2$ for some $x \in h(R_2)$, then I_1 is a graded (m,n)-closed ideal of R_1.

Proof. (1) $\Rightarrow$ (2). Since J is not a graded (m,n)-closed ideal of R, by combining Theorem 4.10 with Remark 4.11(b), we have $J = I_1 \times I_2$, where I_1 is a graded weakly (m,n)-closed ideal of R_1 and I_2 is a graded weakly (m,n)-closed ideal of R_2 and at least one of them is not graded (m,n)-closed. Assume that I_1 is a graded weakly (m,n)-closed ideal of R_1 that is not graded (m,n)-closed. Thus I_1 has a (m,n)-unbreakable-zero element $a \in h(R_1)$. Assume that $y^m \in I_2$ for some $y \in h(R_2)$. Now, assume that $a \in (R_1)_g$ and $y \in (R_2)_h$. Since R_2 is G-crossed product, choose a unit $u \in (R_2)_{gh^{-1}}$ and $(a, uy)^m \in J$ and $(a, uy) \in h(R)$, we have $(a, uy)^m = (0,0)$. Hence $y^m = 0$. Now, assume that $0 \neq x^m \in I_1$ for some $x \in h(R_1)$. Let $y \in h(R_2)$ with $y^m \in I_2$. Assume that $x \in (R_1)_g$ and $y \in (R_2)_h$. Since R_2 is G-crossed product, choose a unit $u \in (R_2)_{gh^{-1}}$. Then $(0,0) \neq (x, uy)^m \in J$. The fact that J is a graded weakly (m,n)-closed ideal of R gives $(uy)^n \in I_2$. Since u is unit, $y^n \in I_2$. Hence I_2 is a graded (m,n)-closed ideal of R_2. In a similar way, if I_2 is a graded weakly (m,n)-closed ideal of R_2 that is not graded (m,n)-closed, then $y^m = 0$ whenever $y^m \in I_1$ for $y \in h(R_1)$ and if $0 \neq x^m \in I_2$ for some $x \in h(R_2)$, then I_1 is a graded (m,n)-closed ideal of R_1.

(2) $\Rightarrow$ (1). Due to symmetry, it suffices to prove (2)(a) $\Rightarrow$ (1).

Suppose that I_1 is a graded weakly (m,n)-closed ideal of R_1 that is not graded (m,n)-closed, $y^m = 0$ whenever $y^m \in I_2$ for $y \in h(R_2)$, and if $0 \neq x^m \in I_1$ for some $x \in h(R_1)$, then I_2 is a graded (m,n)-closed ideal of R_2. Let $a \in h(R_1)$ a (m,n)–unbreakable-zero element of I_1, since $(a,0) \in h(R)$ we have $(a,0)$ is an (m,n)-unbreakable-zero element of J. Thus J is not a graded (m,n)-closed ideal of R. Now assume for some $(x,y) \in h(R)$ that $(0,0) \neq (x,y)^m \in J$. So, by assumption, $y^m = 0$, therefore $x^m \neq 0$ and then I_2 is a graded (m,n)-closed ideal of R_2. Hence $x^n \in I_1$ and $y^n \in I_2$ and consequently $(x,y)^n \in J$. Thus J is a graded weakly (m,n)-closed ideal of R. □

We conclude this section by considering when certain ideals of the graded trivial extension $R(+)M$ are graded weakly (m,n)-closed ideals but not graded (m,n)-closed.

Theorem 4.13. *Let R be a G-graded ring, M a G-graded R module and I a graded ideal of R. Then the following statements are equivalent.*

(1) $I(+)M$ is a graded weakly (m,n)-closed ideal of $R(+)M$ that is not graded (m,n)-closed.

(2) I is a graded weakly (m,n)-closed ideal of R that is not graded (m,n)-closed and for every (m,n)-unbreakable-zero element a of I, we have $m(a^{m-1}M_g) = 0$ for some $g \in G$.

Proof. (1) $\Rightarrow$ (2). Let $J = I(+)M$. Assume that $0 \neq a^m \in I$ for some $a \in h(R)$. Then $(a,0) \in h(R(+)M)$ and $(0,0) \neq (a,0)^m \in J$. Hence $(a,0)^n = (a^n,0) \in J$, as a consequence $a^n \in I$. Thus I is a graded weakly (m,n)-closed ideal of R, that is, by Theorem 3.14(2), not graded (m,n)-closed. Now, let $a \in h(R)$ be an (m,n)-unbreakable-zero element of I. So, there exists $g \in G$ with $a \in R_g$, let $x \in M_g$. We have $(a,x) \in h(R(+)M)$ and $(a,x)^m = (a^m, ma^{m-1}x) \in J$. Since $a^n \notin I$, we have $(a^m, ma^{m-1}x) = (0,0)$. Thus $m(a^{m-1}M_g) = 0$, as we desired.

(2) $\Rightarrow$ (1). Firstly, by Theorem 3.14, it is clear that J is not a graded (m,n)-closed ideal of $R(+)M$. Now, in order to prove that $I(+)M$ is a graded weakly (m,n)-closed ideal of $R(+)M$, assume that $(0,0) \neq (a,x)^m = (a^m, ma^{m-1}x) \in J = I(+)M$ for some $(a,x) \in h(R(+)M)$. Then $a \in h(R)$ and $a^m \in I$. If $a^m \neq 0$, then by assumption, $a^n \in I$ and therefore $(a,x)^n = (a^n, na^{n-1}x) \in J$. If $a^m = 0$ and $a^n \notin I$, then a is an (m,n)-unbreakable-zero element of I. On the other hand, there exists $g \in G$ such that $a \in R_g$ and $x \in M_g$. By assumption we have $ma^{m-1}x = 0$ and hence $(a,x)^m = (0,0)$, which is not the case. We conclude that J is a graded weakly (m,n)-closed ideal of $R(+)M$.

□

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