Moroccan Journal of Algebra and Geometry with Applications
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco

Volume 1, Issue 2 (2022), pp 380-391

ISSN: 2820-7114

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Moroccan Journal of Algebra and Geometry with Applications
Vol. 1(2) (2022), 380–391
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco
ISSN: 2820-7114

Chain bundles and category of chains

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Communicated by Mohammed Tamekkante
(Received 26 April 2022, Revised 20 July 2022, Accepted 26 July 2022)

Abstract. In [8], we introduced the category of chain bundles and it is shown that this category is set based having subobjects and admitting factorization of morphisms. In this paper we consider chains in the chain bundle category which is a category in it's own right and discuss some interesting categorical properties of these categories.
Key Words: Category, Subobject, Chainbundle, Chain bundle map, Factorization, Category of chains, Chain complexes.
2010 MSC: Primary 18D70; Secondary 18G35.

1 Introduction

Category theory, in general is a mathematical theory of structures and system of structures which has come to occupy a central position in present day mathematics. Eilenberg and Mac Lane provide a purely abstract definition of a category in 1945 (cf. [2] and [5]). Accordingly a category $\mathcal{C}$ can be described as a set **Ob** whose members are objects of $\mathcal{C}$ and for every pair X, Y of objects of $\mathcal{C}$ there is $Hom(X, Y)$ called morphisms from X to Y both satisfying certain specified conditions. There are many situations where one has to deal with chains (sequences) of objects and hence it is natural to consider categories whose objects are chains in some category $\mathcal{C}$ and morphisms are appropriate morphisms between such chains. In order to discuss such situations, in [8] we introduced the category of chain bundles as a category whose objects are sequence in a category $\mathcal{C}$ with zero such that for any two objects X and Y any subset of $Hom(X, Y)$ constitute the morphisms between X and Y. A category whose objects are chain bundles and morphisms appropriate chain bundle maps is termed as a category of chain bundles written as $\mathfrak{CB}_{\mathcal{C}}$.
K. S. S. Nambooripad introduced certain categories whose objects are ideal categories of semigroups and morphisms are appropriate translations and he characterized these categories as categories with subobjects and factorization (cf. [7]). In [8], it is shown that the category of chain bundles is a category with subobjects and factorization. In this paper we provide provide the examples of the category of chain bundles of directed graph and augmented simplicial categories. Further we describe the category of chains in chain bundle categories and discuss the properties of these categories.

2 Preliminaries

In the following we briefly recall basic notions such as categories, category with subobjects, factorization of morphisms and the like which are needed in the sequel.

Definition 2.1. *A category $\mathcal{C}$ consists of*

1. *a class $v\mathcal{C}$ called the class of vertices or objects*
2. *a class $\mathcal{C}$ of disjoint sets $\mathcal{C}(a, b)$, one for each pair $(a, b) \in v\mathcal{C} \times v\mathcal{C}$, an element $f \in \mathcal{C}(a, b)$ called a morphism (arrow) from a to b, where $a = dom\, f$ and $b = cod\, f$*

3. *For $a, b, c \in \nu\mathcal{C}$, a map*

$$\circ : \mathcal{C}(a,b) \times \mathcal{C}(b,c) \to \mathcal{C}(a,c)$$

$$(f,g) \to f \circ g \quad \textit{is the composition of morphisms in } \mathcal{C}$$

4. *For each $a \in \nu\mathcal{C}$, a unique $1_a \in \mathcal{C}(a,a)$ is the identity morphism on a.*

These must satisfy the following axioms:

Cat 1 for $f \in \mathcal{C}(a,b)$, $g \in \mathcal{C}(b,c)$ and $h \in \mathcal{C}(c,d)$; the composition is associative

Cat 2 for each $a \in \nu\mathcal{C}$, $f \in \mathcal{C}(a,b)$, $g \in \mathcal{C}(c,a)$, $1_a \circ f = f$ and $g \circ 1_a = g$.

$\nu\mathcal{C}$ may identifed as a subclass of $\mathcal{C}$ and with this identification, categories may regard in terms of morphisms(arrows) alone. The category $\mathcal{C}$ is said to be small if the class $\mathcal{C}$ is a set. For any category $\mathcal{C}$ an opposite category denoted as $\mathcal{C}^{op}$ is the category with

$$\nu\mathcal{C}^{op} = \nu\mathcal{C}, \quad \mathcal{C}^{op}(a,b) = \mathcal{C}(b,a) \text{ for all } a, b \in \nu\mathcal{C}$$

ie., for arrows f^{op} there is a one-one correspondence $f \mapsto f^{op}$ with arrow f of $\mathcal{C}$ and the composite $f^{op}g^{op} = (gf)^{op}$ is defined in $\mathcal{C}^{op}$ exactly when gf is defined in $\mathcal{C}$ (see cf.[2]).

A structure preserving map between categories is called a functor.

Definition 2.2. *A category $\mathcal{D}$ is a subcategory of a category $\mathcal{C}$ if the class $\mathcal{D}$ is a subclass of $\mathcal{C}$ and the composition in $\mathcal{D}$ is the restriction of the composition in $\mathcal{C}$ to $\mathcal{D}$. In this case, the inclusion $\mathcal{D} \subseteq \mathcal{C}$ preserves composition and identities and so represents a functor of $\mathcal{D}$ to $\mathcal{C}$.*

A morphism f in a category $\mathcal{C}$ is a monomorphism if

$$gf = hf \Rightarrow g = h \; \forall g, h \in \mathcal{C}$$

and a morphism f in $\mathcal{C}$ is called a split monomorphism if it has a right inverse. Every morphism in a concrete category whose underlying function is an injection is a monomorphism. A morphism f in a category $\mathcal{C}$ is an *epimorphism* if

$$fg = fh \Rightarrow g = h \; \forall g, h \in \mathcal{C}$$

and a morphism f in $\mathcal{C}$ is called a split epimorphism if it has a left inverse. Every morphism in a concrete category whose underlying function is an surjection is an epimorphism.

Let $\mathbb{M}\mathcal{C}$ denote the class of all monomorphisms in $\mathcal{C}$. On $\mathbb{M}\mathcal{C}$, define the relation

$$f \leq g \Leftrightarrow f = hg \quad \text{for some } h \in \mathcal{C}$$

$\leq$ is a quasi- order and

$$\sim = \leq \cap \leq^{-1} \tag{1}$$

is an equivalence relation on $\mathbb{M}\mathcal{C}$.

Definition 2.3. *A preorder $\mathcal{P}$ is a category such that for any $p, p' \in \mathcal{P}$, $\mathcal{P}(p,p')$ contains at most one morphism. In this case, the relation $\subseteq$ on the class $\nu\mathcal{P}$ defined by*

$$p \subseteq p' \Leftrightarrow \mathcal{P}(p,p') \neq \emptyset \tag{2}$$

is a quasiorder. When $\mathcal{P}$ is a preorder, $\nu\mathcal{P}$ will stand for the quasiordered class $(\nu\mathcal{P}, \subseteq)$. Conversely given a quasiorder $\leq$ on the class X, the subset

$$\mathcal{P} = \{(x,y) \in X \times X : x \leq y\}$$

of $X \times X$ is a preorder such that the quasiordered class $\nu\mathcal{P}$ defined above is order isomorphic with $(X, \leq)$. If the relation $\subseteq$ on $\mathcal{P}$ is antisymmetric then we shall say that $\mathcal{P}$ is a strict preorder.

Definition 2.4. *Let $\mathcal{C}$ be a category. A choice of subobjects in $\mathcal{C}$ is a subcategory $\mathcal{P} \subseteq \mathcal{C}$ satisfying the following:*

(a) $\mathcal{P}$ is a strict preorder with $v\mathcal{P} = v\mathcal{C}$.

(b) Every $f \in \mathcal{P}$ is a monomorphism in $\mathcal{C}$.

(c) If $f, g \in \mathcal{P}$ and if $f = hg$ for some $h \in \mathcal{C}$ then $h \in \mathcal{P}$.

When $\mathcal{P}$ satisfies these conditions, the pair $(\mathcal{C}, \mathcal{P})$ is called a category with subobjects.

When $\mathcal{C}$ has subobjects, unless explicitly stated otherwise, $v\mathcal{C}$ will denote the choice of subobjects in $\mathcal{C}$. The partial order defined by equation (2) is called the preorder of inclusions or subobject relation in $\mathcal{C}$ and is denoted by $\subseteq$. If $c, d \in v\mathcal{C}$ and $c \subseteq d$ the unique morphism from c to d is the inclusion $j_c^d : c \to d$. Any monomorphism f equivalent to an inclusion (with respect to the equivalence relation $\sim$ defined by equation (1)) is called an *embedding*.

Example 2.5. *In categories $Set, Grp, Vect_K, Mod_R$ the relation on objects induced by the usual set inclusion is a subobject relation.*

Definition 2.6. *A morphism f in a category $\mathcal{C}$ with subobjects is said to have factorization if f can be expressed as $f = pm$ where p is an epimorphism and m is an embedding.*

Factorization of a morphism need not be unique. Every morphism f with factorization has at least one factorization of the form $f = qj$ where q is an epimorphism and j is an inclusion. Such factorizations are called *canonical factorization*. A category $\mathcal{C}$ is *category with factorization* if $\mathcal{C}$ has subobjects and if every morphism in $\mathcal{C}$ has factorization. The category has *unique factorization property* if every morphism in $\mathcal{C}$ has unique canonical factorization.

Example 2.7. *If $f : X \to Y$ is a mapping of sets and $f(X) = Imf$ the $f(X) \subseteq Y$ and we can write $f = f^0 j_{f(X)}^Y$, where f^0 denote the mapping of X onto $f(X)$ determined by f. Since surjective mappings are epimorphisms in Set, this gives a canonical factorization of f which is clearly unique. ie., Set is a category with unique factorization.*

Proposition 2.8. *Let $\mathcal{C}$ be category with factorization. Suppose that the morphism $f \in \mathcal{C}$ has the following property:*

(Im) f has a canonical factorization $f = xj$ such that for any canonical factorization $f = yj'$ of f, there is an inclusion j'' with $y = xj''$

then the factorization $f = xj$ is unique.

A morphism f in a category with factorization is said to have *image* if f satisfies the condition (Im) of the proposition above.

Definition 2.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. A universal arrow from $d \in v\mathcal{D}$ to the functor F is a pair (c, g) where $c \in vC$ and $g \in \mathcal{D}(d, F(c))$ such that given any pair (c', g') with $c' \in v\mathcal{C}$ and $g \in \mathcal{D}(d, F(c'))$, there is a unique $f \in C(c, c')$ such that $g' = g \circ F(f)$. In this case, the morphism g is universal from d to F. A universal arrow from F to d is defined dually.*

Example 2.10. *$Vect_K$ be the category of all vector spaces over a field K, with arrows linear transformations, then $U : Vect_K \to Set$ sending each vector space V to the set of its elements is a functor and is called the forgetful functor. For any set X there is a familiar vector space V_X with X as the set of basis vectors, consists of all formal K-linear combinations of the elements of X. Then V which sends each $x \in X$ into the same x regarded as a vector of V_X is also a functor. Consider the arrow $j : X \to U(V_X)$. For any other vector space W, it is a fact that each function $f : X \to U(W)$ can be extended to a unique linear transformation $f' : V_X \to W$ with $Uf' \circ j = f$. This familiar fact states that j is a universal arrow from X to U.*

Definition 2.11. *Let $\mathcal{C}$ be a category and $X_1, X_2 \in v\mathcal{C}$. An object X is a product of X_1 and X_2, denoted $X_1 \times X_2$, if it satisfies this universal property such that: there exist morphisms $\pi_1 : X \to X_1$, $\pi_2 : X \to X_2$ and for every object Y and pair of morphisms $f_1 : Y \to X_1$, $f_2 : Y \to X_2$ there exists a unique morphism $f : Y \to X$ such that the following diagram commutes:*

$$\begin{array}{ccccc} & & Y & & \\ & \swarrow^{f_1} & \downarrow f & \searrow^{f_2} & \\ X_1 & \xleftarrow[\pi_1]{} & X_1 \times X_2 & \xrightarrow[\pi_2]{} & X_2 \end{array}$$

The unique morphism f is called the product of morphisms f_1 and f_2 and is denoted $< f_1, f_2 >$. The morphisms π_1 and π_2 are called the canonical projections.

Example 2.12. *In the category of sets, the product (in the category theoretic sense) is the cartesian product and in the category of groups, the product is the direct product of groups given by the cartesian product with multiplication defined component wise.*

Definition 2.13. *Let $\mathcal{C}$ be a category with zero morphisms. A kernel of a morphism $f : X \to Y$ in $\mathcal{C}$ is an object K together with a morphism $k : K \to X$ such that $f \circ k$ is the zero morphism from K to Y. Given any morphism $k' : K' \to X$ such that $f \circ k'$ is the zero morphism, there is a unique morphism $u : K' \to K$ such that $k \circ u = k'$.*

Kernel of homomorphisms are familiar concept in many categories such as the category of groups or the category of (left) modules over a fixed ring (including vector spaces over a fixed field). To make this explicit, if $f : X \to Y$ is a homomorphism in one of these categories, and K is its kernel in the usual algebraic sense, then K is a subalgebra of X and the inclusion homomorphism from K to X is a kernel in the categorical sense.

3 Category of Chain Bundles

We introduced the category of chain bundles in the paper entitled " Category of Chain Bundles" (cf. [8]). However the following definition is a slightly generalized one.

Definition 3.1. *Let $\mathcal{C}$ be category with zero. A chain bundle c in the category $\mathcal{C}$ is a sequence*

$$\cdots M_3 \xrightarrow{S_3} M_2 \xrightarrow{S_2} M_1 \xrightarrow{S_1} M_0 = \boldsymbol{0}$$

where $M_i \in v\mathcal{C}$ and S_i be any subset of $Hom(M_{i+1}, M_i)$ for all i, which also include homsets of the form $Hom(M_i, M_i)$ and all possible composite of morphisms. Let d be the chain bundle

$$\cdots N_3 \xrightarrow{T_3} N_2 \xrightarrow{T_2} N_1 \xrightarrow{T_1} N_0 = \boldsymbol{0}$$

morphisms $m : c \to d$ of chain bundles is the sequence $m = (f_i, x_i, y_i)$ where $f_i : M_i \to N_i$, $x_i \in S_i$ and $y_i \in T_i$ be such that diagram commutes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M_3 & \xrightarrow{S_3} & M_2 & \xrightarrow{S_2} & M_1 & \xrightarrow{S_1} & \boldsymbol{0} \\ & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\ \cdots & \longrightarrow & N_3 & \xrightarrow{T_3} & N_2 & \xrightarrow{T_2} & N_1 & \xrightarrow{T_1} & \boldsymbol{0} \end{array}$$

ie., $x_i \circ f_{i-1} = f_i \circ y_i$.

Generally a bundle is a triple (E,p,B) where E,B are sets and $p: E \to B$ is a map. Each $M_{i+1} \xrightarrow{S_i} M_i$ in chain bundle can be regarded as a collection of bundles and hence $\cdots M_3 \xrightarrow{S_3} M_2 \xrightarrow{S_2} M_1 \xrightarrow{S_1} M_0 = \mathbf{0}$ is called a chain bundle.

Let $\mathcal{C}$ be a category with zero. A category whose objects are chain bundles in $\mathcal{C}$ and morphism are the morphisms of chain bundles called chain bundle maps is the category of chain bundles $\mathfrak{CB}_{\mathcal{C}}$. Note that all objects and morphisms in a chain bundle category $\mathfrak{CB}_{\mathcal{C}}$ are objects and morphisms in $\mathcal{C}$ as well and so $\mathfrak{CB}_{\mathcal{C}}$ may regard as a subcategory of $\mathcal{C}$, also it is easy to observe that given any category $\mathcal{C}$ one can always obtain chain bundle categories $\mathfrak{CB}_{\mathcal{C}}$ of $\mathcal{C}$. The cochain bundles and the category of cochain bundles may be described dually.

Next we proceed to describe the subobject relation and factorization of morphisms in the category $\mathfrak{CB}_{\mathcal{C}}$, whenever the category $\mathcal{C}$ admits subobjects and factorization of morphisms.

Definition 3.2. *Consider the chain bundle c:*

$$\cdots M_3 \xrightarrow{S_3} M_2 \xrightarrow{S_2} M_1 \xrightarrow{S_1} M_0 = \mathbf{0} \in \mathfrak{CB}_{\mathcal{C}}$$

then, the chain bundle c':

$$\cdots M_3' \xrightarrow{S_3'} M_2' \xrightarrow{S_2'} M_1' \xrightarrow{S_1'} M_0' = \mathbf{0}$$

with M_i' a subobject of M_i and for each $x' \in S_i'$ there is an $x \in S_i$ such that $(j^{M_i}_{M_i'} x)^0 = (x_i')^0$ is a subchain bundle of c.

Consider chain bundles $c: \cdots \to M_i \xrightarrow{S_i} M_j \to \cdots$, $\quad d: \cdots \to N_i \xrightarrow{T_i} N_j \to \cdots \in \nu\mathfrak{CB}_{\mathcal{C}}$ and morphism $m: c \to d$. Since each $f_i: M_i \to N_i$, admits a factorization of the form $f_i^0 j^{N_i}_{N_i'}$ where $N_i' = cod\ f_i^0$ in $\mathcal{C}$, we obtain a factorization of $m = m^\circ J$. For,

$$\begin{array}{cccccccccc}
c: & \cdots \longrightarrow & M_i & \xrightarrow{S_i} & M_j & \longrightarrow \cdots \longrightarrow & M_1 & \longrightarrow & \mathbf{0} \\
\downarrow m & & \downarrow f_i & & \downarrow f_j & & \downarrow f_1 & & \downarrow 0 \\
d: & \cdots \longrightarrow & N_i & \xrightarrow{T_i} & N_j & \longrightarrow \cdots \longrightarrow & N_1 & \longrightarrow & \mathbf{0}
\end{array}$$

each $g \in S_i$ and $m(g) \in T_i$, $(j^{N_i}_{N_i'} m(g))^0 \in Hom(N_i', Nj')$, take

$$S_i' = \{(j^{N_i}_{N_i'} m(g))^0 : g \in S_i\}.$$

Define $m^0: g \mapsto (j^{N_i}_{N_i'} m(g))^0 = m(g)'$, then $m(g) = J((j^{N_i}_{N_i'} m(g))^0) = J(m^0(g))$ and so $m = m^0 J$ is a factorization. i.e., the category $\mathfrak{CB}_{\mathcal{C}}$ admits a factorization as below.

$$\begin{array}{cccccccccc}
c: & \cdots \longrightarrow & M_i & \xrightarrow{g} & M_j & \longrightarrow \cdots \longrightarrow & M_1 & \longrightarrow & \mathbf{0} \\
\downarrow m^0 & & \downarrow f_i^0 & & \downarrow f_j^0 & & \downarrow f_1^0 & & \downarrow 0 \\
c': & \cdots \longrightarrow & N_i' & \xrightarrow{m(g)'} & N_j' & \longrightarrow \cdots \longrightarrow & N_1' & \longrightarrow & \mathbf{0} \\
\downarrow J & & \downarrow j_i & & \downarrow j_j & & \downarrow j_1 & & \downarrow 0 \\
d: & \cdots \longrightarrow & N_i & \xrightarrow{m(g)} & N_j & \longrightarrow \cdots \longrightarrow & N_1 & \longrightarrow & \mathbf{0}
\end{array}$$

Summarizing the above discussions we have the following theorem.

Theorem 3.3. Let $\mathcal{C}$ be a category with zero object. If $\mathcal{C}$ is a category with factorization then so is $\mathfrak{CB}_{\mathcal{C}}$

Example 3.4. *Consider the following chain bundle map $m : c \to d$ in $\mathfrak{CB}_{\mathcal{C}}$ where $\mathcal{C}$ is the category of finite abelian groups*

$$
\begin{array}{ccccccccc}
c: & & \mathbb{Z}_3 & \xrightarrow{2a} & \mathbb{Z}_6 & \xrightarrow{b} & \mathbb{Z}_2 & \xrightarrow{0} & \mathbf{0} \\
\downarrow m & & \downarrow 4 & & \downarrow 4 & & \downarrow 1 & & \downarrow 0 \\
d: & & \mathbb{Z}_4 & \xrightarrow{2a} & \mathbb{Z}_8 & \xrightarrow{b} & \mathbb{Z}_2 & \xrightarrow{0} & \mathbf{0}
\end{array}
$$

then m admits the following factorization:

$$
\begin{array}{ccccccccc}
c: & & \mathbb{Z}_3 & \xrightarrow{2a} & \mathbb{Z}_6 & \xrightarrow{b} & \mathbb{Z}_2 & \xrightarrow{0} & \mathbf{0} \\
\downarrow m^0 & & \downarrow 4 & & \downarrow 4 & & \downarrow 1 & & \downarrow 0 \\
d': & & \{0\} & \xrightarrow{2a} & \{0,4\} & \xrightarrow{b} & \mathbb{Z}_2 & \xrightarrow{0} & \mathbf{0} \\
\downarrow J & & \downarrow j & & \downarrow j & & \downarrow j & & \downarrow j \\
d: & & \mathbb{Z}_4 & \xrightarrow{2a} & \mathbb{Z}_8 & \xrightarrow{b} & \mathbb{Z}_2 & \xrightarrow{0} & \mathbf{0}
\end{array}
$$

3.1 Functors between chain bundle categories

Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories with zero object and $\mathfrak{CB}_{\mathcal{C}}$ and $\mathfrak{CB}_{\mathcal{C}'}$ are the associated chain bundle categories. A functor $\bar{\mathrm{F}} : \mathfrak{CB}_{\mathcal{C}} \to \mathfrak{CB}_{\mathcal{C}'}$ consists of two related maps: the object map $\bar{\mathrm{F}}$ which assigns to each chain bundle c of $\mathfrak{CB}_{\mathcal{C}}$ a chain bundle $\bar{\mathrm{F}}c$ of $\mathfrak{CB}_{\mathcal{C}'}$ and the morphism map which assigns to each morphism $F : c \to c'$ of $\mathfrak{CB}_{\mathcal{C}}$ a morphism $\bar{\mathrm{F}}(F) : \bar{\mathrm{F}}c \to \bar{\mathrm{F}}c'$ of $\mathfrak{CB}_{\mathcal{C}'}$, in such a way that $\bar{\mathrm{F}}(1_c) = 1_{\bar{\mathrm{F}}c}$, $\bar{\mathrm{F}}(F \circ G) = \bar{\mathrm{F}}F \circ \bar{\mathrm{F}}G$.

Example 3.5. *Consider the categories $\mathcal{C} = \textbf{Grp}$ and $\mathcal{D} = \textbf{Set}_*$. The functor from $\mathfrak{CB}_{\mathcal{C}}$ to $\mathfrak{CB}_{\mathcal{D}}$ maps the chain bundle c :*

$$\cdots G_3 \xrightarrow{S_3} G_2 \xrightarrow{S_2} G_1 \xrightarrow{S_1} G_0 = \mathbf{0}$$

in $\mathfrak{CB}_{\mathcal{C}}$ to the chain bundle d :

$$\cdots (G_3, e_{G_3}) \xrightarrow{S_3} (G_2, e_{G_2}) \xrightarrow{S_2} (G_1, e_{G_1}) \xrightarrow{S_1} (G_0, e_{G_0}) = \mathbf{0}$$

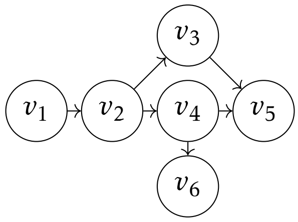
in $\mathfrak{CB}_{\mathcal{D}}$, where each G_i in c is mapped to corresponding pointed set (G_i, e_{G_i}) in d (e_{G_i} is the identity element in group G_i) and homomorphisms in S_i of c are mapped to underlying set map is a forgetful functor from $\mathfrak{CB}_{\mathcal{C}}$ to $\mathfrak{CB}_{\mathcal{D}}$.

3.2 Chain Bundle Category from Directed Graph and Augmented Simplex Category

G be a directed graph (with loop at each vertex), then G is regarded as a category with vertices as objects and path between two vertices as morphism. For a vertex v in G, the set of all paths ending at v is denoted as c_v and any subset of c_v is a chain bundle in G with end vertex v. Let v and w be two vertices in G and c_1, c_2 be chain bundles in G such that $c_1 = \{c_1^i\} \subseteq c_v$ and $c_2 = \{c_2^i\} \subseteq c_w$. A morphism from c_1 to c_2 is a path p from v to w such that $c_1^i p \in c_2, \forall c_1^i \in c_1$. Thus we can obtain a category $\mathfrak{CB}_G$

of chain bundles from a graph G by choosing chain bundles in G as objects and chain bundle maps as morphisms. It is easy to see that this category is a category with subobjects where usual inclusion of sets is the subobject relation.

Example 3.6. *Consider the graph G given below:*



$c_{v_5} = \{v_1v_2v_4v_5, v_1v_2v_3v_5, v_2v_4v_5, v_4v_5, v_3v_5, v_5\}$, $c_{v_2} = \{v_1v_2, v_2\}$, $\{v_1v_2v_3, v_3\} \subset c_{v_3}$, $c_{v_6} = \{v_1v_2v_4v_6, v_2v_4v_6, v_4v_6, v_6\}$, $c_{v_1} = \{v_1\}$ *are chain bundles in G. Then*

$$p_1 = v_2v_4v_5 : c_{v_2} \to c_{v_5}$$

and

$$p_2 = v_2v_3v_5 : c_{v_2} \to c_{v_5}$$

are two chain maps from c_{v_2} *to* c_{v_5}.

Definition 3.7. *The simplex category* Δ *has objects the set of nonempty finite linearly ordered sets* $\{[n] | n \geq 0\}$ *and morphisms from* $[m]$ *to* $[n]$ *are given by weakly monotone maps. A map* $f : \{0, 1, \cdots, m\} \to \{0, 1, \cdots, n\}$ *is called weakly monotone if, for every* $0 \leq i \leq i' \leq m$, *we have* $f(i) \leq f(i')$.

The augmented simplex category, denoted by Δ_+ is the category of all finite ordinals and order-preserving maps, thus $\Delta_+ = \Delta \cup [-1]$, where$[-1] = \emptyset$. $[-1]$ is an initial object and $[0]$ is a terminal object in this category. Consider the augmented simplex category Δ_+. A chain bundle in Δ_+ is of the form

$$\cdots [m_3] \overset{S_3}{\to} [m_2] \overset{S_2}{\to} [m_1] \to [0]$$

where $[m_i]$ are ordinals and S_i is any subset of $Hom([m_i], [m_{i-1}])$.
The chain bundle map between two chain bundles can be defined in a similar way as in Definition 3.1. The chain bundle category thus obtained from Δ_+ is denoted as $\mathfrak{CB}_{\Delta_+}$

4 Categorical Properties of chain bundles

When $\mathcal{C}$ is a category with the categorical properties like product, coproduct, kernel and cokernel then $\mathfrak{CB}_{\mathcal{C}}$ also admits the same.

Consider the chain bundles c and d:

$$c : \cdots M_3' \overset{S_3'}{\to} M_2' \overset{S_2'}{\to} M_1' \overset{S_1'}{\to} M_0' = \mathbf{0}$$

$$d : \cdots M_3 \overset{S_3}{\to} M_2 \overset{S_2}{\to} M_1 \overset{S_1}{\to} M_0 = \mathbf{0}$$

in a category $\mathcal{C}$ with product, then the product $c \times d$ is the chain bundle

$$c \times d : \cdots M_3' \times M_3 \overset{S_3' \times S_3}{\longrightarrow} M_2' \times M_2 \overset{S_2' \times S_2}{\longrightarrow} M_1' \times M_1 \overset{S_1' \times S_1}{\longrightarrow} M_0' \times M_0 = \mathbf{0} \times \mathbf{0}$$

For any $F : l \to c$ and $G : l \to d$ where l is the chain bundle

$$l : \cdots N_3 \xrightarrow{T_3} N_2 \xrightarrow{T_2} N_1 \xrightarrow{T_1} N_0 = \mathbf{0},$$

there exists a chain bundle map $L : l \to c{\times}d$ such that for any $k \in T_i$ and $L(k) \in S'_i{\times}S_i$ there corresponds $F(k) \in S'_i$ and $G(k) \in S_i$ such that $L(k) = (F(k), G(k))$.
In a similar manner one can define coproducts in $\mathfrak{CB}_{\mathcal{C}}$.

Example 4.1. *$\mathcal{C}$ be the category of submodules of $\mathbb{Z}$ and $\mathfrak{CB}_{\mathcal{C}}$ be category of chain bundles in $\mathcal{C}$. Consider the two chain bundles*

$$c : 3\mathbb{Z} \xrightarrow{\frac{2}{3}a} 2\mathbb{Z} \xrightarrow{\frac{5}{2}b} 5\mathbb{Z} \xrightarrow{0} \mathbf{0} \quad \text{and} \quad d : 6\mathbb{Z} \xrightarrow{\frac{2}{3}a'} 4\mathbb{Z} \xrightarrow{\frac{1}{4}b'} \mathbb{Z} \xrightarrow{0} \mathbf{0}$$

The product $c \times d$ is the chain bundle

$$c \times d : 3\mathbb{Z} \times 6\mathbb{Z} \xrightarrow{\frac{2}{3}a, \frac{2}{3}a'} 2\mathbb{Z} \times 4\mathbb{Z} \xrightarrow{\frac{5}{2}b, \frac{1}{4}b'} 5\mathbb{Z} \times \mathbb{Z} \xrightarrow{0,0} \mathbf{0} \times \mathbf{0}$$

For any $l : m\mathbb{Z} \xrightarrow{\frac{n}{m}a''} n\mathbb{Z} \xrightarrow{\frac{p}{n}b''} p\mathbb{Z} \xrightarrow{0} \mathbf{0}$ in $\nu\mathfrak{CB}_{\mathcal{C}}$ and for any $F : l \to c$ and $G : l \to d$ there exists a chain bundle map $L : l \to c \times d$ such that the following diagram commutes

$$\begin{array}{ccccc} & & l & & \\ & \swarrow^{F} & \downarrow^{L} & \searrow^{G} & \\ c & \xleftarrow[\pi_1]{} & c \times d & \xrightarrow[\pi_2]{} & d \end{array}$$

Suppose $\mathcal{C}$ be a category having kernels. For $c, d \in \nu\mathfrak{CB}_{\mathcal{C}}$, let $F : c \to d$ be a morphism in $\mathfrak{CB}_{\mathcal{C}}$ whose vertex mapping is $\{f_i\}$. Then kernel of F is a morphism $K : a \to c$ where a is a chain bundle in $\mathfrak{CB}_{\mathcal{C}}$ whose vertex mappings are $\{ker(f_i)\}$ and morphism map of K maps each morphism in T_i to the zero morphism in S'_i.

$$\begin{array}{ccccccccc} a : \cdots & \longrightarrow & A_i & \xrightarrow{T_i} & A_j & \longrightarrow & \cdots & \longrightarrow & \mathbf{0} \\ \downarrow K & & {\scriptstyle ker f_i}\downarrow & & \downarrow{\scriptstyle ker f_j} & & & & \downarrow{\scriptstyle ker f_0} \\ c : \cdots & \longrightarrow & M_i & \xrightarrow{S'_i} & M_j & \longrightarrow & \cdots & \longrightarrow & \mathbf{0} \\ \downarrow F & & {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} & & & & \downarrow{\scriptstyle f_0} \\ d : \cdots & \longrightarrow & D_i & \xrightarrow{S_i} & D_j & \longrightarrow & \cdots & \longrightarrow & \mathbf{0} \end{array}$$

Similary one can define cokernels in $\mathfrak{CB}_{\mathcal{C}}$.

Example 4.2. *Let C be category of finite abelian groups. In $\mathfrak{CB}_{\mathcal{C}}$, consider the chain bundles $c : \mathbb{Z}_4 \xrightarrow{a} \mathbb{Z}_2 \to \mathbf{0}$, $d : \mathbb{Z}_{12} \xrightarrow{a} \mathbb{Z}_6 \to \mathbf{0}$ and the chain bundle map $F : c \to d$ given by*

$$\begin{array}{cccccc} c : & \mathbb{Z}_4 & \xrightarrow{a} & \mathbb{Z}_2 & \longrightarrow & \mathbf{0} \\ \downarrow F & \downarrow 3 & & \downarrow 3 & & \downarrow 0 \\ d : & \mathbb{Z}_{12} & \xrightarrow{a} & \mathbb{Z}_6 & \longrightarrow & \mathbf{0} \end{array}$$

then kernel of F is the following:

$$\begin{array}{ccccccc} a: & & \mathbb{Z}_8 & \xrightarrow{a} & \mathbb{Z}_4 & \longrightarrow & \mathbf{0} \\ \big\downarrow{\scriptstyle KerF} & & \big\downarrow{\scriptstyle 4} & & \big\downarrow{\scriptstyle 2} & & \big\downarrow{\scriptstyle 0} \\ c: & & \mathbb{Z}_4 & \xrightarrow{a} & \mathbb{Z}_2 & \longrightarrow & \mathbf{0} \end{array}$$

Let $\mathfrak{CB}_{\mathcal{C}}$ and $\mathfrak{CB}_{\mathcal{D}}$ be two chain bundle categories and $\bar{S}$: $\mathfrak{CB}_{\mathcal{D}} \to \mathfrak{CB}_{\mathcal{C}}$ be a functor. For a chain bundle c in $\mathfrak{CB}_{\mathcal{C}}$, a universal arrow from c to $\bar{S}$ is a pair $< r, G >$; r is a chain bundle in $\mathfrak{CB}_{\mathcal{D}}$ and $G : c \to \bar{S}r$ such that for every pair $< d, F >$; $d \in \nu\mathfrak{CB}_{\mathcal{D}}$ and $F : c \to \bar{S}d$ there exists a unique arrow $F' : r \to d$ of $\mathfrak{CB}_{\mathcal{D}}$ with $G \circ \bar{S}F' = F$.
Consider the category $Vect_K$ of all vector spaces over a field K, the category Set of all sets and the forgetful functor $U : Vect_K \to Set$. For $X \in \nu Set$, universal arrow from X to U is the pair (V_X, j) where V_X is the vector space generated by X and $j : X \to U(V_X)$ is the inclusion morphism.

Example 4.3. *Consider forgetful functor $\bar{U} : \mathfrak{CB}_{\mathcal{D}} \to \mathfrak{CB}_{\mathcal{C}}$ where $\mathcal{C} = Vect_K$ and $\mathcal{D} = Set_*$. Let $c : \cdots \to (X, x) \to (Y, y) \to \cdots \to \mathbf{0} \in \nu\mathfrak{CB}_{\mathcal{C}}$. A universal arrow from c to $\bar{U}$ is the pair $< r, \bar{J} >$ where $r : \cdots \to V_X \to V_Y \to \mathbf{0}$ and $\bar{J} : c \to \bar{U}(r)$ is defined as follows:*

$$\begin{array}{ccccccccc} c & & \cdots & \longrightarrow & (X,x) & \longrightarrow & (Y,y) & \longrightarrow & \mathbf{0} \\ \big\downarrow{\scriptstyle \bar{J}} & & & & \big\downarrow{\scriptstyle j_X} & & \big\downarrow{\scriptstyle j_Y} & & \big\downarrow{\scriptstyle 0} \\ \bar{U}(r) & & \cdots & \longrightarrow & (\bar{U}(V_X, 0_{V_X})) & \longrightarrow & (\bar{U}(V_Y, 0_{V_Y})) & \longrightarrow & \mathbf{0} \end{array}$$

where $j_X : (X, x) \to \bar{U}(V_X, 0_{V_X})$ is defined by $j_X(x) = 0, j_X(X - x) = I(X - x)$ and for $f : X \to Y$ with$f(x) = y$, $\bar{J}(f) : \bar{U}(V_X) \to \bar{U}(V_Y)$ is given by

$$\bar{J}(f)(j_X(x)) = j_Y(y)$$

$$\bar{J}(f)(\bar{U}(V_X) - X) = 0_{V_Y}$$

$$\bar{J}(f)(X - j_X(x)) = f(X - j_X(x))$$

Let $S : \mathcal{D} \to \mathcal{C}$ be a functor and $\bar{S}$ be the functor from $\mathfrak{CB}_{\mathcal{D}}$ to $\mathfrak{CB}_{\mathcal{C}}$ induced by S. Then the universal arrow in the base categories can be used to define the universal arrow in the corresponding chain bundle categories. Suppose that for each $C_i \in \nu\mathcal{C}$ there is a universal arrow $< D_i, g_i >$ form C_i to S, where $D_i \in \nu\mathcal{D}$ and $g_i : C_i \to S(D_i)$ is a retraction. Now, for $c : \cdots \to C_i \xrightarrow{A_i} C_j \to \cdots \to \mathbf{0} \in \nu\mathfrak{CB}_{\mathcal{C}}$, we have $< d, G >$ where $d : \cdots \to D_i \xrightarrow{Hom(D_i,D_j)} D_j \to \cdots \to \mathbf{0} \in \nu\mathfrak{CB}_{\mathcal{D}}$ and $G : c \to \bar{S}d$ with $\nu G = \{g_i\}$ and for $x \in A_i$, $G(x) = g_i^{-1} x g_j$ is a universal arrow from c to $\bar{S}$.

5 Category of Chains

Next we proceed to discuss the chains in the category of chain bundles $\mathfrak{CB}_{\mathcal{C}}$. A chain is a chain bundle having exactly one morphism in each homset. A category whose objects are chains and morphisms are appropriate chain maps is called category of chains.

Definition 5.1. *Let $\mathcal{C}$ be a category with zero object. A chain in $\mathcal{C}$ is*

$$\cdots M_3 \xrightarrow{s_3} M_2 \xrightarrow{s_2} M_1 \xrightarrow{s_1} M_0 = \mathbf{0}$$

where $M_i \in \nu\mathcal{C}$ and s_i is a morphism in $Hom(M_{i+1}, M_i)$ $\forall i$, which also consists of morphisms 1_{M_i} and all possible composite of morphisms.

Definition 5.2. *A chain map between two chains in $\mathcal{C}$ is a functor F between the two whose vertex map $\nu F = \{f_i : M_i \to N_i\}$ is a sequence of morphisms in $\mathcal{C}$ such that the below diagram commutes.*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M_3 & \xrightarrow{s_3} & M_2 & \xrightarrow{s_2} & M_1 & \xrightarrow{s_1} & \mathbf{0} \\ & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ \cdots & \longrightarrow & N_3 & \xrightarrow{s_3'} & N_2 & \xrightarrow{s_2'} & N_1 & \xrightarrow{s_1'} & \mathbf{0} \end{array}$$

Let $\mathcal{C}$ be a category with zero. If we take chains as objects and chain maps as morphisms, we get a category called category of chains which is actually subcategory of $\mathfrak{CB}_{\mathcal{C}}$.

Definition 5.3. *Given a chain c in the category $\mathcal{C}$*

$$c : \cdots M_3 \xrightarrow{s_3} M_2 \xrightarrow{s_2} M_1 \xrightarrow{s_1} M_0 = \mathbf{0}$$

then the chain

$$c' : \cdots M_3' \xrightarrow{s_3'} M_2' \xrightarrow{s_2'} M_1' \xrightarrow{s_1'} M_0' = \mathbf{0}$$

with M_i' a subobject of M_i and $(s_i')^0 = (j^{M_i}_{M_{i'}} s_i)^0$ is a subchain of c.

Definition 5.4. *A chain complex (C, ∂) is a graded abelian group $C = \{C_n\}$ together with an endomorphism $\partial = \{\partial_n\}$ of degree -1, called boundary homomorphism $\partial = \{\partial_n : C_n \to C_{n-1}\}$, such that $\partial^2 = 0$. (cf. [3])*

Example 5.5. *Consider the category $\mathcal{C}$ of graded abelian groups and the category $\mathfrak{CB}_{\mathcal{C}}$ of chain bundles in $\mathcal{C}$. Let the condition for choosing chain from a chain bundle in $\mathfrak{CB}_{\mathcal{C}}$ be as follows:*
For $c : \cdots C_{n+1} \to C_n \to C_{n-1} \to \cdots \to \mathbf{0}$ a chain bundle in $\mathfrak{CB}_{\mathcal{C}}$, choose one ∂_i from each $Hom(C_{i+1}, C_i)$ such that $\partial_{i+1} \circ \partial_i = 0$. By choosing such a ∂_i from each $Hom(C_i, C_{i+1})$ $\forall i$ we obtain a chain of the form

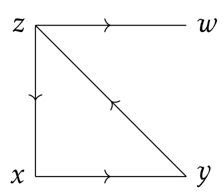
$$c_1 : \cdots C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to \mathbf{0}$$

and such chains in the chain bundle category together with chain maps is a category which we call category of chains.

Let $\mathcal{C}$ be a category with factorization and Γ be a category of chains obtained from $\mathfrak{CB}_{\mathcal{C}}$. Consider the chains $c : \cdots \to M_i \xrightarrow{s_i} M_j \to \cdots$ and $d : \cdots \to N_i \xrightarrow{t_i} N_j \to \cdots \in \nu\Gamma$, and the chain map $F : c \to d$ with $\nu F = \{f_i : M_i \to N_i\}$. Since each f_i admits a factorization in $\mathcal{C}$, F may admit a factorization of the form $F = F^0 J$ as in case of factorization of chain bundle map depending on the choice of category of chains.

In the following we provide an example of a category of chains with subobjects and factorization.

Example 5.6. *Let **Csim** denote category of simplicial complexes and simplicial maps. Consider the following simplicial complex.*



*Let $\mathcal{C}'$ be the subcategory of **Csim** whose objects are set of all subcomplexes of above simplicial complex and morphisms are simplicial maps between them. Consider the homology functor $H_*(-,\mathbb{Z}) : \textbf{Csim} \to \textbf{Ab}^{\mathbb{Z}}$, where $\textbf{Ab}^{\mathbb{Z}}$ denotes category of graded abelian groups and related morphisms (see cf.[3]). $Im(j_{\mathcal{C}'}^{\textbf{Csim}} H_*(-,\mathbb{Z}))$ is a subcategory of $\textbf{Ab}^{\mathbb{Z}}$ denoted by $\mathcal{C}$. Let $\mathfrak{CB}_{\mathcal{C}}$ be category of chain bundles in $\mathcal{C}$. By choosing chain complexes from $\mathfrak{CB}_{\mathcal{C}}$ we obtain a category Γ with subobjects and factorization. Being a (chain) subcomplex is a subobject relation and any chain map F,*

$$
\begin{array}{ccccccccc}
c:\cdots & \longrightarrow & C_2' & \xrightarrow{\partial_2} & C_1' & \xrightarrow{\partial_1} & C_0' & \xrightarrow{\partial_0} & \mathbf{0} \\
\downarrow{\scriptstyle F} & & {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle 0} \\
d:\cdots & \longrightarrow & C_2 & \xrightarrow{\partial_2'} & C_1 & \xrightarrow{\partial_1'} & C_0 & \xrightarrow{\partial_0'} & \mathbf{0}
\end{array}
$$

admits a factorization given by

$$
\begin{array}{ccccccccc}
c:\cdots & \longrightarrow & C_2' & \xrightarrow{\partial_2} & C_1' & \xrightarrow{\partial_1} & C_0' & \xrightarrow{\partial_0} & \mathbf{0} \\
\downarrow{\scriptstyle F^0} & & {\scriptstyle f_2^0}\downarrow & & \downarrow{\scriptstyle f_1^0} & & \downarrow{\scriptstyle f_0^0} & & \downarrow{\scriptstyle 0} \\
c':\cdots & \longrightarrow & F^0(C_2') & \xrightarrow{\partial'_{2|F^0(C_2')}} & F^0(C_1') & \xrightarrow{\partial'_{1|F^0(C_1')}} & F^0(C_0') & \xrightarrow{\partial'_{0|F^0(C_0')}} & \mathbf{0} \\
\downarrow{\scriptstyle J} & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle 0} \\
d:\cdots & \longrightarrow & C_2 & \xrightarrow{\partial_2'} & C_1 & \xrightarrow{\partial_1'} & C_0 & \xrightarrow{\partial_0'} & \mathbf{0}
\end{array}
$$

In particular, consider $K_1, K_2 \in \nu\mathcal{C}'$ where

$$K_1 = \{\{x,y,z,w\},\{(xyzw),(xyz),(xy),(yz),(zx),(zw),(x),(y),(z),(w)\}\}$$

and

$$K_2 = \{\{x,y,z,w\},\{(yz),(zx),(zw),(x),(y),(z),(w)\}\}$$

Let $f : K_1 \to K_2$ be the simplicial map given by

$$x \mapsto y, y \mapsto z, z \mapsto z, w \mapsto z$$

$C(K_1) = \{C_n(K_1)\}$ and $C(K_2) = \{C_n(K_2)\}$ be chain complexes corresponding to K_1 and K_2. $C_n(f) : C_n(K_1) \to C_n(K_2)$ is given by

$$C_n(f)(\{x,y,z,w\}) = \begin{cases} \{f(x), f(y), f(z), f(w)\}, & \textit{if images are distinct.} \\ 0, & \textit{otherwise.} \end{cases}$$

Consider the chain map $C(f) : C(K_1) \to C(K_2)$ in Γ induced by f:

$$
\begin{array}{ccccccccccc}
C(K_1): & & 0 & \xrightarrow{0} & <xyz> & \xrightarrow{\partial_3'} & <xy,yz,zx,zw> & \xrightarrow{\partial_2'} & <x,y,z,w> & \xrightarrow{\partial_1'} & \mathbf{0} \\
\downarrow{\scriptstyle C(f)} & & {\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle C_2(f)}\downarrow & & \downarrow{\scriptstyle C_1(f)} & & \downarrow{\scriptstyle 0} \\
C(K_2): & & 0 & \xrightarrow{0} & 0 & \xrightarrow{\partial_3} & <yz,zx,zw> & \xrightarrow{\partial_2} & <x,y,z,w> & \xrightarrow{\partial_1} & \mathbf{0}
\end{array}
$$

Factorization of $C(f) : C(K_1) \to C(K_2)$ is given by:

$$
\begin{array}{ccccccccccc}
C(K_1): & & 0 & \xrightarrow{0} & <xyz> & \xrightarrow{\partial'_3} & <xy,yz,zx,zw> & \xrightarrow{\partial'_2} & <x,y,z,w> & \xrightarrow{\partial'_1} & \boldsymbol{0} \\
\big\downarrow{\scriptstyle C(f)^0} & & {\scriptstyle 0}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & {\scriptstyle C_2(f)^0}\big\downarrow & & \big\downarrow{\scriptstyle C_1(f)^0} & & \big\downarrow{\scriptstyle 0} \\
C(K_3): & & 0 & \xrightarrow{0} & 0 & \xrightarrow{\partial''_3} & <yz> & \xrightarrow{\partial''_2} & <y,z> & \xrightarrow{\partial''_1} & \boldsymbol{0} \\
\big\downarrow{\scriptstyle J} & & {\scriptstyle 0}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle 0} \\
C(K_2): & & 0 & \xrightarrow{0} & 0 & \xrightarrow{\partial_3} & <yz,zx,zw> & \xrightarrow{\partial_2} & <x,y,z,w> & \xrightarrow{\partial_1} & \boldsymbol{0}
\end{array}
$$

where $C(K_3)$ *is the chain complex corresponding to simplicial complex* $K_3 = \{\{y,z\},\{(yz),(y),(z)\}\}$

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