Moroccan Journal of Algebra and Geometry with Applications
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco
Volume 1, Issue 2 (2022), pp 286-360
ISSN: 2820-7114

Title :

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Cotorsion theory and its application to ring structures - a book chapter

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Communicated by Najib Mahdou
(Received 02 April 2022, Revised 20 July 2022, Accepted 24 July 2022)

Abstract. In this survey article, we introduce the properties of the cotorsion theory and show how to construct the homology theory for all cotorsion theories. This can be considered the final chapter of the author's book entitled "Foundations of Commutative Rings and Their Modules" published by Springer in 2016.

Key Words: cotorsion theory, hereditary cotorsion theory, complete cotorsion theory, perfect cotorsion theory, pure submodule, Tor-orthocomplement, (pre)cover, (pre)envelope, resolving class, coresolving class, weak w-projective module, Kaplansky's theorem, w-split module, n-cotorsion module, n-torsion-free module, n-Warfield cotorsion module, weak finitistic dimension, Matlis cotorsion module, Matlis domain, Almost perfect domain, n-perfect ring, strongly flat module, Prüfer domain, G-Dedekind domain.

2010 MSC: Primary 13C10, 13DXX; Secondary 13C11, 13C12. 13C13, 13F05, 13F30, 13G05, 13H99.

0. Introduction

Cotorsion theory is a hot topic in homological algebra. Validation of the cotorsion theory shows that classical homological methods can be transferred to many classes of modules. Therefore many scholars believe that this is a relative homology theory, and Gorenstein homology theory is only a special form of the relative homology theory. The formation and completion of the cotorsion theory lies in solving the so-called flat cover conjecture (FCC) in [7] in 2001. The FCC is committed to Enochs, who has been worked since 1984. Since flatness and injectivity are linked through character modules, the researchers believe that each module has a flat cover. Finally this conclusion is solved by proving that the class of flat modules and the class of cotorsion modules form a cotorsion theory. In this survey article, we introduce the properties of the cotorsion theory and show how to construct the homology theory for all cotorsion theories.

The class of modules mentioned in this article refers to a full subcategory, which is closed isomorphisms, of the R-module category $\mathfrak{M}$. We always denote by $\mathcal{P}$ the class of projective modules, by $\mathcal{I}$ the class of injective modules, by $\mathcal{F}$ the class of flat modules. We also represent by $\mathcal{P}_n$, $\mathcal{I}_n$ and $\mathcal{F}_n$ respectively the class of modules whose projective dimension, injective dimension and flat dimension is at most n.

1 Generalized purity of modules

1.1 Pure exact sequences

A flat module over a ring R is an R-module M such that taking the tensor product over R with M preserves exact sequences. Below we discuss the converse property.

Definition 1.1. Let $\mathcal{L}$ be a class of modules.

(1) An exact sequence $\xi : 0 \to A \to B \to C \to 0$ is called an $\mathcal{L}$-**pure exact sequence** if for any $M \in \mathcal{L}$,

$$M \otimes_R \xi : 0 \longrightarrow M \otimes_R A \longrightarrow M \otimes_R B \longrightarrow M \otimes_R C \longrightarrow 0$$

is also an exact sequence.

(2) A submodule A of an R-module B is called an $\mathcal{L}$-**pure submodule** if the exact sequence $0 \to A \to B \to B/A \to 0$ is $\mathcal{L}$-pure.

(3) A monomorphism $f : A \to B$ is called an $\mathcal{L}$-**pure monomorphism** if the exact sequence $0 \to A \xrightarrow{f} B \to \mathrm{Coker}(f) \to 0$ is $\mathcal{L}$-pure.

(4) When $\mathcal{L} = \mathfrak{M}$, an $\mathcal{L}$-pure exact sequence is called a pure exact sequence, an $\mathcal{L}$-pure submodule is called a pure submodule, and an $\mathcal{L}$-pure monomorphism is called a pure monomorphism.

Example 1.2. Let $\mathcal{L}$ be any class of modules. Then every split exact sequence and every pure exact sequence are $\mathcal{L}$-pure.

Lemma 1.3. *Let $x_1, \ldots, x_m$ be a basis of R^m, $e_1, \ldots, e_n$ be a basis of R^n, and $\alpha : R^m \to R^n$ be a homomorphism. Write $\alpha(x_i) = \sum_{j=1}^{n} r_{ij} e_j$, $r_{ij} \in R$, $i = 1, \ldots, m$. Let $R^m \xrightarrow{\alpha} R^n \xrightarrow{\beta} N \to 0$ be an exact sequence and let B be an R-module. If in $N \otimes_R B$ we have*

$$\sum_{j=1}^{n} \beta(e_j) \otimes b_j = 0, \quad b_j \in B,$$

then there exist $u_1, \ldots, u_m \in B$ such that $b_j = \sum_{i=1}^{n} r_{ij} u_i$, $j = 1, \ldots, n$.

Proof. Let $K = \mathrm{Ker}(\beta)$. Then K is a submodule of R^n generated by $\{\sum_{j=1}^{n} r_{ij} e_j\}_{i=1}^{m}$ and $0 \to K \to R^n \xrightarrow{\beta} N \to 0$ is an exact sequence. Thus $K \otimes_R B \to F \otimes_R B \to N \otimes_R B \to 0$ is an exact sequence. Hence there is $u_i \in B$, $i = 1, \ldots, m$, such that

$$\sum_{j=1}^{n} e_j \otimes b_j = \sum_{i=1}^{m} (\sum_{j=1}^{n} r_{ij} e_j) \otimes u_i = \sum_{j=1}^{n} e_j \otimes (\sum_{i=1}^{m} r_{ij} u_i).$$

By [22, Exercise 2.7], $b_j = \sum_{i=1}^{n} r_{ij} u_i$, $j = 1, \ldots, n$. □

For a finitely presented module N, we can assume that there exist a free module F with its basis $e_1, \ldots, e_n$ and an epimorphism $\beta : F \to N$. Set $K = \mathrm{Ker}(\beta)$. Then K is finitely generated with its generating system $y_i = \sum_{j=1}^{n} r_{ij} e_j$, $i = 1, \ldots, m$. Moreover for any module X, like [22, Exercise 3.27], its character module is denoted by $X^+ = \mathrm{Hom}_R(X, \mathbb{Q}/\mathbb{Z})$. As in [22, Example 2.1.27], there exists an evaluation map $\rho : X \to X^{++}$:

$$\rho(x)(f) = f(x), \qquad x \in X, \quad f \in X^+.$$

Theorem 1.4. The following are equivalent for an exact sequence $\xi : 0 \to A \to B \xrightarrow{g} C \to 0$:

(1) ξ is a pure exact sequence.

(2) For any finitely presented module N, the induced sequence $N \otimes_R \xi$ is exact.

(3) Let $a_j = \sum\limits_{i=1}^{m} r_{ij} b_i$, $r_{ij} \in R$, $a_j \in A$, $b_i \in B$, $j = 1, \dots, n$. Then there exist $v_i \in A$ such that $a_j = \sum\limits_{i=1}^{m} r_{ij} v_i$. In other words, for any integers m, n, if any system of linear equations

$$(S_{mn}): \sum_{i=1}^{m} r_{ij} x_i = a_j, \qquad r_{ij} \in R, \quad a_j \in A, i = 1, \dots,, n \tag{1.1}$$

has a solution in B, the equations must have a solution in A.

(4) For any commutative diagram of the form:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & F & \xrightarrow{\beta} & N & \longrightarrow & 0 \\
 & & \sigma\downarrow & \overset{\gamma}{\swarrow} & \downarrow\tau & & \downarrow h & & \\
0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{g} & C & \longrightarrow & 0
\end{array}$$

where N is a finitely presented module and F is a finitely generated free module, there is a homomorphism $\gamma : F \to A$ such that $\gamma|_K = \sigma$.

(5) For any finitely presented module N, the induced sequence

$$\mathrm{Hom}_R(N, \xi) : 0 \to \mathrm{Hom}_R(N, A) \to \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C) \to 0$$

is exact.

(6) The induced exact sequence $\xi^+ : 0 \to C^+ \to B^+ \to A^+ \to 0$ is split.

Proof. (1)⇒(3) Let F be a free module with its basis $e_1, \dots, e_n$, K be a submodule of F generated by $\{\sum\limits_{j=1}^{n} r_{ij} e_j \mid i = 1, \dots, m\}$ and $N := F/K$. Let $\beta : F \to N$ be the natural homomorphism. Then in $N \otimes_R B$, we have

$$\sum_{j=1}^{n} \beta(e_j) \otimes a_j = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta(e_j) \otimes r_{ij} b_i = \sum_{i=1}^{m} (\sum_{j=1}^{n} r_{ij} \beta(e_j)) \otimes b_i = 0.$$

Since A is a pure submodule of B, we also have, in $N \otimes_R A$, $\sum\limits_{j=1}^{n} \beta(e_j) \otimes a_j = 0$. By Lemma 1.3, there exist $v_i \in A$, $i = 1, \dots, m$, such that $a_j = \sum\limits_{i=1}^{m} r_{ij} v_i$.

(3)⇒(2) Let $a_j \in A$ such that in $N \otimes_R B$, we have $\sum\limits_{j=1}^{n} \beta(e_j) \otimes a_j = 0$. Also assume $N = F/K$. By Lemma 1.3, there exist $b_i \in B$, $i = 1, \dots, m$, such that $a_j = \sum\limits_{i=1}^{m} r_{ij} b_i$. By the hypothesis, there exist $v_i \in A$, $i = 1, \dots, n$, such that $a_j = \sum\limits_{i=1}^{n} r_{ij} v_i$. Hence in $N \otimes_R A$, we have

$$\sum_{j=1}^{n} \beta(e_j) \otimes a_j = \sum_{i=1}^{m} (\sum_{j=1}^{n} r_{ij} \beta(e_j)) \otimes v_i = 0.$$

Therefore $N \otimes_R A \to N \otimes_R B$ is a monomorphism.

(2)⇒(1) Let M be any R-module. By [22, Theorem 2.6.20], $M = \varinjlim\{N_i \mid i \in \Gamma\}$, where Γ is a directed set and $\{N_i \mid i \in \Gamma\}$ is a direct system of finitely presented modules over Γ. By the hypothesis, $0 \to N_i \otimes_R A \to N_i \otimes_R B \to N_i \otimes_R C \to 0$ is an exact sequence for any $i \in \Gamma$. By [22, Theorem 2.5.33] and [22, Theorem 2.5.34], $M \otimes_R \xi$ is an exact sequence.

(3)⇒(4) Still let $e_1, \dots, e_n$ be a basis of F and set $y_i := \sum_{j=1}^{n} r_{ij} e_j$ for each $j = 1, \dots, m$ and let K be generated by $\{y_i\}_{i=1}^{m}$. Write $\tau(e_j) = b_j$ and $a_i = \sigma(y_i)$. Then $a_i = \sum_{j=1} r_{ij} b_j$. By the hypothesis, there exist $v_j \in A$, $j = 1, \dots, n$, such that $a_i = \sum_{j=1}^{n} r_{ij} v_j$, $i = 1, \dots, m$. Set $\gamma(e_j) = v_j$, $j = 1, \dots, n$. Then $\gamma(y_i) = \gamma(\sum_{j=1} r_{ij} e_j) = \sum_{j=1} r_{ij} v_j = a_i = \sigma(y_i)$. Therefore $\gamma|_K = \sigma$.

(4)⇒(3) Let $a_j = \sum_{i=1}^{m} r_{ij} b_i$, $r_{ij} \in R$, $a_j \in A$, $b_i \in B$, $j = 1, \dots, n$. Let F be a free module with its basis $x_1, \dots, x_m$, and set $z_j := \sum_{i=1}^{m} r_{ij} x_i$ for each $i = 1, \dots, n$ and let K be a submodule of F generated by $\{z_j\}_{j=1}^{n}$, and $N = F/K$. Set $\tau(x_i) = b_i$ and $\sigma(z_j) = a_j$. If $c_j \in R$, $\sum_{j=1}^{n} c_j z_j = 0$, then for any i, $\sum_{j=1} c_j r_{ij} = 0$. Thus $\sum_{j=1}^{n} c_j a_j = \sum_{i=1}^{m} \sum_{j=1}^{n} c_j r_{ij} b_i = 0$. Hence σ is a well-defined homomorphism. It is easy to see that σ and τ give the hypothesis of the commutative diagram. By the hypothesis, there exists a homomorphism $\gamma : F \to A$ such that $\gamma|_K = \sigma$. Write $v_i = \gamma(x_i)$. Then $a_j = \gamma(z_j) = \sum_{i=1}^{m} r_{ij} \gamma(x_i) = \sum_{i=1}^{m} r_{ij} v_i$.

(4)⇒(5) Let $h \in \mathrm{Hom}_R(N, C)$. It is easy to construct conditions given in the commutative diagram. By [22, Exercise 1.60], there exists a homomorphism $h' : N \to B$ such that $h = gh' = g_*(h')$. Therefore $g_* : \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C)$ is an epimorphism.

(5)⇒(4) This follows from [22, Exercise 1.60].

(1)⇒(6) Since ξ is a pure exact sequence, for any module M, $(M \otimes_R B)^+ \to (M \otimes_R A)^+ \to 0$ is an exact sequence. Taking $M = A^+$, by [22, Theorem 2.2.16], $\mathrm{Hom}_R(A^+, B^+) \to \mathrm{Hom}_R(A^+, A^+) \to 0$ is an exact sequence. Since $\mathbf{1}_{A^+} \in \mathrm{Hom}_R(A^+, A^+)$, the exact sequence ξ^+ is split.

(6)⇒(1) Since ξ^+ is a split exact sequence, $\mathrm{Hom}_R(M, \xi^+)$ is an exact sequence for any module M. By [22, Theorem 2.2.16], we get that ξ is a pure exact sequence. □

Theorem 1.5. (**Bican-Bashir-Enochs**) Let M be an R-module and κ be an infinite cardinal number with $\kappa \geqslant |R|$.

(1) Let X be a submodule of M with $|X| \leqslant \kappa$. Then there exists a pure submodule N of M containing X such that $|N| \leqslant \kappa$.

(2) There exists a continuous ascending chain of pure submodules of M

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\alpha \subset M_{\alpha+1} \subset \cdots \subset M_\tau = M,$$

such that $|M_{\alpha+1}/M_\alpha| \leqslant \kappa$.

Proof. (1) Let N_0 be a submodule generated by X. Then trivially $|N_0| \leqslant \kappa$. Inductively, let i be a natural number and let a module N_i be given. Let N_{i+1} be a submodule of M generated by the solution x_i of a system of linear equations (1.1) (S_{mn}) (where $a_j \in N_i$) and N_i. Since $|R| \leqslant \kappa$, we have $|N_{i+1}| \leqslant \kappa$. Set $N = \bigcup_{i=0}^{\infty} N_i$. Then $|N| \leqslant \kappa$, and it follows from Theorem 1.4 that N is a pure submodule of M with $X \subseteq N$.

(2) Without loss of generality, we assume that $M \neq 0$. Set $M_0 := 0$. By (1), there exists a nonzero pure submodule M_1 of M such that $|M_1| \leqslant \kappa$. Again by (1), there exists a nonzero pure submodule M_2/M_1 of M/M_1 such that $|M_2/M_1| \leqslant \kappa$. By Exercise 5, M_2 is a pure submodule of M. If this goes on, it is proved by transfinite induction. □

Definition 1.6. Let $\mathcal{L}$ be a class of modules and let L be an R-module.
(1) L is called an **$\mathcal{L}$-injective module** if $\mathrm{Ext}_R^1(X,L) = 0$ for any $X \in \mathcal{L}$.
(2) L is called an **$\mathcal{L}$-flat module** if $\mathrm{Tor}_1^R(L,X) = 0$ for any $X \in \mathcal{L}$.

Example 1.7. (1) If $\mathcal{L}$ denotes the class of finitely generated modules, then an $\mathcal{L}$-injective module is just injective.

(2) If $\mathcal{L}$ denotes the class of finitely presented modules, then an $\mathcal{L}$-injective module is just an FP-injective module.

Theorem 1.8. Let $\mathcal{L}$ be a class of modules. Then a module L is $\mathcal{L}$-injective if and only if any exact sequence of the form $\xi : 0 \to L \to B \to C \to 0$ is split, where $C \in \mathcal{L}$.

Proof. Assume that L is an $\mathcal{L}$-injective module. Since $C \in \mathcal{L}$, $\mathrm{Hom}_R(\xi, L)$ is an exact sequence. It follows from the fact that $\mathbf{1} \in \mathrm{Hom}_R(L,L)$ that this exact sequence is split.

Assume that the converse condition is satisfied. For any $C \in \mathcal{L}$, by [22, Theorem 3.3.5], $\mathrm{Ext}_R^1(C,L) = 0$. Therefore L is $\mathcal{L}$-injective. □

1.2 Pure injective modules

Definition 1.9. Let $\mathcal{L}$ be a class of modules and let L be an R-module. Then L is called an **$\mathcal{L}$-pure injective module** if for any $\mathcal{L}$-pure exact sequence $\xi: 0 \to A \to B \to C \to 0$, the induced sequence $\mathrm{Hom}_R(\xi, L): 0 \to \mathrm{Hom}_R(C,L) \to \mathrm{Hom}_R(B,L) \to \mathrm{Hom}_R(A,L) \to 0$ is also exact. When $\mathcal{L} = \mathfrak{M}$, an $\mathcal{L}$-pure injective module is just a pure injective module.

Example 1.10. Let $\mathcal{L}$ be a class of modules.

(1) Since every pure exact sequence is an $\mathcal{L}$-pure exact sequence, every $\mathcal{L}$-pure injective module is pure injective.

(2) Every injective module is $\mathcal{L}$-pure injective. We denote by $\mathcal{PI}$ the class of pure injective modules. Then $\mathcal{I} \subseteq \mathcal{PI}$.

(3) Any direct product of $\mathcal{L}$-pure injective modules is also $\mathcal{L}$-pure injective.

(4) For any $A \in \mathcal{L}$, its character module A^+ is $\mathcal{L}$-pure injective, and thus A^+ is pure injective.

Proof. We only prove (4). Let $0 \to X \to Y \to Z \to 0$ be an $\mathcal{L}$-pure exact sequence. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (A\otimes_R Z)^+ & \longrightarrow & (A\otimes_R Y)^+ & \longrightarrow & (A\otimes_R X)^+ & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(Z,A^+) & \longrightarrow & \mathrm{Hom}_R(Y,A^+) & \longrightarrow & \mathrm{Hom}_R(X,A^+) & \longrightarrow & 0 \end{array}$$

By the hypothesis, the top row is exact. By [22, Theorem 2.2.16], three vertical arrows are isomorphisms. By [22, Exercise 1.59], the bottom row is exact. Therefore A^+ is $\mathcal{L}$-pure injective. □

Theorem 1.11. Let $\mathcal{L}$ be a class of modules. Then the following are equivalent for a module L:

(1) L is an $\mathcal{L}$-pure injective module.

(2) Any $\mathcal{L}$-pure exact sequence of the form $\xi : 0 \to L \to M \to C \to 0$ is split.

Proof. (1)⇒(2) This follows from the facts that $\mathrm{Hom}_R(\xi, L)$ is an exact sequence and $\mathbf{1}_L \in \mathrm{Hom}_R(L, L)$.

(2)⇒(1) Let $0 \to A \to B \to C \to 0$ be an $\mathcal{L}$-pure exact sequence and let $h : A \to L$ be a homomorphism. By [22, Theorem 1.9.19], we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & h\downarrow & & \downarrow\beta & & \| & & \\
0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\gamma} & C & \longrightarrow & 0
\end{array}$$

Let N be a finitely presented module. Use the functor $\mathrm{Hom}_R(N, -)$ to act on the above commutative diagram to get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(N,A) & \longrightarrow & \mathrm{Hom}_R(N,B) & \xrightarrow{g_*} & \mathrm{Hom}_R(N,C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Hom}_R(N,L) & \longrightarrow & \mathrm{Hom}_R(N,M) & \xrightarrow{\gamma_*} & \mathrm{Hom}_R(N,C) & &
\end{array}$$

From the right side of the above diagram, we get that γ_* is an epimorphism, and thus $0 \to L \to M \to C \to 0$ is a pure exact sequence. By hypothesis, there exists a homomorphism $\sigma : M \to L$ such that $\sigma\alpha = \mathbf{1}$. Thus $\tau := \sigma\beta : B \to L$, and $\tau f = \sigma\beta f = \sigma\alpha h = h$. Therefore L is an $\mathcal{L}$-pure injective module. □

Corollary 1.12. *Let $\mathcal{L}$ be a class of modules and let L be an $\mathcal{L}$-pure injective module. If L is an $\mathcal{L}$-pure submodule of an injective module, then L is injective.*

Corollary 1.13. *The following are equivalent for a module L:*

(1) L is a pure injective module.

(2) Any pure exact sequence of the form $0 \to L \to B \to C \to 0$ is split.

(3) The natural homomorphism $\rho : L \to L^{++}$ is a split monomorphism, that is, L is a direct summand of L^{++}.

(4) L is a direct summand of some X^+.

Let $\mathcal{L}$ be a class of modules. Define:

$$\mathcal{L}^{\top} = \{D \in \mathfrak{M} \mid \mathrm{Tor}_1^R(L, D) = 0 \text{ for any } L \in \mathcal{L}\},$$

which is called a Tor-**orthocomplement** of $\mathcal{L}$. Trivially each $D \in \mathcal{L}^{\top}$ is an $\mathcal{L}$-flat module. Note that, since we assume that R is a commutative ring, we don't need to define ${}^{\top}\mathcal{L}$.

Theorem 1.14. Let $\mathcal{L}$ be a class of R-modules. Then the following are equivalent for an R-module D:

(1) D is an $\mathcal{L}$-flat module.

(2) D^+ is an $\mathcal{L}$-injective module, that is, $\mathrm{Ext}_R^1(A, D^+) = 0$ for any $A \in \mathcal{L}$.

(3) Any exact sequence of the form $\eta : 0 \to X \to Y \to D \to 0$ is $\mathcal{L}$-pure.

(4) If $\xi : 0 \to X \to Y \to A \to 0$ is an exact sequence with $A \in \mathcal{L}$, then the induced sequence $\xi \otimes_R D$ is exact.

(5) Any $\mathcal{L}$-pure exact sequence of the form $\eta: 0 \to X \to F \to D \to 0$ is $\mathcal{L}$-pure, where F is projective.

(6) Any $\mathcal{L}$-pure exact sequence of the form $\eta: 0 \to X \to F \to D \to 0$ is $\mathcal{L}$-pure, where F is flat.

(7) There exists an $\mathcal{L}$-pure exact sequence $\eta: 0 \to K \to F \to D \to 0$, where F is flat.

(8) There exists an $\mathcal{L}$-pure exact sequence $\eta: 0 \to K \to F \to D \to 0$, where F is $\mathcal{L}$-flat.

Proof. We only prove (1)⇔(2)⇔(3), and leave the rest to the reader.

(1)⇒(2) By [22, Theorem 3,4,11], for any $A, B \in \mathfrak{M}$, we have the natural isomorphism:

$$\mathrm{Ext}_R^1(A, B^+) \cong \mathrm{Tor}_1^R(B, A)^+. \tag{1.2}$$

When $A \in \mathcal{L}$ and $D \in \mathcal{L}^\top$, we have $\mathrm{Tor}_1^R(A, D) = 0$. It follows from (1.2) that $\mathrm{Ext}_R^1(A, D^+) = 0$.

(2)⇒(3) It can be obtained by backward deduction from the isomorphism (1.2).

(1)⇒(3) For any $A \in \mathcal{L}$, this follows from the exact sequence

$$0 = \mathrm{Tor}_1^R(A, D) \to A \otimes_R X \to A \otimes_R Y \to A \otimes_R D \to 0.$$

(3)⇒(1) Let F be a flat module and let $\eta: 0 \to X \to F \to D \to 0$ be an exact sequence. By the hypothesis, η is an $\mathcal{L}$-pure exact sequence. Thus $A \otimes_R \eta$ is an exact sequence for any module $A \in \mathcal{L}$. Since

$$0 \longrightarrow \mathrm{Tor}_1^R(A, D) \longrightarrow A \otimes_R X \longrightarrow A \otimes_R F \longrightarrow A \otimes_R D \longrightarrow 0$$

is also an exact sequence, $\mathrm{Tor}_1^R(A, D) = 0$. Therefore D is $\mathcal{L}$-flat. □

Corollary 1.15. *The following are equivalent for an R-module D:*

(1) D is a flat module.

(2) D^+ is an injective module.

(3) Any exact sequence of the form $0 \to X \to Y \to D \to 0$ is pure.

(4) Any exact sequence of the form $0 \to X \to F \to D \to 0$ is pure, where F is flat.

Definition 1.16. Let $\mathcal{L}$ be a class of modules.

(1) $\mathcal{L}$ is said to be **closed under extensions** if for any exact sequence $0 \to A \to B \to C \to 0$, $A, C \in \mathcal{L}$ imply that $B \in \mathcal{L}$.

(2) $\mathcal{L}$ is said to be **closed under direct sums** (resp., **direct products**) if whenever $\{C_i\}$ is a family of modules in $\mathcal{L}$, we have $\bigoplus_i C_i \in \mathcal{L}$ (resp., $\prod_i C_i \in \mathcal{L}$).

(3) $\mathcal{L}$ is said to be **closed under direct summands** if $C_1 \oplus C_2 \in \mathcal{L}$ implies that $C_1, C_2 \in \mathcal{L}$.

(4) $\mathcal{L}$ is said to be **closed under direct limits** if whenever $\{(C_i, \varphi_{ij}) \mid, i \in \Gamma\}$ is a direct system, where Γ is a directed set and $C_i \in \mathcal{L}$, we have $\varinjlim C_i \in \mathcal{L}$.

(5) $\mathcal{L}$ is said to be **closed under kernels of epimorphisms** if whenever $0 \to A \to B \to C \to 0$ is an exact sequence, where $B, C \in \mathcal{L}$, we have $A \in \mathcal{L}$.

(6) $\mathcal{L}$ is said to be **closed under cokernels of monomorphisms** if whenever $0 \to A \to B \to C \to 0$ is an exact sequence, where $A, B \in \mathcal{L}$, we have $C \in \mathcal{L}$.

Proposition 1.17. *Let $\mathcal{L}$ be a class of modules.*

(1) *If $\mathcal{P} \subseteq \mathcal{L}$ and $\mathcal{L}$ is closed under kernels of epimorphisms, then a module D is an $\mathcal{L}$-flat module if and only if* $\operatorname{Tor}_k^R(L,D) = 0$ *for any $L \in \mathcal{L}$ and any $k > 0$.*

(2) *Let F be an $\mathcal{L}$-flat module and X be an $\mathcal{L}$-pure submodule of F. Then F/X is an $\mathcal{L}$-flat module. In addition, if $\mathcal{P} \subseteq \mathcal{L}$ and $\mathcal{L}$ is closed under kernels of epimorphisms, then X is also an $\mathcal{L}$-flat module. In particular, let F be a flat module and X be a pure submodule of F. Then F/X and X are both flat modules.*

Proof. Exercise. □

Theorem 1.18. Let $\mathcal{L}$ be a class of R-modules and let $\xi : 0 \to A \to B \to C \to 0$ be an exact sequence. Then the following are equivalent:

(1) ξ is $\mathcal{L}$-pure.

(2) $\operatorname{Hom}_R(\xi, L)$ is an exact sequence for any $\mathcal{L}$-pure injective module L.

(3) $\operatorname{Hom}_R(\xi, M^+)$ is an exact sequence for any $M \in \mathcal{L}$.

(4) $0 \to \operatorname{Hom}_R(M, C^+) \to \operatorname{Hom}_R(M, B^+) \to \operatorname{Hom}_R(M, A^+) \to 0$ is an exact sequence for any $M \in \mathcal{L}$.

Proof. (1)⇒(2) Trivial.

(2)⇒(3) By Example 1.10, M^+ is $\mathcal{L}$-pure injective. Now the assertion follows from the hypothesis.

(3)⇒(4) For any module X, it follows from [22, Theorem 2.2.16] that there is an isomorphism

$$\operatorname{Hom}_R(M, X^+) \cong (M \otimes_R X)^+ \cong \operatorname{Hom}_R(X, M^+).$$

Therefore the assertion is true.

(4)⇒(1) By [22, Theorem 2.2.16], $(M \otimes_R \xi)^+$ is an exact sequence. By [22, Exercise 3.27], $M \otimes_R \xi$ is an exact sequence. Therefore ξ is an $\mathcal{L}$-pure exact sequence. □

Corollary 1.19. *Let $\xi : 0 \to A \to B \to C \to 0$ be an exact sequence. Then ξ is pure if and only if the induced sequence* $\operatorname{Hom}_R(\xi, L)$ *is exact for any pure injective module L..*

Theorem 1.20. Let $\mathcal{L}$ be a class of R-modules. Then every $\mathcal{L}$-pure injective module is $\mathcal{L}^\top$-injective.

Proof. Let E be an $\mathcal{L}$-pure injective module. For any $\mathcal{L}$-flat module D, by Theorem 1.14, there is an $\mathcal{L}$-pure exact sequence $\xi : 0 \to K \to P \to D \to 0$, where P is a projective module. Since $\operatorname{Hom}_R(\xi, E)$ is an exact sequence, $\operatorname{Ext}_R^1(D, E) = 0$. Therefore E is $\mathcal{L}^\top$-injective. □

Theorem 1.21. Let $\mathcal{L}$ be a class of R-modules. Then the following are equivalent for an R-module D:

(1) D is an $\mathcal{L}$-flat module.

(2) $\operatorname{Ext}_R^1(D, U) = 0$ for any $\mathcal{L}^\top$-injective module U.

(3) $\operatorname{Ext}_R^1(D, V) = 0$ for any $\mathcal{L}$-pure injective module V.

Proof. Exercise. □

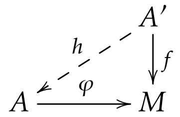
2 Approximation theory of a class of modules

Approximation theory of module classes is known as a cover and an envelope theory of modules. It can be traced back to: In 1940, Baer *et al.* constructed a theory of injective envelopes and in 1960, Bass construct a projective cover and characterized perfect rings. By good properties of the cover and the envelope, after giving impetus by Enochs *et al.*, approximation theory of classes of modules has been already demonstrated important applications in many problem solving.

2.1 Precovers and covers of modules

Definition 2.1. Let $\mathcal{A}$ be a class of modules, M be an R-modules, $A \in \mathcal{A}$, and $\varphi : A \to M$ be a homomorphism.

(1) (A, φ) is called an **$\mathcal{A}$-precover** of M if for any $A' \in \mathcal{A}$, the following diagram can be completed into a commutative diagram,



equivalently $\mathrm{Hom}_R(A', A) \xrightarrow{\varphi_*} \mathrm{Hom}_R(A', M) \to 0$ is an exact sequence for any $A' \in \mathcal{A}$.

(2) Let (A, φ) be an $\mathcal{A}$-precover of M. Then (A, φ) is called an **$\mathcal{A}$-cover** of M provided that $A' = A$ and $f = \varphi$ imply that h is an automorphism of A.

Theorem 2.2. Let $\mathcal{A}$ be a class of modules and let M be an R-module. If an $\mathcal{A}$-cover of M exists, then $\mathcal{A}$-covers of M are isomorphic, in other words, an $\mathcal{A}$-cover of M is unique up to isomorphism if it exists.

Proof. Let (A_1, φ_1) and (A_2, φ_2) be $\mathcal{A}$-covers of M. Then there exist a homomorphism $f : A_1 \to A_2$ such that $\varphi_2 f = \varphi_1$, and a homomorphism $g : A_2 \to A_1$ such that $\varphi_1 g = \varphi_2$. It follows from equalities $\varphi_1 g f = \varphi_1$ and $\varphi_2 f g = \varphi_2$ that fg and gf are isomorphisms, which imply that f and g are isomorphisms. □

Theorem 2.3. Let $\mathcal{A}$ be a class of modules, M be an R-module, and (A, φ) be an $\mathcal{A}$-precover of M. If $\mathcal{P} \subseteq \mathcal{A}$, then φ is an epimorphism.

Proof. Let $P \in \mathcal{P}$ and let $f : P \to M$ be an epimorphism. By the definition of precovers, there exists a homomorphism $h : P \to A$ such that $\varphi h = f$. Therefore φ is an epimorphism. □

Let $\mathcal{L}$ be a class of modules. Define:

$$^{\perp}\mathcal{L} = \{A \in \mathfrak{M} \mid \mathrm{Ext}^1_R(A, C) = 0 \text{ for any } C \in \mathcal{L}\}$$

and

$$\mathcal{L}^{\perp} = \{B \in \mathfrak{M} \mid \mathrm{Ext}^1_R(C, B) = 0 \text{ for any } C \in \mathcal{L}\},$$

which are called respectively the **left orthocomplement** and **right orthocomplement** of $\mathcal{L}$.

The simplest case is $\mathcal{L} = \{R\}$, at this time $\mathcal{L}^{\perp} = {}_R\mathfrak{M}$, $^{\perp}(\mathcal{L}^{\perp}) = \mathcal{P}$. For any class $\mathcal{L}$ of modules, obviously $\mathcal{L} \subseteq {}^{\perp}(\mathcal{L}^{\perp})$, $\mathcal{L} \subseteq ({}^{\perp}\mathcal{L})^{\perp}$, and $\mathcal{L}^{\perp}$ and $^{\perp}\mathcal{L}$ are closed under extensions.

Definition 2.4. Let $\mathcal{A}$ be a class of modules, M be an R-module. Then M is said to have a **special $\mathcal{A}$-precover** of M (also ξ or (A, φ) is called a special $\mathcal{A}$-precover of M) if there exists an exact sequence $\xi : 0 \to K \to A \xrightarrow{\varphi} M \to 0$, where $A \in \mathcal{A}$ and $\mathrm{Ker}(\varphi) \in \mathcal{A}^{\perp}$.

Theorem 2.5. Let $\mathcal{A}$ be a class of modules.

(1) Every special $\mathcal{A}$-precover of M is necessarily $\mathcal{A}$-precover of M.

(2) Let $\mathcal{A}$ be closed under extensions. Let (A, φ) be an $\mathcal{A}$-cover of a module M and φ be an epimorphism. Then (A, φ) is a special $\mathcal{A}$-precover of M. In particular, if $\mathcal{P} \subseteq \mathcal{A}$, then every $\mathcal{A}$-cover of a module is a special $\mathcal{A}$-precover.

Proof. (1) Let $\xi : 0 \to K \to A \stackrel{\varphi}{\to} M \to 0$ be a special $\mathcal{A}$-precover of M. For any $A' \in \mathcal{A}$, since $\mathrm{Ext}^1_R(A', K) = 0$, $\mathrm{Hom}_R(A', \xi)$ is an exact sequence. So (A, φ) is an $\mathcal{A}$-precover of M.

(2) Write $K = \mathrm{Ker}(\varphi)$. Then $0 \to K \to A \stackrel{\varphi}{\to} M \to 0$ is an exact sequence. By (1), it is enough to prove that $K \in \mathcal{A}^{\perp}$. Let $A' \in \mathcal{A}$ and let $0 \to K \stackrel{\sigma}{\to} X \stackrel{\alpha}{\to} A' \to 0$ be an exact sequence. Consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{\varphi} & M & \longrightarrow & 0 \\ & & {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle h} & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow{f} & Q & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & & & \\ & & A' & = & A' & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

where the left upper corner is a pushout. Since $A, A' \in \mathcal{A}$, we have $Q \in \mathcal{A}$. Since $(\mathcal{A}, \varphi)$ is an $\mathcal{A}$-cover of M, there exists $\delta : Q \to A$ such that $\varphi\delta = g$. Hence $\varphi = \varphi\delta h$. Therefore δh is an isomorphism.

Define $\eta(x) = (\delta h)^{-1}\delta f(x)$, $x \in X$. Since $\varphi = \varphi(\delta h)^{-1}$, we have $\varphi\eta(x) = \varphi\delta f(x) = gf(x) = 0$. Thus $\eta(x) \in K$. Hence $\eta : X \to K$. Since $\eta\sigma = (\delta h)^{-1}\delta f\sigma = (\delta h)^{-1}(\delta h) = \mathbf{1}$, it follows that $0 \to K \to X \to A' \to 0$ is a split exact sequence. By [22, Theorem 3.3.5], $\mathrm{Ext}^1_R(A', K) = 0$. Therefore $K \in \mathcal{A}^{\perp}$. □

Theorem 2.6. Let $\mathcal{A}$ be a class of modules, M be an R-module, and (A, φ) be an $\mathcal{A}$-precover of M. If M has an $\mathcal{A}$-cover, then

(1) $A = D_1 \oplus D_2$, where $D_1 \subseteq \mathrm{Ker}(\varphi)$ and $\varphi|_{D_2}$ is an $\mathcal{A}$-cover of M.

(2) (A, φ) is an $\mathcal{A}$-cover of M if and only if A has a nonzero direct summand contained in $\mathrm{Ker}(\varphi)$.

Proof. (1) Let (A', φ') be an $\mathcal{A}$-cover of M. By the definition, there exist homomorphisms $f' : A' \to A$ such that $\varphi f = \varphi'$, and $g : A \to A'$ such that $\varphi' g = \varphi$. Thus $\varphi' gf = \varphi'$. By the definition of covers, gf is an isomorphism. Thus f is a monomorphism, and there is an isomorphism $h : A' \to A'$ such that $gfh = \mathbf{1}_{A'}$. By [22, Exercise 1.23], $A = \mathrm{Ker}(g) \oplus \mathrm{Im}(fh)$. Set $D_1 = \mathrm{Ker}(g)$ and $D_2 = \mathrm{Im}(fh)$. Then obviously $D_1 \subseteq \mathrm{Ker}(\varphi)$ and $D_2 \cong \mathrm{Im}(f) \cong A'$.

(2) This follows from (1). □

Example 2.7. (1) Let $\varphi : P \to M \to 0$ be an epimorphism, where P is a projective module. Then obviously (P, φ) is a special $\mathcal{P}$-precover of a module M. So every module has a special $\mathcal{P}$-precover.

(2) The $\mathcal{P}$-cover of a module is the same as the projective cover. In fact, assume (P, φ) is the projective cover of a module M and let $h : P \to P$ such that $\varphi h = \varphi$. By [22, Theorem 2.7.13(1)], h is an epimorphism. Since $\mathrm{Ker}(h) \subseteq \mathrm{Ker}(\varphi)$, it follows that $\mathrm{Ker}(h)$ is a superfluous submodule of P. So (P, h) is a projective cover of P. From [22, Theorem 2.7.13(3)], h is an isomorphism. Therefore a projective cover is a $\mathcal{P}$-cover.

Conversely, assume (P, φ) is a $\mathcal{P}$-cover of a module M. Suppose that $\mathrm{Ker}(\varphi) + A = P$ and take a homomorphism $f : F \to P$ such that $\mathrm{Im}(f) = A$, where F is a free module. For any $x \in M$, since φ is an epimorphism, there is $y \in P$ such that $\varphi(y) = x$. Write $y = z + a$, where $z \in \mathrm{Ker}(\varphi)$ and $a \in A$. Then $x = \varphi(y) = \varphi(a)$. So $\varphi f : F \to M$ is an epimorphism, that is, $(F, \varphi f)$ is a $\mathcal{P}$-precover of M. So there is a homomorphism $g : P \to F$ such that $\varphi fg = \varphi$. So fg is an isomorphism, and thus f is an epimorphism, that is, $A = P$. So $\mathrm{Ker}(\varphi)$ is a superfluous submodule of P. So every $\mathcal{P}$-cover is a projective cover.

(3) Since each module has a projective cover if and only if R is a perfect ring. Thus, although a special $\mathcal{A}$-precover exists, an $\mathcal{A}$-cover does not necessarily exist.

2.2 Preenvelopes and envelopes of modules

The definition of preenvelopes and envelopes of the module and the related conclusions can be obtained by the dual method of precovers and covers of the module. Therefore here we only make only the corresponding statements without proofs.

Definition 2.8. Let $\mathcal{B}$ be a class of modules, N be an R-module, $B \in \mathcal{B}$, and $\varphi : N \to B$ be a homomorphism.

(1) (B,φ) is called a $\mathcal{B}$**-preenvelope** of N if for any $B' \in \mathcal{B}$, the following can be completed into a commutative diagram:

$$\begin{array}{ccc} B' & \overset{h}{\dashleftarrow} & \\ f\uparrow & & \\ N & \xrightarrow{\varphi} & B \end{array}$$

equivalently $\mathrm{Hom}_R(B,B') \xrightarrow{\varphi^*} \mathrm{Hom}_R(N,B') \to 0$ is an exact sequence for any $B' \in \mathcal{B}$.

(2) Let (B,φ) be a $\mathcal{B}$-preenvelope of N. Then (B,φ) is called a $\mathcal{B}$**-envelope** of N provided that $B' = B$ and $f = \varphi$ imply that h is an automorphism of B.

Theorem 2.9. Let $\mathcal{B}$ be a class of modules and let N be an R-module. If a $\mathcal{B}$-preenvelope of N exists, then $\mathcal{B}$-preenvelopes of N are isomorphic, in other word, a $\mathcal{B}$-preenvelope of N is unique up to isomorphism if it exists.

Theorem 2.10. Let $\mathcal{B}$ be a class of modules, N be an R-module, and (B,φ) be a $\mathcal{B}$-preenvelope of N. If $\mathcal{I} \subseteq \mathcal{B}$, then φ is a monomorphism.

Definition 2.11. Let $\mathcal{B}$ be a class of modules, N be an R-module. Then (B,φ) is called a **special** $\mathcal{B}$**-preenvelope** of N (also ξ or (B,φ) is a special $\mathcal{B}$-preenvelope of N) if there exists an exact sequence $\xi : 0 \to N \xrightarrow{\varphi} B \to L \to 0$, where $B \in \mathcal{B}$ and $L \in {}^{\perp}\mathcal{B}$

Theorem 2.12. Let $\mathcal{B}$ be a class of modules.

(1) Every special $\mathcal{B}$-preenvelope of a module N is necessarily a $\mathcal{B}$-preenvelope of N.

(2) Let $\mathcal{B}$ be closed under extensions. Let (B,φ) be an $\mathcal{B}$-envelope of a module N and φ be a monomorphism. Then (B,φ) is a special $\mathcal{A}$-preenvelope of N. In particular, if $\mathcal{I} \subseteq \mathcal{B}$, then every $\mathcal{B}$-envelope of a module is a special $\mathcal{B}$-preenvelope.

Theorem 2.13. Let $\mathcal{B}$ be a class of modules, N be an R-module, and (B,φ) be a $\mathcal{B}$-preenvelope of N. If N has a $\mathcal{B}$-envelope, then:

(1) $B = C_1 \oplus C_2$, where $\mathrm{Im}(\varphi) \subseteq C_1$ and $\varphi : N \to C_1$ is a $\mathcal{B}$-envelope of N.

(2) (B,φ) is a $\mathcal{B}$-envelope of N if and only if B has a direct summand properly containing $\mathrm{Im}(\varphi)$.

Example 2.14. By [22, Theorem 2.4.19(2)], an injective envelope is an $\mathcal{I}$-envelope. Conversely, assume that (E,φ) is an $\mathcal{I}$-envelope of N. By Theorem 2.10, φ is a monomorphism. Let $g : E(N) \to E$ be a homomorphism such that $g|_N = \varphi$. By [22, Lemma 2.4.15], g is a monomorphism. By Theorem 2.13, $E \cong E(N)$. Thus an $\mathcal{I}$-envelope is also an injective envelope. Therefore $\mathcal{I}$-envelopes and injective envelopes are identical.

2.3 Minimal approximation

An extension generator and a minimal extension generator of a class of modules play an important role in determining a special precover and a special preenvelope of a module.

Definition 2.15. Let $\mathcal{A}$ be a class of modules, N be an R-module, and $\xi : 0 \to N \to G \to L \to 0$ be an exact sequence.

(1) ξ is called an **Ext-generator of** N **relative to** $\mathcal{A}$ (below, abbreviated by an Ext-generator of N) if for any exact sequence $\xi' : 0 \to N \to G' \to L' \to 0$, where $L' \in \mathcal{A}$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G' & \longrightarrow & L' & \longrightarrow & 0 \\ & & \| & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

(2) Let ξ be an Ext-generator of N. Then ξ is called a **minimal Ext-generator of** N **relative to** $\mathcal{A}$ (below, abbreviated by a minimal Ext-generator of N) if h (and hence g) of the following commutative diagram is an isomorphism:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

Note that if ξ is an Ext-generator of N, $\xi' : 0 \to N \to G' \to L' \to 0$ is an exact sequence, and we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & N & \longrightarrow & G' & \longrightarrow & L' & \longrightarrow & 0 \end{array}$$

then ξ' is also an Ext-generator of N.

Example 2.16. Let $\mathcal{A} = {}_R\mathfrak{M}$, N be an R-module, and E be an injective module containing N. Then $\xi : 0 \to N \to G \to L \to 0$ is an Ext-generator of N (relative to ${}_R\mathfrak{M}$). When E is the injective envelope of N, ξ is also a minimal Ext-generator of N.

Lemma 2.17. *Let $\mathcal{A}$ be a class of modules which is closed under direct limits and let N be an R-module. If $\xi : 0 \to N \to G \to L \to 0$ is an Ext-generator, then there exists an Ext-generator $\xi' : 0 \to N \to G' \to L' \to 0$ such that the corresponding commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & N & \longrightarrow & G' & \longrightarrow & L' & \longrightarrow & 0 \\ & & \| & & \downarrow g' & & \downarrow h' & & \\ 0 & \longrightarrow & N & \longrightarrow & G'' & \longrightarrow & L'' & \longrightarrow & 0 \end{array}$$

satisfies: For any Ext-generator $\xi'' : 0 \to N \to G'' \to L'' \to 0$ in the corresponding commutative diagram, $\mathrm{Ker}(g'g) = \mathrm{Ker}(g)$.

Proof. Assume on the contrary that an Ext-generator which satisfies the given property does not exist. Set $G_0 = G$ and $L_0 = L$. Then there exists an Ext-generator $\xi_1 : 0 \to N \to G_1 \to L_1 \to 0$ such that in the following commutative diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & G_0 & \longrightarrow & L_0 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g_{10}} & & \downarrow{\scriptstyle h_{10}} & & \\
0 & \longrightarrow & N & \longrightarrow & G_1 & \longrightarrow & L_1 & \longrightarrow & 0
\end{array}$$

where $0 = \mathrm{Ker}(\mathbf{1}_G) \subset \mathrm{Ker}(g_{10})$. Thus g_{10} is not a monomorphism.

Since ξ_1 does not have the required properties, there exists an Ext-generator $\xi_2 : 0 \to N \to G_2 \to L_2 \to 0$ such that the corresponding commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & G_1 & \longrightarrow & L_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g_{21}} & & \downarrow{\scriptstyle h_{21}} & & \\
0 & \longrightarrow & N & \longrightarrow & G_2 & \longrightarrow & L_2 & \longrightarrow & 0
\end{array}$$

has $\mathrm{Ker}(g_{10}) \subset \mathrm{Ker}(g_{20})$, where $g_{20} := g_{21}g_{10}$.

For any ordinal α, by induction for any ordinal number $\beta < \alpha$, an Ext-generator has been constructed $\xi_\beta : 0 \to N \to G_\beta \to L_\beta \to 0$ such that the corresponding commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & G_\beta & \longrightarrow & L_\beta & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g_{\beta+1,\beta}} & & \downarrow{\scriptstyle h_{\beta+1,\beta}} & & \\
0 & \longrightarrow & N & \longrightarrow & G_{\beta+1} & \longrightarrow & L_{\beta+1} & \longrightarrow & 0
\end{array}$$

has the property $g_{\beta+1,\beta'} = g_{\beta+1,\beta}g_{\beta,\beta'}$, $h_{\beta+1,\beta'} = h_{\beta+1,\beta}g_{\beta,\beta'}$, $\beta' \leqslant \beta$, (Regard $g_{\beta,\beta} = \mathbf{1}$.) and $\mathrm{Ker}(g_{\beta,0}) \subset \mathrm{Ker}(g_{\beta+1,0})$. If α is not a limit ordinal, take $\beta = \alpha - 1$. Thus $\alpha = \beta + 1$. By the way as stated above, we can construct ξ_α, and the corresponding commutative diagram (in the above diagram, take $\beta = \alpha - 1$). If α is a limit ordinal, then set $G_\alpha = \varinjlim_{\beta<\alpha} G_\beta$ and $L_\alpha = \varinjlim_{\beta<\alpha} L_\beta$. By the hypothesis, $L_\alpha \in \mathcal{A}$. By [22, Theorem 2.5.33], $\xi_\alpha : 0 \to N \to G_\alpha \to L_\alpha \to 0$ is an Ext-generator.

By our construction, for any ordinal $\beta < \alpha$, $\mathrm{Ker}(g_{\beta,0}) \subset \mathrm{Ker}(g_{\alpha,0}) \subseteq G$. So for any non-limit ordinal α, there exists an element $x_\alpha \in \mathrm{Ker}(g_{\alpha,0})$ such that $x_\alpha \notin \mathrm{Ker}(g_{\alpha-1,0})$. Since the cardinality $|G|$ of G is fixed, but α is arbitrary, there is an ordinal α such that $|\alpha| > |G|$, a contradiction. □

Lemma 2.18. *Let $\mathcal{A}$ be a class of modules which is closed under direct limits and let N be an R-module. If $\xi : 0 \to N \to G \to L \to 0$ is an Ext-generator, then:*

(1) *There exists an Ext-generator $\xi' : 0 \to N \to G' \to L' \to 0$ such that for any Ext-generator $\xi'' : 0 \to N \to G'' \to L'' \to 0$, g' of the following corresponding commutative diagram is a monomorphism:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & N & \longrightarrow & G' & \longrightarrow & L' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h'} & & \\
0 & \longrightarrow & N & \longrightarrow & G'' & \longrightarrow & L'' & \longrightarrow & 0
\end{array}$$

(2) *ξ' is a minimal Ext-generator of N.*

Proof. (1) Set $G_0 = G$ and $L_0 = L$. By Lemma 2.17, we may assume that $\xi_0 := \xi$ has the concluding properties of Lemma 2.17. For any $n \in \mathbb{N}$, recursively construct an Ext-generator $\xi_n : 0 \to N \to G_n \to L_n \to 0$ to have the property: In the corresponding commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G_n & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \downarrow g_{n+1,n} & & \downarrow h_{n+1,n} & & \\ 0 & \longrightarrow & N & \longrightarrow & G_{n+1} & \longrightarrow & L_{n+1} & \longrightarrow & 0 \\ & & \| & & \downarrow g'' & & \downarrow h'' & & \\ 0 & \longrightarrow & N & \longrightarrow & G'' & \longrightarrow & L'' & \longrightarrow & 0 \end{array}$$

we have $\mathrm{Ker}(g''g_{n+1,n}) = \mathrm{Ker}(g_{n+1,n})$. Set $G' = \varinjlim_n G_n$ and $L' = \varinjlim_n L_n$. By the hypothesis, $\xi' : 0 \to N \to G' \to L' \to 0$ is an Ext-generator. Let $g_n : G_n \to G'$ and $h_n : L_n \to L'$ be the direct limits defined by companion maps. For any Ext-generator $\xi'' : 0 \to N \to G'' \to L'' \to 0$, let the corresponding commutative diagram be

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G' & \longrightarrow & L' & \longrightarrow & 0 \\ & & \| & & \downarrow g' & & \downarrow h' & & \\ 0 & \longrightarrow & N & \longrightarrow & G'' & \longrightarrow & L'' & \longrightarrow & 0 \end{array}$$

Let $x \in G'$ with $g'(x) = 0$. By [22, Theorem 2.5.31], there is $x_n \in G_n$ such that $x = g_n(x_n)$. Thus $g'g_n(x_n) = 0$. Since $g'g_n = g'g_{n+1}g_{n+1,n}$, it follows that $\mathrm{Ker}(g'g_n) = \mathrm{Ker}(f_{n+1,n})$. Thus $x_n \in \mathrm{Ker}(g_{n+1,n})$. Hence $x = g_n(x_n) = g_{n+1}g_{n+1,n}(x_n) = 0$. Therefore g' is a monomorphism.

(2) Assume on the contrary that the assertion is not true. Set $G_1 = G$ and $L_1 = L$. Then in the corresponding commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \downarrow g_{10} & & \downarrow h_{10} & & \\ 0 & \longrightarrow & N & \longrightarrow & G_1 & \longrightarrow & L_1 & \longrightarrow & 0 \end{array}$$

g_{10} is a monomorphism, but is not an epimorphism. Again set $G_2 = G$ and $L_2 = L$. Then g_{21} is a monomorphism, but is not an epimorphism.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G_1 & \longrightarrow & L_1 & \longrightarrow & 0 \\ & & \| & & \downarrow g_{21} & & \downarrow h_{21} & & \\ 0 & \longrightarrow & N & \longrightarrow & G_2 & \longrightarrow & L_2 & \longrightarrow & 0 \end{array}$$

For any ordinal α, inductively assume that if $\beta < \alpha$. Then an Ext-generator $\xi_\beta : 0 \to N \to G_\beta \to L_\beta \to 0$ satisfies $G_\beta = G$, $N_\alpha = L$, and in the corresponding commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G_\beta & \longrightarrow & L_1\beta & \longrightarrow & 0 \\ & & \| & & \downarrow g_{\beta+1,\beta} & & \downarrow h_{\beta+1,\beta} & & \\ 0 & \longrightarrow & N & \longrightarrow & G_{\beta+1} & \longrightarrow & L_{\beta+1} & \longrightarrow & 0 \end{array}$$

$g_{\beta+1,\beta}$ is a monomorphism, but is not an epimorphism. As Lemma 2.17 above, for $\beta' \leqslant \beta$, we have $g_{\beta+1,\beta'} = g_{\beta+1,\beta}g_{\beta,\beta'}$. When α is the limit ordinal number, set $G_\alpha = \varinjlim_{\beta<\alpha} G_\beta = G$ and $L_\alpha = \varinjlim_{\beta<\alpha} L_\beta = L$. By [22, Exercise 2.52], in the commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G_\beta & \longrightarrow & L_\beta & \longrightarrow & 0 \\ & & \| & & \downarrow g_{\alpha,\beta} & & \downarrow h_{\alpha,\beta} & & \\ 0 & \longrightarrow & N & \longrightarrow & G_\alpha & \longrightarrow & L_\alpha & \longrightarrow & 0 \end{array}$$

$g_{\alpha,\beta}$ is a monomorphism. If such a $g_{\alpha,\beta}$ is an epimorphism (and so an isomorphism), there is nothing to prove. Now assume that such $g_{\alpha,\beta}$'s are not epimorphisms. Take an ordinal number α such that $|\alpha| > |G|$. When $\beta' < \beta < \alpha$, $\mathrm{Im}(g_{\alpha,\beta'}) \subset \mathrm{Im}(g_{\alpha,\beta})$, which induces a contradiction. □

Theorem 2.19. Let $\mathcal{A}$ be a class of modules which is closed under extensions and direct limits and let N be an R-module. If N has a special $\mathcal{A}^{\perp}$–preenvelope (G,λ) and $L := \mathrm{Coker}(\lambda) \in \mathcal{A}$, then N has a special $\mathcal{A}^{\perp}$-envelope.

Proof. By the hypothesis, N has an Ext-generator. By Lemma 2.18, we may assume that $\xi : 0 \to N \to G \to L \to 0$ is a minimal Ext-generator. Let $A \in \mathcal{A}$ and let $0 \to G \to Q \to A \to 0$ be an exact sequence. Consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\lambda} & G & \xrightarrow{\sigma} & L & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & N & \xrightarrow{\mu} & Q & \xrightarrow{\tau} & M & \longrightarrow & 0 \end{array}$$

where the right square is a pushout. Since $\mathrm{Coker}(h) \cong A \in \mathcal{A}$ and $L \in \mathcal{A}$, we have $M \in \mathcal{A}$. Since ξ is an Ext-generator, there exist homomorphisms $\alpha : Q \to G$ and $\beta : M \to L$ such that $\alpha\mu = \lambda$ and $\sigma\alpha = \beta\tau$. It follows from the minimality that $g\alpha$ and $h\beta$ are isomorphisms. So the left vertical exact sequence of the above diagram is split. By [22, Theorem 3.3.5], $\mathrm{Ext}^1_R(A,G) = 0$. Thus $G \in \mathcal{A}^{\perp}$. By the minimality, G is a special $\mathcal{A}^{\perp}$-envelope of N. □

Dually we have:

Theorem 2.20. Let $\mathcal{A}$ be a class of modules which is closed under extensions and direct limits and let M be an R-module. If M has a special $\mathcal{A}$-precover (A,φ), then M has a special $\mathcal{A}$-cover.

3 Cotorsion theory

3.1 Basic properties of cotorsion theory

Definition 3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of modules. Then $\mathfrak{G} := (\mathcal{A},\mathcal{B})$ is called a **cotorsion pair** or **cotorsion theory** if $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$. In this case, we write $\mathcal{K} = \mathcal{A} \cap \mathcal{B}$, which is called the **kernel of** $\mathfrak{G}$. Naturally $B \in \mathcal{B}$ is called an $\mathcal{A}$**-injective module**, $A \in \mathcal{A}$ is also called a $\mathcal{B}$**-projective module**.

Example 3.2. (1) $(\mathcal{P},\mathfrak{M})$ and $(\mathfrak{M},\mathcal{I})$ are cotorsion theories.

(2) For any class $\mathcal{L}$ of modules, by Exercise 7, $\mathfrak{L}_{\mathcal{L}} := ({}^{\perp}(\mathcal{L}^{\perp}),\mathcal{L}^{\perp})$ and $\mathfrak{R}_{\mathcal{L}} := ({}^{\perp}\mathcal{L},({}^{\perp}\mathcal{L})^{\perp})$ are cotorsion theories, which are called the **cotorsion theory generated by** $\mathcal{L}$ and the **cotorsion theory cogenerated by** $\mathcal{L}$ respectively.

Trivially if $(\mathcal{A},\mathcal{B})$ is a cotorsion theory, then $\mathcal{P} \subseteq \mathcal{A}$ and $\mathcal{I} \subseteq \mathcal{B}$. In addition, $\mathcal{A}$ is closed under direct sums and $\mathcal{B}$ is closed under direct products, as well as $\mathcal{A}$ and $\mathcal{B}$ are closed under extensions and direct summands.

Definition 3.3. Let $\mathfrak{G} = (\mathcal{A},\mathcal{B})$ be a cotorsion theory.

(1) $\mathfrak{G}$ is called a **complete cotorsion theory** if every module has a special $\mathcal{A}$-precover.

(2) $\mathfrak{G}$ is called a **perfect cotorsion theory** if every module has an $\mathcal{A}$-cover and a $\mathcal{B}$-envelope.

(3) $\mathfrak{G}$ is called a **hereditary cotorsion theory** if $\mathcal{A}$ is closed under kernels of epimorphism.

By Theorem 2.5, every perfect cotorsion theory is necessarily complete.

Theorem 3.4. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion theory. Then $\mathfrak{C}$ is complete if and only if every module has a special $\mathcal{B}$-preenvelope.

Proof. Assume that $\mathfrak{C}$ is complete. Let N be an R-module. Then there is an exact sequence $0 \to N \to E \xrightarrow{g} F \to 0$, where E is an injective module. Let (A, φ) be a special $\mathcal{A}$-precover of F. Construct the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{\sigma} & Q & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \varphi & & \\
0 & \longrightarrow & N & \longrightarrow & E & \xrightarrow{g} & F & \longrightarrow & 0
\end{array}$$

where the diagram on the right square is a pullback. Set $B := \mathrm{Ker}(\varphi)$. By the hypothesis, $B, E \in \mathcal{B}$, and so $Q \in \mathcal{B}$. Thus (Q, σ) is a special $\mathcal{B}$-preenvelope of N.

The conclusion of the converse follows dually from the above proof. □

Theorem 3.5. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion theory. Then the following are equivalent:

(1) $\mathfrak{C}$ is hereditary.

(2) $\mathcal{B}$ is closed under cokernels of monomorphisms.

(3) $\mathrm{Ext}_R^k(A, B) = 0$ for any $A \in \mathcal{A}$, $B \in \mathcal{B}$, and any $k \geqslant 1$.

Proof. (1)⇒(3) Let $0 \to K \to P \to A \to 0$ be an exact sequence, where $P \in \mathcal{P}$, $A \in \mathcal{A}$. By the hypothesis, $K \in \mathcal{A}$. Thus $\mathrm{Ext}_R^{k+1}(A, B) \cong \mathrm{Ext}_R^k(K, B)$ for any $B \in \mathcal{B}$. Now the assertion follows by the induction.

(3)⇒(1) Let $0 \to A_1 \to A \to A_2 \to 0$ be an exact sequence, where $A, A_2 \in \mathcal{A}$. For any $B \in \mathcal{B}$, there is an exact sequence $0 = \mathrm{Ext}_R^1(A, B) \to \mathrm{Ext}_R^1(A_1, B) \to \mathrm{Ext}_R^2(A_2, B) = 0$. Thus $A_1 \in \mathcal{A}$.

(2)⇔(3) This follows dually from the above proof. □

Example 3.6. (1) $(\mathcal{P}, \mathfrak{M})$ is a hereditary complete cotorsion theory. Note that $(\mathcal{P}, \mathfrak{M})$ is a perfect cotorsion theory if and only if R is a perfect ring. Therefore a complete cotorsion theory is not necessarily perfect.

(2) $(\mathfrak{M}, \mathcal{I})$ is a hereditary perfect cotorsion theory.

Theorem 3.7. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete cotorsion theory. If $\mathcal{A}$ is closed under direct limits, then $\mathfrak{C}$ is perfect.

Proof. This follows from Theorem 2.19 and Theorem 2.20. □

3.2 Structures of $\mathcal{B}$-envelopes of modules

When $(\mathcal{A}, \mathcal{B})$ constitutes a perfect cotorsion theory, each module has a $\mathcal{A}$-cover and a $\mathcal{B}$-envelope. For any given module N, the specific structure of $\mathcal{B}$-envelope of N is given below.

Theorem 3.8. Let $(\mathcal{A}, \mathcal{B})$ be a perfect cotorsion theory and let $N \subseteq L$ be an extension of R-modules. Then L is a $\mathcal{B}$-envelope of N if and only if L satisfies the following conditions:

(1) $L \in \mathcal{B}$;

(2) $L/N \in \mathcal{A}$. ((1) and (2) show that L is a special $\mathcal{B}$-preenvelope of N);

(3) there is no nonzero submodules X of L such that $N \cap X = 0$, but $L/(N + X) \in \mathcal{A}$.

Proof. Assume that L is a $\mathcal{B}$-envelope of N and let $i : N \to L$ be the inclusion map. Then the conditions (1) and (2) trivially hold true. Let X be a submodule of L such that $X \cap N = 0$ and set $A := L/(N+X) \in \mathcal{A}$. Let $\pi : L \to L/X$ be the natural homomorphism. Then the natural homomorphism $f = \pi i : N \to L/X$ is a monomorphism and $\mathrm{Coker}(f) = A$. Let (B, φ) be the $\mathcal{B}$-envelope of L/X and set $\phi = \varphi f$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{f} & L/X & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & N & \xrightarrow{\phi} & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where the right square is a pushout. Write $Y = \mathrm{Coker}(\varphi) \cong \mathrm{Coker}(p)$. Since $Y, A \in \mathcal{A}$, we obtain that $C \in \mathcal{A}$. Thus (B, ϕ) is a special $\mathcal{B}$-envelope of N. Since $L \in \mathcal{B}$, there exists a homomorphism $g : B \to L$ such that $g\phi = i$. Thus $g\varphi\pi i = i$, and so $g\varphi\pi$ is an isomorphism, and thus π is a monomorphism. It follows that $X = 0$.

Conversely, assume that L satisfies conditions (1), (2), and (3). Let (E, φ) be the $\mathcal{B}$-envelope of N. Then $A := E/N \in \mathcal{A}$. Thus there exist homomorphisms $f : E \to L$ and $g : L \to E$ such that $f\varphi = i$ and $gi = \varphi$. Hence $gf : E \to E$ satisfies $(gf)\varphi = \varphi$. Thus gf is an isomorphism. Hence $L = \mathrm{Im}(f) \oplus \mathrm{Ker}(g)$. Set $X := \mathrm{Ker}(g)$. Since φ is a monomorphism, $N \cap X = 0$. Note that we have an exact sequence $0 \to X \to L/N \to E/N \to 0$. Thus $L/(N+X) \cong E/N \in \mathcal{A}$. It follows from the condition (3) that $X = 0$. Therefore $f : E \to L$ is an isomorphism. □

3.3 Test method of complete cotorsion theory

Given a class $\mathcal{A}$ of modules, when $\mathcal{A}$ is generated by a class $\mathcal{L}$ of modules, by Exercise 11 we know that $(\mathcal{A}, \mathcal{A}^{\perp})$ formed a cotorsion theory. But it is very difficult to determine whether $(\mathcal{A}, \mathcal{A}^{\perp})$ becomes a complete cotorsion theory. The following cardinal number method is currently used to determine the validity of the complete cotorsion theory.

Definition 3.9. Let $\mathcal{A}$ be a class of modules and let M be an R-module. If there exist an ordinal number λ and a continuous chain of submodules:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\alpha \subset M_{\alpha+1} \subset \cdots \subset M_\lambda = M$$

such that whenever $\alpha < \lambda$, one has $M_{\alpha+1}/M_\alpha \in \mathcal{A}$, then M is called an **$\mathcal{A}$-filtered module** and a continuous ascending chain $\{M_\alpha \mid \alpha \leqslant \lambda\}$ is called an **$\mathcal{A}$-filtration** of M.

Furthermore, let κ be a given cardinal number. If for any ordinal number α, M_α is a pure submodule of M and $|M_{\alpha+1}/M_\alpha| \leqslant \kappa$, then this $\mathcal{A}$-filtration is called a **κ-refinement**.

Lemma 3.10. *Let $\mathcal{A}$ be a class of modules and let M be an $\mathcal{A}$-filtered module. If $N \in \mathcal{A}^{\perp}$, then* $\mathrm{Ext}^1_R(M, N) = 0$.

Proof. This follows by applying [22, Lemma 11.7.2]. □

Countably generated modules, finitely generated modules, finitely presented modules, and super-finitely presented modules are all sets. More generally, let κ be an infinite cardinal number and let X and Y be two cardinal numbers at most κ. By set theory, we know that the number of mapping from X to Y does not exceed 2^κ. Now let M be a set with $|M| \leqslant \kappa$. A binary operation on M is a mapping from the Cartesian product $M \times M$ to M. Therefore, in the sense of isomorphism, the number of Abelian groups that M can be made into does not exceed $2^{(}\kappa)$, in other words, the number of non-isomorphic Abelian groups whose cardinality does not exceed κ does not exceed 2^κ. In particular, in the category $\mathfrak{M}$ of modules, the totality of non-isomorphic R-modules whose cardinality does not exceed κ is a set whose cardinality does not exceed 2^κ. This is an important fact for judging whether a cotorsion theory is complete.

In order to determine when $(\mathcal{L}, \mathcal{L}^\perp)$ is a complete cotorsion theory, the following theorem is very effective.

Theorem 3.11. (**Eklof-Trlifaj**) Let $\mathcal{S}$ be a set of modules.

(1) Let M be an R-module. Then there exists an exact sequence $0 \to M \to P \to A \to 0$, where $P \in \mathcal{S}^\perp$ and A is an $\mathcal{S}$-filtered module, and thus $A \in {}^\perp(\mathcal{S}^\perp)$.

(2) $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$ is a complete cotorsion theory.

Proof. (1) Write $X = \bigoplus_{S \in \mathcal{S}} S$. Then $X^\perp = \mathcal{S}^\perp$. Thus we may assume that $\mathcal{S}$ is the class of modules constituting of a module S and direct sums of some copies of S. Let $0 \to K \xrightarrow{\mu} F \to S \to 0$ be an exact sequence, where F is a free module. Take a fixed cardinal number λ such that K has a generation system X with $|X| < \lambda$.

Set $P_0 = M$. Using cardinal numbers and ordinal numbers without distinction, for any ordinal number $\alpha < \lambda$, if P_α has been given, then set $I_\alpha := \mathrm{Hom}_R(K, P_\alpha)$ as a new index set. Let $\mu_\alpha : K^{(I_\alpha)} \to F^{(I_\alpha)}$ be a homomorphism of direct sums. Then μ_α is a monomorphism and $\mathrm{Coker}(\mu_\alpha) = S^{(I_\alpha)}$.

Define $\varphi_\alpha : K^{(I_\alpha)} = \bigoplus_{f \in I_\alpha} K_f \to P_\alpha$, where $K_f = K$, by $\varphi_\alpha([x_f]) = \sum_{f \in I_\alpha} f(x_f)$. Furthermore, for any $f \in I_\alpha$, let $i_f : K \to K^{(I_\alpha)}$ and $j_f : F \to F^{(I_\alpha)}$ be the natural embedding maps. Then there is a relation

$$f = \varphi_\alpha i_f, \qquad j_f \mu = \mu_\alpha i_f. \tag{3.1}$$

Now assume that when $\beta \leqslant \alpha$, P_β has been constructed (Note that when α is a limit ordinal number, set $P_\alpha := \bigcup_{\beta < \alpha} P_\beta$), in particular P_α has been constructed. Construction a pushout diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^{(I_\alpha)} & \xrightarrow{\mu_\alpha} & F^{(I_\alpha)} & \longrightarrow & S^{(I_\alpha)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_\alpha} & & \downarrow{\scriptstyle \psi_\alpha} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & P_\alpha & \xhookrightarrow{h_\alpha} & P_{\alpha+1} & \longrightarrow & P_{\alpha+1}/P_\alpha & \longrightarrow & 0
\end{array}$$

we get $P_{\alpha+1}$. Set $P := \bigcup_{\alpha < \lambda} P_\alpha = \varinjlim_{\alpha < \lambda} P_\alpha$. Set $A := P/M$ and $A_\alpha := P_\alpha/M$. Then $A_{\alpha+1}/A_\alpha \cong P_{\alpha+1}/P_\alpha \cong S^{(I_\alpha)}$. Since $P = \bigcup_{\alpha < \lambda} P_\alpha$, we have $A = \bigcup_{\alpha < \lambda} A_\alpha$. Thus A is an $\mathcal{S}$-filtered module.

Finally we prove that $P \in \mathcal{S}^\perp$. To do this, it suffices to prove that $\mu^* : \mathrm{Hom}_R(F, P) \to \mathrm{Hom}_R(K, P)$ is an epimorphism.

Let $g : K \to P$ be a homomorphism. Since K has a generating system X with $|X| < \lambda$ and $P = \bigcup_{\alpha < \lambda} P_\alpha$, there exists an ordinal number $\alpha < \lambda$ such that $\mathrm{Im}(g) \subseteq P_\alpha$. Hence there exists a homomorphism $f : K \to P_\alpha$ such that for any $x \in K$, we have $g(x) = f(x)$. By the pushout diagram and (3.1), we have

$$\psi_\alpha j_f \mu = \psi_\alpha \mu_\alpha i_f = h_\alpha \varphi_\alpha i_f = h_\alpha f. \tag{3.2}$$

Define $\sigma : F \to P$ such that $\sigma(z) = \psi_\alpha j_f(z) \in P_{\alpha+1} \subseteq P$. Then a direct verification shows that $g = \sigma\mu = \mu^*(\sigma)$. Therefore σ is an epimorphism.

(2) Since $S \in {}^\perp(\mathcal{S}^\perp)$, this follows from (1), Lemma 3.10, and Theorem 2.12. □

Definition 3.12. Let $\mathcal{L}$ be a class of modules.

(1) We say that $\mathcal{L}$ is a **resolving class** if $\mathcal{L}$ is closed under both extensions and kernels of epimorphisms, and $\mathcal{P} \subseteq \mathcal{L}$.

(2) We say that $\mathcal{L}$ is a **coresolving class** if $\mathcal{L}$ is closed under both extensions and cokernels of monomorphisms, and $\mathcal{I} \subseteq \mathcal{L}$.

(3) It is said that $\mathcal{L}$ has the property (P) if for any $L \in \mathcal{L}$, there is an epimorphism $P \to L$, where P is a projective module and $P \in \mathcal{L}$.

(4) It is said that $\mathcal{L}$ has the property (I) if for any $L \in \mathcal{L}$, there is a monomorphism $L \to I$, where I is an injective module and $I \in \mathcal{L}$.

Definition 3.13. Let B be an R-module.

(1) B is called a **strong $\mathcal{L}$-injective module** if $\mathrm{Ext}^i_R(L, B) = 0$ for any $L \in \mathcal{L}$ and any $i > 0$.

(2) B is called a **strong $\mathcal{L}$-projective module** if $\mathrm{Ext}^i_R(B, L) = 0$ for any $L \in \mathcal{L}$ and any $i > 0$.

Theorem 3.14. Let $\mathcal{L}$ be a class of modules. Use $\mathcal{L}^{\perp_\infty}$ to denote the class of strong $\mathcal{L}$-injective modules and ${}^{\perp_\infty}\mathcal{L}$ to denote the class of strong $\mathcal{L}$-projective modules.

(1) $\mathcal{L}^{\perp_\infty}$ is a coresolving class.

(2) ${}^{\perp_\infty}\mathcal{L}$ is a resolving class.

Proof. (1) Obviously $\mathcal{L}^{\perp_\infty}$ is closed under extensions, and $\mathcal{I} \subseteq \mathcal{L}^{\perp_\infty}$.

Let $0 \to A \to B \to C \to 0$ be an exact sequence with $A, B \in \mathcal{L}^{\perp_\infty}$. Let $L \in \mathcal{L}$ and $i > 0$. Then there is an exact sequence $0 = \mathrm{Ext}^i_R(L, B) \to \mathrm{Ext}^i_R(L, C) \to \mathrm{Ext}^{i+1}_R(L, A) = 0$. From this we get $\mathrm{Ext}^i_R(L, C) = 0$, that is, $C \in \mathcal{L}^{\perp_\infty}$. So $\mathcal{L}^{\perp_\infty}$ is closed under cokernels of monomorphisms. Thus $\mathcal{L}^{\perp_\infty}$ is a coresolving class.

(2) It can be proved similarly to (1). □

Theorem 3.15. (1) Let $\mathcal{L}$ be a class of modules closed under kernels of epimorphisms, and have the property (P). Then $\mathcal{L}^{\perp} = \mathcal{L}^{\perp_\infty}$.

(2) Let $\mathcal{L}$ be a class of modules closed under cokernels of monomorphisms, and have the property (I). Then ${}^{\perp}\mathcal{L} = {}^{\perp_\infty}\mathcal{L}$.

Proof. (1) Obviously $\mathcal{L}^{\perp_\infty} \subseteq \mathcal{L}^{\perp}$. Conversely, let $B \in \mathcal{L}^{\perp}$. For any $L \in \mathcal{L}$, take an exact sequence $0 \to M \to F \to L \to 0$, where F is a free module. From the condition, $M \in \mathcal{L}$. Let $i > 0$. Then $\mathrm{Ext}^{i+1}_R(L, B) \cong \mathrm{Ext}^i_R(M, B)$. So $B \in \mathcal{L}^{\perp_\infty}$ can be obtained by induction.

(2) It can be proved similarly to (1). □

Lemma 3.16. *Let $\mathcal{L}$ be a class of modules. Then:*

(1) $\mathcal{L} \subseteq {}^{\perp}(\mathcal{L}^{\perp_\infty})$.

(2) $({}^{\perp}(\mathcal{L}^{\perp_\infty}))^{\perp} = \mathcal{L}^{\perp_\infty}$.

(3) $({}^{\perp}(\mathcal{L}^{\perp_\infty}), \mathcal{L}^{\perp_\infty})$ is a hereditary cotorsion theory.

(4) For each $M \in \mathcal{L}$, select a projective resolution $\mathbf{P}(M)$ of M. Let $\mathcal{A}_M$ be the set of all syzygies (including M itself) of $\mathbf{P}(M)$. Set $\mathcal{A} = \bigcup\limits_{M \in \mathcal{L}} \mathcal{A}_M$. Then $\mathcal{A}^{\perp} = \mathcal{L}^{\perp_\infty}$.

Proof. (1) Let $L \in \mathcal{L}$. Then obviously $\mathrm{Ext}_R^1(L,A) = 0$ for any $A \in \mathcal{L}^{\perp_\infty}$, and so $L \in {}^{\perp}(\mathcal{L}^{\perp_\infty})$.

(2) Apparently $\mathcal{L}^{\perp_\infty} \subseteq ({}^{\perp}(\mathcal{L}^{\perp_\infty}))^{\perp}$. By Theorem 3.14, $\mathcal{L}^{\perp_\infty}$ contains all injective modules and it is closed under cokernels of monomorphisms. From Theorem 3.15(2) and Theorem 3.14, we know ${}^{\perp}(\mathcal{L}^{\perp_\infty})$ contains all projective modules and closed under kernels of epimorphisms. Applying Theorem 3.15(1), we know that $({}^{\perp}(\mathcal{L}^{\perp_\infty}))^{\perp_\infty} = ({}^{\perp}(\mathcal{L}^{\perp_\infty}))^{\perp} = \mathcal{L}^{\perp_\infty}$.

(3) This is obtained by (2) and Theorem 3.14.

(4) Let $N \in \mathcal{L}^{\perp_\infty}$. For any $A \in \mathcal{A}$, there exist $M \in \mathcal{L}$ and an exact sequence

$$0 \longrightarrow A \longrightarrow P_k \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \tag{3.2}$$

where $P_0, P_1, \ldots, P_k$ are projective modules. So $\mathrm{Ext}_R^1(A,N) \cong \mathrm{Ext}_R^{k+2}(M,N) = 0$. Therefore $N \in \mathcal{A}^{\perp}$.

On the other hand, let $N \in \mathcal{A}^{\perp}$. For any $M \in \mathcal{L}$ and any $k \geqslant -1$, consider the exact sequence (3.2). Then $\mathrm{Ext}_R^{k+2}(M,N) \cong \mathrm{Ext}_R^1(A,N) = 0$. So $N \in \mathcal{L}^{\perp_\infty}$. □

Theorem 3.17. Let $\mathcal{S}$ be a set of modules and set $\mathcal{B} = \mathcal{S}^{\perp_\infty}$. Then $({}^{\perp}\mathcal{B}, \mathcal{B})$ is a hereditary and complete cotorsion theory. Thus each module has a strong $\mathcal{S}$-injective special preenvelope.

Proof. By Lemma 3.16, $(\mathcal{B}^{\perp}, \mathcal{B})$ is a hereditary cotorsion theory. Construct $\mathcal{A}$ as in Lemma 3.16. Then $\mathcal{L}$ is a set. It is easy to see that $\mathcal{B} = \mathcal{A}^{\perp}$. It follows from Theorem 3.11 that $({}^{\perp}\mathcal{B}, \mathcal{B})$ is a complete cotorsion theory. □

The following example is a specific application of Theorem 3.11, Theorem 3.17, and other results.

Example 3.18. (1) Let $\mathcal{L}$ denote the class of finitely presented modules. Then $\mathcal{L}$-injective modules are FP-injective modules. Use $\mathcal{FPI}$ to denote FP-injective modules. Then $\mathcal{FPI} = \mathcal{L}^{\perp}$. From Theorem 3.11, $({}^{\perp}\mathcal{FPI}, \mathcal{FPI})$ is a complete cotorsion theory, so that each module has an FP-injective special preenvelope. Similarly, by Theorem 3.17, each module has a strong FP-injective special preenvelope.

(2) Let $\mathcal{L}$ denote the class of super-finitely presented modules. Then $\mathcal{L}$ is a set, at this time $\mathcal{L}$-injective modules are exactly weak injective modules. Use $\mathcal{WI}$ to denote the class of weak injective modules. Then $\mathcal{WI} = \mathcal{L}^{\perp}$. According to Theorem 3.11, $({}^{\perp}\mathcal{WI}, \mathcal{WI})$ is a hereditary and complete cotorsion theory, so that each module has a weak injective special preenvelope.

(3) Let n be a non-negative integer and $\mathcal{L}$ represent the class of finitely generated modules. For each $M \in \mathcal{L}$, take the $(n-1)$th syzygy K_M of a projective resolution. Then the module class $\mathcal{S}$ formed by these K_M is also a set, and $\mathcal{I}_n = \mathcal{S}^{\perp}$. By Theorem 3.17, $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n)$ is a hereditary and complete cotorsion theory, so that each module has an $\mathcal{I}_n$-special preenvelope.

Use $\mathcal{P}_1$ to denote the class of modules whose projective dimension does not exceed 1.

Lemma 3.19. *Let $M \in \mathcal{P}_1$.*

(1) M has a **tight system**, *that is, a family $\mathcal{T}$ of submodules satisfying:*

(a) $0, M \in \mathcal{T}$, and $\mathcal{T}$ is closed under unions of chains;

(b) If $A, B \in \mathcal{T}$, and $A \subseteq B$, then $B/A \in \mathcal{P}_1$;

(c) Let $A \in \mathcal{T}$ and X be a countable subset of M. Then there exists $B \in \mathcal{T}$ such that $A, X \subseteq B$, and B/A is countably generated.

(2) Let $\mathcal{S}$ be the class of modules that can be generated countably and whose projective dimension does not exceed 1. Then $\mathcal{S}$ is a set and M has $\mathcal{S}$-filtration.

Proof. (1) See [11, Proposition VI.5.1]. (2) See [11, Proposition VI.6.1]. □

Example 3.20. Let $\mathcal{L}$ denote the module class of 1-cosyzygies. Note that for an R-module M and a positive integer n, $\mathrm{pd}_R M \leqslant n$ if and only if $\mathrm{Ext}_R^1(M,X) = 0$ for any $(n-1)$-th cosyzygy module X. Thus $\mathcal{P}_1 = {}^{\perp}\mathcal{L}$. Then $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is a cotorsion theory. From Lemma 3.19, ${}^{\perp}(\mathcal{S}^{\perp}) = {}^{\perp}(\mathcal{P}_1^{\perp}) = {}^{\perp}(({}^{\perp}\mathcal{L})^{\perp}) = {}^{\perp}\mathcal{L} = \mathcal{P}_1$. By Theorem 3.11, $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is a hereditary and complete cotorsion theory. Thus each module has a $\mathcal{P}_1$-special precover.

3.4 Test method of perfect cotorsion theory

Now we define κ as follows: When $|R|$ is finite, set $\kappa = \aleph_0$; when $|R|$ is infinite, set $\kappa = |R|$.

Lemma 3.21. *Let $\mathfrak{G} = (\mathcal{A},\mathcal{B})$ be the hereditary cotorsion theory generated by a class $\mathcal{L}$ of modules, where $\mathcal{L} \subseteq \mathcal{PI}$. Then $A \in \mathcal{A}$. if and only if there exists an ordinal number λ such that A has a κ-refinement of an $\mathcal{A}$-filtration $\{A_\alpha \mid \alpha \leqslant \lambda\}$.*

Proof. Assume that $A \in \mathcal{A}$. If $|A| \leqslant \kappa$, then the assertion follows by setting $A_0 = 0$ and $A_1 = A$. Now assume that $\lambda = |A| > \kappa$. By Theorem 1.5(1), there exists a nonzero pure submodule A_1 of A such that $|A_1| \leqslant \kappa$. For any $L \in \mathcal{L}$, since $\mathrm{Ext}_R^1(A,L) = 0$ and L is a pure injective module, $\mathrm{Ext}_R^1(A/A_1, L) = 0$, that is, $A/A_1 \in \mathcal{A}$. By the heredity, $A_1 \in \mathcal{A}$. Trivially $A_1 \neq A$. Thus there exists a submodule A_2 of A containing A_1 such that A_2/A_1 is a nonzero pure submodule of A/A_1, $|A_2/A_1| \leqslant \kappa$, and $A_2/A_1 \in \mathcal{A}$. Similarly we have $A/A_2 \in \mathcal{A}$. For any ordinal number α, inductively we assume that when $\beta < \alpha$, β is a non-limit ordinal, A_β has been given, and satisfies the condition: A_β is a pure submodule of A, $A/A_\beta \in \mathcal{A}$, $A_\beta/A_{\beta-1} \in \mathcal{A}$, and $|A_\beta/A_{\beta-1}| \leqslant \kappa$. When α is a non-limit ordinal, repeating the above-mentioned process for $A/A_{\alpha-1}$, construct A_α to meet the requirements. When α is a limit ordinal, set $A_\alpha := \bigcup_{\beta<\alpha} A_\beta$. Since the functors $\otimes$ and lim commute, A_α is a pure submodule of A. Thus $A/A_\alpha \in \mathcal{A}$. Let $\lambda = |A|$. Using cardinal numbers and ordinal numbers without distinction, we can recursively get the required κ-refinement of A.

Conversely, assume that A has a κ-refinement. For any $N \in \mathcal{L}$, by Lemma 3.10, $\mathrm{Ext}_R^1(A,N) = 0$. Therefore $A \in {}^{\perp}\mathcal{L} = \mathcal{A}$. □

Theorem 3.22. Let $\mathfrak{G} = (\mathcal{A},\mathcal{B})$ be the hereditary cotorsion theory generated by a class $\mathcal{L}$ of modules, where $\mathcal{L} \subseteq \mathcal{PI}$. Then $\mathfrak{G}$ is a perfect cotorsion theory.

Proof. Let $\mathcal{S}$ be the module class $\{A \in \mathfrak{M} \mid A \in \mathcal{A} \text{ and } |A| \leqslant \kappa\}$. Then $\mathcal{S}$ is a set. Since $\mathcal{S} \subseteq \mathcal{A}$, we have $\mathcal{B} = \mathcal{A}^{\perp} \subseteq \mathcal{S}^{\perp}$. Now let $B \in \mathcal{S}^{\perp}$ and $A \in \mathcal{A}$. By Lemma 3.21, A has a κ-refinement $\{A_\alpha\}$. Since $|A_{\alpha+1}/A_\alpha| \leqslant \kappa$, by the choice of $\mathcal{S}$, we have $\mathrm{Ext}_R^1(A_{\alpha+1}/A_\alpha, N) = 0$. By Lemma 3.10, $\mathrm{Ext}_R^1(A,N) = 0$, that is, $N \in \mathcal{A}^{\perp} = \mathcal{B}$. Thus $\mathcal{B} = \mathcal{S}^{\perp}$. By Theorem 3.11, $\mathfrak{G} = (\mathcal{A},\mathcal{B}) = ({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ is a complete cotorsion theory.

In order to prove that $\mathfrak{G}$ is a perfect cotorsion theory, by Theorem 3.7 it is enough to show that $\mathcal{A}$ is closed under direct limits. Let $\{A_i, \varphi_{ij}\}$ be a direct system in $\mathcal{A}$ over a directed set Γ. By Exercise 11, there is an exact sequence $0 \to N \to \bigoplus_i A_i \to \varinjlim A_i \to 0$. For any $C \in \mathcal{L}$, we have $\mathrm{Ext}_R^1(\bigoplus_i A_i, C) \cong \prod_i \mathrm{Ext}_R^1(A_i, C) = 0$. Thus $\mathrm{Ext}_R^1(\varinjlim A_i, C) = 0$. Therefore $\mathcal{A}$ is closed under direct limits. □

Remark 3.1 Although, in Theorem 3.22 and Theorem 3.26 behind, hereditary conditions are not added, we still get that $\mathfrak{G}$ is a perfect cotorsion theory, see reference [13]. But to reduce the space, in Lemma 3.21 the hereditary condition is added.

Definition 3.23. Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of modules. Then $(\mathcal{A},\mathcal{B})$ is called a **Tor-torsion pair** (or **Tor-torsion theory**) if $\mathcal{A} = \mathcal{B}^{\top}$ and $\mathcal{B} = \mathcal{A}^{\top}$. If $\mathcal{A}$ is closed under kernels of epimorphisms, then $(\mathcal{A},\mathcal{B})$ is said to be a **hereditary Tor-torsion theory**.

Theorem 3.24. Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a Tor-torsion theory. Then the following are equivalent:

(1) $\mathfrak{G}$ is hereditary.

(2) $\mathcal{B}$ is closed under kernels of epimorphisms.

(3) $\mathrm{Tor}_k^R(A, B) = 0$ for any $A \in \mathcal{A}$, $B \in \mathcal{B}$, and any $k \geqslant 1$.

Proof. The proof is similar to that of Theorem 3.5. □

Example 3.25. (1) $(\mathcal{F}, \mathfrak{M})$ is a hereditary Tor-torsion theory.

(2) For any class $\mathcal{L}$ of modules, by Exercise 7, $\mathfrak{T}_{\mathcal{L}} := (\mathcal{L}^{\top}, (\mathcal{L}^{\top})^{\top})$ is a Tor-torsion theory.

Theorem 3.26. Let $(\mathcal{A}, \mathcal{B})$ be a hereditary Tor-torsion theory. Then:

(1) $\mathcal{A}$ is closed under direct limits.

(2) Set $\mathcal{L} = \{B^+ \mid B \in \mathcal{B}\}$. Then $\mathcal{L} \subseteq \mathcal{A}^{\perp}$.

(3) ${}^{\perp}\mathcal{L} = \mathcal{A}$, and thus ${}^{\perp}(\mathcal{A}^{\perp}) = \mathcal{A}$.

(4) $\mathfrak{R}_{\mathcal{L}} = (\mathcal{A}, \mathcal{A}^{\perp})$ is a perfect cotorsion theory.

Proof. (1) This follows from [22, Theorem 2.5.34].

(2) By [22, Theorem 3.4.11], for any $A, B \in \mathfrak{M}$, we have the natural isomorphism:

$$\mathrm{Ext}_R^1(A, B^+) \cong \mathrm{Tor}_1^R(A, B)^+.$$

When $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have $\mathrm{Tor}_1^R(A, B) = 0$, and so $\mathrm{Ext}_R^1(A, B^+) = 0$. It follows that $\mathcal{L} \subseteq \mathcal{A}^{\perp}$.

(3) Trivially $\mathcal{A} \subseteq {}^{\perp}(\mathcal{A}^{\perp}) \subseteq {}^{\perp}\mathcal{L}$. Now let $A \in {}^{\perp}\mathcal{L}$. Then $\mathrm{Ext}_R^1(A, B^+) = 0$ for any $B \in \mathcal{B}$. It follows from (1.2) and [22, Exercise 3.27] that $\mathrm{Tor}_1^R(A, B) = 0$. Thus $A \in \mathcal{B}^{\top} = \mathcal{A}$. Therefore ${}^{\perp}(\mathcal{A}^{\perp}) = \mathcal{A}$.

(4) This follows from Example 1.10 and Theorem 3.22. □

Example 3.27. (1) Let $\mathcal{A} = {}_R\mathfrak{M}$. Then $\mathcal{B} = \mathfrak{M}^{\top} = \mathcal{F}$ and $\mathcal{A}^{\perp} = \mathcal{I}$. Thus $(\mathcal{A}, \mathcal{B})$ is a Tor-torsion theory. It follows from Theorem 3.26 that each module has an injective envelope.

(2) Let $\mathcal{A} = \mathcal{F}$ and $\mathcal{B} = {}_R\mathfrak{M}$. Then $(\mathcal{A}, \mathcal{B})$ is a Tor-torsion theory. It follows from Theorem 3.26 that each module has a flat cover.

(3) (FCC) Note that for a module M, M is w-flat if and only if $\mathrm{Tor}_1^R(M, B) = 0$ for any $B \in \mathcal{F}_w^{\top}$. Thus $(\mathcal{F}_w, \mathcal{F}_w{}^{\top})$ is a Tor-torsion theory. By Theorem 3.26, each module has a w-flat cover.

Enochs' Conjecture [9]: For any ring R, every R-module has a flat cover (FCC). Enochs *et al.* made a long-term effort, in 2001 [7] solved this problem.

Now let's consider the dual approach of strong $\mathcal{L}$-injective modules, in which the proof methods of some results are similar, and so the proof is omitted.

Definition 3.28. Let $\mathcal{L}$ be a class modules. Then a module A is called **strong $\mathcal{L}$-flat moduleif** $\mathrm{Tor}_i^R(L, A) = 0$ for any $L \in \mathcal{L}$ and any $i > 0$.

Use $\mathcal{L}^{\top_\infty}$ to represent the class of strong $\mathcal{L}$-flat modules. From [22, Theorem 3.4.14], the module class $\mathcal{L}^{\top_\infty}$ is closed under direct limits.

Theorem 3.29. $\mathcal{L}^{\top_\infty}$ is a resolving class.

Theorem 3.30. Let $\mathcal{L}$ be a class of modules closed under kernels of epimorphisms and have the property (P). Then $\mathcal{L}^{\top} = \mathcal{L}^{\top_\infty}$, namely every $\mathcal{L}$-flat module is a strong $\mathcal{L}$-flat module.

Lemma 3.31. *Let $\mathcal{L}$ be a class of modules. Then:*

(1) $\mathcal{L} \subseteq (\mathcal{L}^{\top_\infty})^{\top}$.

(2) $((\mathcal{L}^{\top_\infty})^{\top})^{\top} = \mathcal{L}^{\top_\infty}$.

(3) $(\mathcal{L}^{\top_\infty}, (\mathcal{L}^{\top_\infty})^{\top})$ *is a hereditary Tor-torsion theory.*

Theorem 3.32. Let $\mathcal{L}$ be a class of modules and set $\mathcal{A} = \mathcal{L}^{\top_\infty}$. Then $(\mathcal{A}, \mathcal{A}^{\perp})$ is a hereditary perfect cotorsion theory. In other words, each module has a special strong $\mathcal{L}$-flat cover and a special $(\mathcal{L}^{\top_\infty})^{\perp}$-injective envelope.

Proof. This is obtained from Lemma 3.31 and Theorem 3.26. □

Corollary 3.33. *Let $\mathcal{L}$ be a class of modules closed under kernels of epimorphisms and have the property (P). Then each module has a special $\mathcal{L}$-flat cover and a special $\mathcal{L}^{\perp}$-injective envelope.*

Example 3.34. If $\mathcal{L}$ represents the class of flat modules, then $\mathcal{A} := \mathcal{L}^{\top} = {}_R\mathfrak{M}$ and $\mathcal{A}^{\perp} = \mathcal{I}$. At this point, Corollary 3.33 points out that each module has an injective envelope.

4 Cotorsion theory of weak w-projective modules

4.1 w-version of Kaplansky's theorem

Kaplansky proved that every projective module is a direct sum of countably generated projective submodules ([22, Corollary 2.3.15]), so that the projective modules over the local ring are all free modules. This section provides the w-version of [22, Corollary 2.3.15]. The content of this subsection is taken from [25].

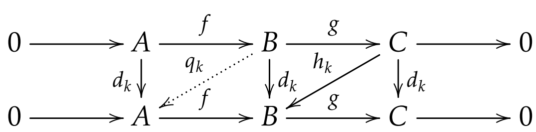
Definition 4.1. (1) Let $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence. Then ξ is said to be ***w*-split** if there exist $J := (d_1, \ldots, d_n) \in \mathrm{GV}(R)$ and $h_1, \ldots, h_n \in \mathrm{Hom}_R(C, B)$ such that $d_k \mathbf{1}_C = gh_k$, $k = 1, \ldots, n$.

(2) Let M be an R-module. Then M is called a ***w*-split module** if there exist a projective module F and an epimorphism $g : F \to M$ such that $0 \to \mathrm{Ker}(g) \to F \xrightarrow{g} M \to 0$ is a w-split exact sequence.

(3) Let $\xi : 0 \to P \xrightarrow{f} F \xrightarrow{g} M \to 0$ be a w-split exact sequence and J and h_k is defined as above. Let F_1 be a submodule of F and set $g_1 := g|_{B_1}$, $C_1 := \mathrm{Im}(g_1)$, $A_1 := f^{-1}(\mathrm{Ker}(g_1))$, and $f_1 = f|_{A_1}$. Then $\xi_1 : 0 \to A_1 \xrightarrow{f_1} B_1 \xrightarrow{g_1} C_1 \to 0$ is an exact sequence. If $h_k(C_1) \subseteq B_1$ for $k = 1, \ldots, n$, then it is obvious that ξ_1 is a w-split exact sequence, which is called a ***w*-split exact sequence induced by** ξ.

Lemma 4.2. *Let $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence. Then ξ is w-split if and only if there exist $J = (d_1, \ldots, d_n) \in \mathrm{GV}(R)$ and $q_1, \ldots, q_n \in \mathrm{Hom}_R(B, A)$ such that $d_k \mathbf{1}_A = q_k f$, $k = 1, \ldots, n$.*

Proof. Assume that ξ is w-split. Then there exist $J := (d_1, \ldots, d_n) \in \mathrm{GV}(R)$ and $h_1, \ldots, h_n \in \mathrm{Hom}_R(C, B)$ such that $d_k \mathbf{1}_C = gh_k$, $k = 1, \ldots, n$. Consider the following commutative diagram with exact rows:



where the vertical arrows are all multiplicative homomorphisms used to multiply. By [22, Exercise 1.60] there exist a homomorphism $q_k : B \to A$ such that $q_k f = d_k \mathbf{1}_A$, $k = 1, \ldots, n$. Similarly the converse can be proved. □

Remark 4.3. In the diagram of the proof of Lemma 4.2, it is easy to see that the equality $d_k \mathbf{1}_B = f q_k + h_k g$ holds, $k = 1, \ldots, n$.

Let $\xi : 0 \to P \xrightarrow{f} F \xrightarrow{g} M \to 0$ be an exact sequence, where $F = \bigoplus_{i\in I} F_i$ with each F_i a projective module. Let $H \subseteq I$. If $H = \emptyset$, then write:

$$F(H) = 0, P(H) = 0, M(H) = 0.$$

And if $H \neq \emptyset$, then write:

$$F(H) = \bigoplus_{i\in H} F_i, \quad g_H = g|_{F(H)}, \quad M(H) = \mathrm{Im}(g_H), \quad P(H) = f^{-1}(\mathrm{Ker}(g_H)), \quad f_H = f|_{P(H)}.$$

Naturally $F = F(I)$, $P = P(I)$, and $M = M(I)$. Obviously we have: if $H_1 \subseteq H_2$, then $F(H_1)$ is a direct summand of $F(H_2)$. In addition, for any $H \subseteq I$, $\xi_H : 0 \to P(H) \xrightarrow{f_H} F(H) \xrightarrow{g_H} M(H) \to 0$ is an exact sequence.

Definition 4.4. Suppose F, ξ, H and other assumptions are as above. If ξ_H is a w-split exact sequence induced by ξ, then $M(H)$ is a w-split module, which is called a **w-split module induced by** ξ.

Lemma 4.5. *Let $\xi : 0 \to P \xrightarrow{f} F \xrightarrow{g} M \to 0$ be a w-split exact sequence, where $F = \bigoplus_{i\in I} F_i$ with each F_i a projective module. Assume that S is a set of subsets H of I, totally ordered by inclusion, satisfying: each $\xi_H : 0 \to P(H) \xrightarrow{f_H} F(H) \xrightarrow{g_H} M(H) \to 0$ is a w-split exact sequence induced by ξ. Then we have*

(1) $\bigcup_{H\in S} F(H)$ is a projective module.

(2) The sequence

$$\xi' : 0 \longrightarrow \bigcup_{H\in S} P(H) \longrightarrow \bigcup_{H\in S} F(H) \longrightarrow \bigcup_{H\in S} M(H) \longrightarrow 0$$

is a w-split exact sequence induced by ξ.

(3) $M(\bigcup_{H\in S} H) = \bigcup_{H\in S} M(H)$ is a w-split module induced by ξ.

Proof. (1) Write $J = \bigcup_{H\in S} H$. We prove $F(J) = \bigcup_{H\in S} F(H)$, and thus $\bigcup_{H\in S} F(H)$ is a projective module.

Trivially $\bigcup_{H\in S} F(H) \subseteq F(J)$. Let $y \in F(J) = \bigoplus_{i\in J} F_i$. Then there exist $i_1, \ldots, i_m \in J$ such that $y = y_1 + \cdots + y_m$, where $y_t \in F_{i_t}$, $t = 1, \ldots, m$. Since S is totally ordered, there exists $H \in S$ such that $i_1, \ldots, i_m \in H$. Thus $y \in F(H)$. Therefore $F(J) = \bigcup_{H\in S} F(H)$.

(2) Let $x \in \bigcup_{H\in S} M(H)$. Then there exists $H_0 \in S$ such that $x \in M(H_0)$. Since ξ_{H_0} is a w-split exact sequence induced by ξ, we have $h_k(x) \in F(H_0) \subseteq \bigcup_{H\in S} F(H), k = 1, \ldots, n$. Thus ξ' a w-split exact sequence induced by ξ.

(3) By (2) we know that $\bigcup_{H\in S} M(H)$ is a w-split module induced by ξ. It is enough to prove that $M(J) = \bigcup_{H\in S} M(H)$.

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigcup_{H\in S} P(H) & \longrightarrow & \bigcup_{H\in S} F(H) & \longrightarrow & \bigcup_{H\in S} M(H) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & P(J) & \longrightarrow & F(J) & \longrightarrow & M(J) & \longrightarrow & 0 \end{array}$$

Trivially $\bigcup_{H\in S} P(H) \subseteq P(J)$. Thus the left vertical arrow is a monomorphism. By [22, Theorem 1.9.10], the right vertical arrow is an epimorphism. Also since $\bigcup_{H\in S} M(H) \subseteq M(J)$, it follows immediately that $M(J) = \bigcup_{H\in S} M(H)$. □

Definition 4.6. An R-module M is called a w**-countably generated module** if there exist a countably generated module M_0 and a w-isomorphism $f: M_0 \to M$; equivalently, there is a countably generated submodule N of M such that $N_{\mathfrak{m}} = M_{\mathfrak{m}}$ for any maximum w-ideal $\mathfrak{m}$.

Let M, N be R-modules. Then M and N are said to be w**-isomorphic** if there exist an R-module A, a w-isomorphism (mapping) $f: A \to M$, and a w-isomorphism (mapping) $g: A \to N$. Obviously, if a homomorphism $f: M \to N$ is a w-isomorphism (mapping), then M and N are w-isomorphic.

Lemma 4.7. *Let F be a w-module and have a direct sum decomposition $F = \bigoplus_{i\in I} F_i$, where F_i is countably generated. Let*

$$\xi: 0 \longrightarrow P \xrightarrow{f} F \xrightarrow{g} M \longrightarrow 0$$

be a w-split exact sequence. If H is a proper subset of I and satisfies that

$$\xi_H: 0 \longrightarrow P(H) \xrightarrow{f_H} F(H) \xrightarrow{g_H} M(H) \longrightarrow 0$$

is a w-split exact sequence induced by ξ, then:

(1) There exists a subset $H_1 \supset H$ of I such that

$$\xi_{H_1}: 0 \longrightarrow P(H_1) \xrightarrow{f_{H_1}} F(H_1) \xrightarrow{g_{H_1}} M(H_1) \longrightarrow 0$$

is also a w-split exact sequence induced by ξ.

(2) $C := M(H_1)/M(H)$ is a countably generated module.

(3) If M is a GV-torsion-free module, then $D := M(H_1)_w/M(H)_w$ is w-isomorphic to C, and so D is a w-countably generated module.

(4) If each F_j is a projective module, then $M(H)$ and $M(H_1)$ are all w-split modules induced by ξ, and C is a w-split module, and so D is a w-countably generated w-projective module.

Proof. (1) Let $J := (d_1, \ldots, d_n)$ and q_k, h_k be as in Lemma 4.2. Then for any $j \in I$, both $fq_k(F_j)$ and $h_k g(F_j)$ are countably generated. If $x \in F_j$, it follows from Remark 4.3 that $d_k x = fq_k(x) + h_k g(x)$. Thus $d_k F_j \subseteq fq_k(F_j) + h_k g(F_j)$, $k = 1, \ldots, n$.

Take an $i_0 \in I \setminus H$. Then there exists a countable subset $I_1 \subseteq I$ such that $d_k F_{i_0} \subseteq fq_k(F_{i_0}) + h_k g(F_{i_0}) \subseteq \bigoplus_{i\in I_1} F_i$, $k+1, \ldots, n$. Since F_i is countably generated and I_1 is a countable subset, $\bigoplus_{i\in I_1} F_i$ is countably generated. Thus there exists a countable subset $I_2 \subseteq I$ such that $I_1 \subseteq I_2$ and $d_k(\bigoplus_{j\in I_1} F_j) \subseteq fq_k(\bigoplus_{j\in I_1} F_j) + h_k g(\bigoplus_{j\in I_1} F_j) \subseteq \bigoplus_{i\in I_2} F_i$, $k = 1,\ldots,n$. Inductively we obtain subsets $I_0 = \{i_0\}, I_1, I_2, \ldots, I_s, \ldots$ satisfying

$$d_k(\bigoplus_{j\in I_s} F_j) \subseteq fq_k(\bigoplus_{j\in I_s} F_j) + h_k g(\bigoplus_{j\in I_s} F_j) \subseteq \bigoplus_{i\in I_{s+1}} F_i, k = 1,\ldots,n \tag{4.1}$$

Hence $J(\bigoplus_{j\in I_s} F_j) \subseteq \bigoplus_{i\in I_{s+1}} F_i$. Since F_i are all w-modules, $\bigoplus_{j\in I_s} F_j \subseteq \bigoplus_{i\in I_{s+1}} F_i$.

Set $L := \bigcup_{k=1}^{\infty} I_k$. Then L is a countable set, and thus $L_1 := L \setminus H$ is a countable set. Set $H_1 := H \cup L$. Then $H_1 = H \cup L_1$. Thus $V := \bigoplus_{i\in L_1} F_i$ is countably generated. Write $U = F(H)$ and $W = F(H_1)$. Then $W = U \oplus V$.

We want to prove that $0 \to P(H_1) \to F(H_1) \to M(H_1) \to 0$ is a w-split exact sequence induced by the exact sequence ξ. It suffices to prove that $h_k(M(H_1)) \subseteq F(H_1)$, that is, for any $j \in H_1$, we have $h_k g(F_j) \subseteq F(H_1), k = 1,\ldots,n$.

If $j \in H$, then $h_k g(F_j) \subseteq F(H) \subseteq F(H_1)$ since ξ_H is a w-split exact sequence. If $j \in L_1$, then there exists s such that $j \in I_s$. It follows from (4.1) that $h_k g(F_j) \subseteq F(H_1)$.

(2) By [22, Theorem 1.9.10], we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P(H) & \longrightarrow & F(H) & \longrightarrow & M(H) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P(H_1) & \longrightarrow & F(H_1) & \longrightarrow & M(H_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & V & \xrightarrow{g'} & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where g' is the homomorphism induced from the right upper square. Since V is countably generated, C is also countably generated.

(3) Observing the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M(H) & \longrightarrow & M(H_1) & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow h & & \\
0 & \longrightarrow & M(H)_w & \longrightarrow & M(H_1)_w & \longrightarrow & D & \longrightarrow & 0
\end{array}$$

it follows by [22, Theorem 1.9.10] that we have an exact sequence $0 \to \mathrm{Ker}(h) \to M(H)_w/M(H) \to M(H_1)_w/M(H_1) \to \mathrm{Coker}(h) \to 0$. Thus $\mathrm{Ker}(h)$ and $\mathrm{Coker}(h)$ are all GV-torsion modules, and so D is w-isomorphic to C.

(4) By the hypothesis, $M(H)$ and $M(H_1)$ are w-split modules induced by ξ. Note that for $k = 1,\ldots,n$, the restrictions of h_k on $M(H)$ and $M(H_1)$ make the upper right dashed diagram a commutative diagram. Thus there exists a homomorphism $\alpha_k : C \to V$ such that $g'\alpha_k = d_k \mathbf{1}_C$. Thus the bottom row of the above diagram is also a w-split exact sequence. Therefore C is a w-split module. □

Let M be a w-split module and $\xi : 0 \to P \to F \to M \to 0$ be a w-split exact sequence, where $F = \bigoplus_{i\in I} F_i$ and each F_i is a countably generated projective module. Let N be a submodule of M. If there is a subset H of I and a continuous ascending chain of subsets of H (if α is a limit ordinal number, then define $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$)

$$\varnothing = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_\alpha \subseteq \cdots \subseteq H_\mu = H$$

such that $N = M(H)$, each $N_\alpha := M(H_\alpha)$ is a w-split module induced by ξ, and in the corresponding continuous ascending chain of submodules

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots \subseteq N_\mu = N, \tag{4.2}$$

where each factor $N_{\alpha+1}/N_\alpha$ is a countably generated w-split module, then N is called a **submodule of** M **with respect to the** w**-split** $\aleph_0$**-continuous ascending chain of** ξ or N **has a** w**-split** $\aleph_0$**-continuous ascending chain of** ξ. When $N = M$, we ignore ξ and simply it is called that M has a w-split $\aleph_0$-continuous ascending chain.

Note that each N_α in (4.2) is a submodule of M with respect to the w-split $\aleph_0$-continuous ascending chain of ξ. In addition, let B be also a submodule of M and have a w-split $\aleph_0$-continuous ascending chain of ξ

$$0 = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_\alpha \subseteq \cdots \subseteq B_\lambda = B.$$

If there exists an ordinal μ such that $\mu \leqslant \lambda$ and $N_\alpha = B_\alpha$ for each $\alpha \leqslant \mu$, then B is called an $\aleph_0$**-filtered extension of** N **with respect to** ξ, simply B is called an $\aleph_0$**-filtered extension of** N.

Lemma 4.8. *Let M be a w-split module and the corresponding ξ, F be assumed as above. Assume that $\{A_i\}$ is a totally ordered family of submodules of M with respect to a w-split $\aleph_0$-continuous ascending chain of ξ, and that if $A_i \subseteq A_j$, then A_j is an $\aleph_0$-filtered extension of A_i. Then $N := \bigcup_i A_i$ is also a w-split $\aleph_0$-continuous ascending chain of submodule of M.*

Proof. Write $A_i = M(H_i)$, where $H_i \subseteq I$. By Lemma 4.5, $N = \bigcup_i A_i = M(\bigcup_i H_i)$ is a w-split module induced by ξ. In the following we use the fact that each A_i has a w-split $\aleph_0$-continuous ascending chain of ξ and the $\aleph_0$-extension property to construct a w-split $\aleph_0$-continuous ascending chain (4.2) of ξ of N. For each index i, there exist an ordinal λ_i and a w-split $\aleph_0$-continuous ascending chain of A_i with respect to ξ:

$$0 = A_{i0} \subseteq A_{i1} \subseteq A_{i2} \subseteq \cdots \subseteq A_{i\alpha} \subseteq \cdots \subseteq A_{i\lambda_i} = A_i, \tag{4.3}$$

If there is an index i such that $N = A_i$, then we have nothing to prove. So we assume that for any i, $A_i \neq N$. Now to initiate our structure, choose arbitrary a w-split $\aleph_0$-continuous ascending chain (4.2) of ξ. Thus for an ordinal number $\alpha \leqslant \lambda_i$, set $N_0 = 0 = A_{i0}, N_1 = A_{i1}, \ldots, N_{\lambda_i} = A_{i\lambda_i} = A_i$. Hence for $\alpha \leqslant \lambda_i$, all N_α are constructed. By the definition of the $\aleph_0$-filtered extension, the choice of N_α has nothing to do with the subscript i that satisfies $\alpha \leqslant \lambda_i$, that is, for any subscript i satisfying $\alpha \leqslant \lambda_i$,

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_\gamma \subseteq \cdots \subseteq N_\alpha$$

is a subchain of (4.3) and $N_{\lambda_i} = A_i$.

Since $A_i = N_{\lambda_i} \neq N$, there exists A_j such that $A_j \not\subseteq A_i$. Thus by the hypothesis, $A_i \subset A_j$. Set $N_{\lambda_i+1} = A_{j(\lambda_i+1)}, N_{\lambda_2+1} = A_{j(\lambda_i+2)}, \ldots, N_{\lambda_j} = A_{j(\lambda_j)}$. By our construction method, for a given ordinal number α and the subscript satisfying $\alpha \leqslant \lambda_j$, all N_α are constructed.

Continue the above process. So we've reached this point: For a given ordinal α, satisfying that for any i, there is always $\lambda_i < \alpha$, and when $\beta < \alpha$, $N_\beta (\neq N)$ has been constructed, and there is a subscript i such that $N_\beta = A_i$. At this time, α is a limit ordinal, otherwise $\beta := \alpha - 1 < \alpha$. Thus there exists a subscript i such that $N_\beta \subseteq A_i$. If $N_\beta \neq A_i$, then $\beta < \lambda_i$, and so $\alpha \leqslant \lambda_i$, a contradiction. If $\beta = \lambda_i$, then there exists a subscript j such that $A_i \subset A_j$ since $A_i \neq N$. Thus $\lambda_i < \lambda_j$, and so $\alpha \leqslant \lambda_j$, a contradiction. Set $N_\alpha := \bigcup_{\beta<\alpha} N_\beta$. Then $N_\alpha \subseteq \bigcup A_i = N$. On the other hand, since $N_{\lambda_i} = A_i$, we have $N \subseteq N_\alpha$. It follows immediately that $N_\alpha = N$. Taking $\mu = \alpha$, we get the required continuous ascending chain (4.2). □

Let M be a w-projective w-module. If there is a continuous ascending chain

$$0 = M'_0 \subseteq M'_1 \subseteq M'_2 \subseteq \cdots \subseteq M'_\alpha \subseteq \cdots \subseteq M'_\mu = M$$

of w-projective w-submodules of M such that each factor $M'_{\alpha+1}/M'_\alpha$ is a w-$\aleph_0$-generated w-projective module, then it is said that M has a **w-projective w-$\aleph_0$-continuous ascending chain**.

Theorem 4.9. (**w-version of Kaplansky's theorem**) Let M be a w-projective w-module. Then we have:

(1) M has a w-split $\aleph_0$-continuous ascending chain.

(2) M has a w-projective w-$\aleph_0$-continuous ascending chain.

Proof. (1) Let $\xi : 0 \to P \to F \to M \to 0$ be an exact sequence, where F is a projective module. By Kaplansky's theorem, we may assume that $F = \bigoplus_{i\in I} F_i$, where each F_i is a countably generated projective module. It follows from [25, Proposition 2.7] that ξ is a w-split exact sequence. Let the notation be as in Lemma 4.7 and let S be a set of subsets H of I satisfying:

(a) If $H \in S$, then $\xi_H : 0 \to P(H) \to F(H) \to M(H) \to 0$ is a w-split exact sequence induced by ξ.

(b) $M(H)$ has a w-split $\aleph_0$-continuous ascending chain with respect to ξ.

When $H = \emptyset$, $F(H) = 0$ and $M(H) = 0$, and so S is not empty. Define a partial order as follows:

$$H_1 \leqslant H_2 \Leftrightarrow H_1 \subseteq H_2 \; mboxand \; M(H_2) \text{ is an } \aleph_0\text{-filtered extension of } M(H_1)$$

Then S is a partially ordered set. Let $S_1 = \{H_s\}$ be a totally ordered subset of S. Set $H := \bigcup_s H_s$. By Lemma 4.5, ξ_H is a w-split exact sequence induced by ξ, and $M(H) = \bigcup_s M(H_s)$. By Lemma 4.8, $M(H)$ has a w-split $\aleph_0$-continuous ascending chain with respect to ξ. Thus $H \in S$. Hence H is an upper bound of S_1. By Zorn's lemma, S has a maximal element, still denoted by H.

If $H \neq I$, then it follows by Lemma 4.7 that there exists $H_1 \supset H$ such that $0 \to M(H) \to M(H_1) \to C \to 0$ is an exact sequence, and $M(H), M(H_1)$, and C are w-split modules induced by ξ, and C is countably generated. Thus $H_1 \in S$, which contradicts the maximality of H. Therefore $H = I$, and so $M(H) = M$. It follows immediately that M has a w-split $\aleph_0$-continuous ascending chain.

(2) Let $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots \subseteq M_\lambda = M$ be a w-split $\aleph_0$-continuous ascending chain of M. For any ordinal number α, set $M'_\alpha := (M_\alpha)_w$. Then M'_α is a w-projective w-submodule of M. Similarly to the proof of Lemma 4.7(3), we can prove that $M'_{\alpha+1}/M'_\alpha$ is w-isomorphic to $M_{\alpha+1}/M_\alpha$. Therefore $M'_{\alpha+1}/M'_\alpha$ is a w-$\aleph_0$-generated w-projective module. □

4.2 Cotorsion theory of weak w-projective modules

The contents of this subsection are excerpts from [17]. Denote by $\mathcal{FI}$ the class of GV-torsion-free modules, by $\mathcal{W}$ the class of w-modules, and by $\mathcal{W}_\infty$ the class of strong w-modules.

Let $\mathcal{S}$ be a class of modules. Define:

$$\begin{aligned} \mathcal{S}^\dagger &:= \mathcal{S}^\perp \cap \mathcal{FT} \\ &= \{N \in \mathfrak{M} \mid N \text{ is GV-torsion-free and } \operatorname{Ext}^1_R(M,N) = 0 \text{ for any } M \in \mathcal{S}\} \end{aligned}$$

Correspondingly define:

$$\begin{aligned} \mathcal{S}^{\dagger_\infty} &:= \mathcal{S}^{\perp_\infty} \cap \mathcal{FT} \\ &= \left\{ N \in \mathfrak{M} \;\middle|\; \begin{array}{l} N \text{ is GV-torsion-free and} \\ \operatorname{Ext}^k_R(M,N) = 0 \text{ for any } M \in \mathcal{S} \text{ and any } k \geqslant 1 \end{array} \right\}. \end{aligned}$$

Set $\mathrm{GV}(R)^* := \{R/J \mid J \in \mathrm{GV}(R)\}$. Obviously $\mathrm{GV}(R)^*$ is a set of modules.

Example 4.10. (1) It is easy to see that $(\mathrm{GV}(R)^*)^{\dagger} = \mathcal{W}$.

(2) $(\mathrm{GV}(R)^*)^{\dagger_\infty} = \mathcal{W}_\infty$. Indeed, this follows from the fact that for a GV-torsion-free module N, N is a strong w-module if and only if $\mathrm{Ext}_R^k(R/J, N) = 0$ for any $J \in \mathrm{GV}(R)$ and any $k \geqslant 1$.

Proposition 4.11. *Let $\mathcal{S}, \mathcal{S}_1$ be classes of modules. Then:*

(1) $\mathcal{S} \subseteq {}^{\perp}(\mathcal{S}^{\dagger_\infty}) \subseteq {}^{\perp}(\mathcal{S}^{\dagger})$.

(2) If $\mathcal{S} \subseteq \mathcal{S}_1$, then $\mathcal{S}_1^{\dagger} \subseteq \mathcal{S}^{\dagger}$ and $\mathcal{S}_1^{\dagger_\infty} \subseteq \mathcal{S}^{\dagger_\infty}$.

(3) $(\mathcal{S} \cup \mathcal{S}_1)^{\dagger} = \mathcal{S}^{\dagger} \cap \mathcal{S}_1^{\dagger}$.

(4) If $\mathrm{GV}(R)^* \subseteq \mathcal{S}$, then $\mathcal{S}^{\dagger} \subseteq \mathcal{W}$ and $\mathcal{S}^{\dagger_\infty} \subseteq \mathcal{W}_\infty$.

Proof. These are obvious. □

In order to make Theorem 3.11 apply to the context of a class of w-modules, we make corresponding modifications to it, but note that the idea belongs to Eklof–Trlifaj essentially.

Lemma 4.12. *Let $\mathcal{S} = \mathrm{GV}(R)^* \cup \mathcal{S}_1$ be a set of modules, where $\mathcal{S}_1 \subseteq \mathcal{FT}$.*

(1) Let N be a GV-torsion-free module. Then there exists an exact sequence

$$0 \to N \to Q \to A \to 0,$$

where $Q \in \mathcal{S}^{\dagger}$ and A is an $\mathcal{S}$-filtered module such that $A \in {}^{\perp}(\mathcal{S}^{\dagger})$.

(2) Let M be an R-module. Then there exists an exact sequence

$$0 \to B \to P \to M \to 0,$$

where $P \in {}^{\perp}(\mathcal{S}^{\dagger})$ and $B \in \mathcal{S}^{\dagger}$.

Proof. (1) Set $X := \bigoplus\limits_{S \in \mathcal{S}_1} S$ and $Y := \bigoplus\limits_{J \in \mathrm{GV}(R)} R/J$. Then X is a GV-torsion-free module and Y is a GV-torsion module. Set $S = X \oplus Y$. Then $\mathcal{S}^{\perp} = \{S\}^{\perp}$. Thus we may assume that $\mathcal{S}$ is the class of modules consists of a specific module S and its direct sums. Let $0 \to K_1 \xrightarrow{\mu_1} F_1 \to X \to 0$ and $0 \to K_2 \xrightarrow{\mu_2} F_2 \to Y \to 0$ be exact sequences, where F_1 and F_2 are free modules. Set $F := F_1 \oplus F_2$ and $K := K_1 \oplus K_2$. Then $0 \to K \xrightarrow{\mu} F \to S \to 0$ is an exact sequence, where $\mu := \mu_1 \oplus \mu_2$. Since X is GV-torsion-free, K_1 is a w-module. Since Y is GV-torsion, $(K_2)_w = F_2$

Take a regular cardinal λ so that K has a generating system X with $|X| < \lambda$.

Set $Q_0 := N$. Then Q_0 is GV-torsion-free. For $\alpha < \lambda$, if Q_α has been constructed, select a free module F_α' and an epimorphism $\delta_\alpha : F_\alpha' \to Q_\alpha$. Set $I_\alpha := \mathrm{Hom}_R(K, Q_\alpha)$ to be a new index set and define $\mu_\alpha : K^{(I_\alpha)} \to F^{(I_\alpha)}$ as the homomorphism of direct sums, which is induced by μ. Then μ_α is a monomorphism and $\mathrm{Coker}(\mu_\alpha) = S^{(I_\alpha)}$.

Define $\varphi_\alpha : K^{(I_\alpha)} \oplus F_\alpha' = (\bigoplus\limits_{f \in I_\alpha} K_f) \oplus F_\alpha' \to Q_\alpha$, where $K_f = K$, by $\varphi_\alpha([u_f], z) = \sum\limits_{f \in I_\alpha} f(u_f) + \delta_\alpha(z)$, where $u_f \in K_f, z \in F_\alpha'$. Since δ_α is an epimorphism, so is φ_α. Now assume that if $\beta \leqslant \alpha$, then Q_β has been constructed (if α is a limit ordinal, set $Q_\alpha := \bigcup\limits_{\beta < \alpha} Q_\beta$), in particular, Q_α has been constructed. Construct the following pushout diagram:

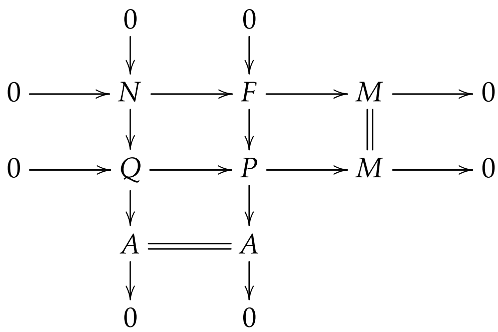
$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^{(I_\alpha)} \oplus F_\alpha' & \xrightarrow{\mu_\alpha \oplus 1} & F^{(I_\alpha)} \oplus F_\alpha' & \longrightarrow & S^{(I_\alpha)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_\alpha} & & \downarrow{\scriptstyle \psi_\alpha} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & Q_\alpha & \xrightarrow{h_\alpha} & Q_{\alpha+1} & \longrightarrow & Q_{\alpha+1}/Q_\alpha & \longrightarrow & 0
\end{array}$$

One gets $Q_{\alpha+1}$. At this time ψ_α is an epimorphism. As you can see from the above diagram, if Q_α is a GV-torsion-free module, then $\mathrm{Ker}(\psi_\alpha) \cong \mathrm{Ker}(\varphi_\alpha)$ is a w-module, and thus $Q_{\alpha+1}$ is also a GV-torsion-free module. Hence by a transfinite induction, we see that each Q_α is a GV-torsion-free module.

Set $Q := \bigcup_{\alpha<\lambda} Q_\alpha = \varinjlim_{\alpha<\lambda} Q_\alpha$. Then Q is a GV-torsion-free module. Set $A := Q/N$ and $A_\alpha := Q_\alpha/N$. Then $A_{\alpha+1}/A_\alpha \cong Q_{\alpha+1}/Q_\alpha \cong S^{(I_\alpha)}$. Since $Q = \bigcup_{\alpha<\lambda} Q_\alpha$, one gets that $A = \bigcup_{\alpha<\lambda} A_\alpha$. Thus A is an $\mathcal{S}$-filtered module, and thus one has $A \in {}^\perp(\mathcal{S}^\perp)$. Since $\mathcal{S}^\dagger \subseteq \mathcal{S}^\perp$, one has $A \in {}^\perp(\mathcal{S}^\dagger)$.

Similarly to the process of Theorem 3.11, one can prove that $Q \in \mathcal{S}^\perp$. Therefore $Q \in \mathcal{S}^\perp \cap \mathcal{FT} = \mathcal{S}^\dagger$.

(2) Take an exact sequence $0 \to N \to F \to M \to 0$, where F is a projective module. Then N is a GV-torsion-free module. By (1), there is an exact sequence $0 \to N \to Q \to A \to 0$, where $Q \in \mathcal{S}^\dagger$ and $A \in {}^\perp(\mathcal{S}^\dagger)$. Consider the following commutative diagram with exact rows:



where the square diagrams in the upper left and lower corners are pushout diagrams. Since $F, A \in {}^\perp(\mathcal{S}^\dagger)$, one has $P \in {}^\perp(\mathcal{S}^\dagger)$. Therefore one gets the desired sequence by taking $B := Q$. □

In order to make Theorem 3.11 suitable for the relevant module classes under the w-module framework, we make corresponding transformations to it.

Theorem 4.13. Let $\mathcal{S} = \mathrm{GV}(R)^* \cup \mathcal{S}_1$ be a set of modules, where $\mathcal{S}_1 \subseteq \mathcal{FT}$. Set $\mathcal{A} := {}^\perp(\mathcal{S}^\dagger)$. If $\mathcal{A}$ is closed under w-isomorphisms, then $(\mathcal{A}, \mathcal{A}^\perp)$ is a complete cotorsion theory.

Proof. Note that $(\mathcal{A}, \mathcal{A}^\perp)$ is the cotorsion theory generated by $\mathcal{S}^\dagger$. In the following, we prove that every module M has a special $\mathcal{A}$-precover.

By Lemma 4.12, there exists an exact sequence $0 \to B \to P \to M \to 0$, where $P \in \mathcal{A}$, $B \in \mathcal{S}^\dagger \subseteq [{}^\perp(\mathcal{S}^\dagger)]^\perp = \mathcal{A}^\perp$. Therefore M has a special $\mathcal{A}$-precover.

□

Proposition 4.14. *Let $\mathcal{S}$ be a class of modules such that* $\mathrm{GV}(R)^* \subseteq \mathcal{S}$. *Set $\mathcal{B} := {}^\perp(\mathcal{S}^{\dagger_\infty})$. Then:*

(1) $\mathcal{S}^{\dagger_\infty}$ is closed under direct products, direct summands, and cokernels of monomorphisms.

(2) $\mathcal{B}$ is closed under direct sums, direct summands, kernels of epimorphisms, and w-isomorphisms.

(3) $\mathcal{B}^\dagger = \mathcal{B}^{\dagger_\infty} = \mathcal{S}^{\dagger_\infty}$.

Proof. (1) Obviously $\mathcal{L}^{\dagger_\infty}$ is closed under direct products and direct summands. By Theorem 3.14, $\mathcal{S}^{\perp_\infty}$ is closed under cokernels of monomorphisms. By [24, Proposition 2.2], $\mathcal{W}_\infty$ is also closed under cokernels of monomorphisms. Since $\mathcal{S}^{\dagger_\infty} = \mathcal{S}^{\perp_\infty} \cap \mathcal{W}_\infty$, it follows that $\mathcal{S}^{\dagger_\infty}$ is closed under cokernels of monomorphisms.

(2) Obviously $\mathcal{B}$ is closed under direct sums and direct summands. I follows by (1) that $\mathcal{B}$ is closed under kernels of epimorphisms.

(3) Obviously $\mathcal{S}^{\dagger_\infty} \subseteq \left({}^{\perp}(\mathcal{S}^{\dagger_\infty})\right)^{\perp} \cap \mathcal{FT} = \mathcal{B}^{\dagger}$. Since $\mathcal{B}$ is closed under kernels of epimorphisms, $\mathcal{B}^{\perp_\infty} = \mathcal{B}^{\perp}$. Thus $\mathcal{B}^{\dagger} = \mathcal{B}^{\perp_\infty} \cap \mathcal{FT} = \mathcal{B}^{\dagger_\infty}$. Since $\mathcal{S} \subseteq \mathcal{B}$, it follows that $\mathcal{B}^{\dagger} = \mathcal{B}^{\dagger_\infty} \subseteq \mathcal{S}^{\dagger_\infty}$. Therefore $\mathcal{B}^{\dagger} = \mathcal{S}^{\dagger_\infty}$. □

Proposition 4.15. *(1) Set* $\mathrm{w}\mathcal{P}_w := {}^{\perp}(\mathcal{P}_w^{\dagger_\infty})$*, that is,* $\mathrm{w}\mathcal{P}_w$ *is the class of weak* w*-projective* R*-modules. Then* ${\mathrm{w}\mathcal{P}_w}^{\dagger} = \mathcal{P}_w^{\dagger_\infty}$.

(2) Let $\mathcal{S} = \mathrm{GV}(R)^* \cup \mathcal{S}_1$ *be a set of modules, where* $\mathcal{S}_1$ *is the class of* w*-projective* w*-*$\aleph_0$*-generated* w*-modules. Then* $\mathcal{S}^{\dagger_\infty} = \mathcal{P}_w^{\dagger_\infty}$.

Proof. (1) This follows immediately by setting $\mathcal{S} := \mathcal{P}_w$ in Proposition 4.14.

(2) Since $\mathcal{S} \subseteq \mathcal{P}_w$, we have $\mathcal{P}_w^{\dagger_\infty} \subseteq \mathcal{S}^{\dagger_\infty}$. Let $N \in \mathcal{S}^{\dagger_\infty}$. For any w-projective w-module P, by Theorem 4.9 P is an $\mathcal{S}_1$-filtered module. Thus $\mathrm{Ext}_R^i(P,N) = 0$ for any $i \geqslant 1$. By Proposition 4.11, N is a strong w-module. Let P be a w-projective module. Then $\mathrm{Ext}_R^i(P,N) = 0$ for any w-projective module P and any $i \geqslant 1$. Thus $N \in \mathcal{P}_w^{\perp_\infty} \cap \mathcal{FT} = \mathcal{P}_w^{\dagger_\infty}$. Therefore $\mathcal{S}^{\dagger_\infty} = \mathcal{P}_w^{\dagger_\infty}$. □

Theorem 4.16. Let $\mathcal{S} = \mathrm{GV}(R)^* \cup \mathcal{S}_1$ be a set of modules, where $\mathcal{S}_1 \subseteq \mathcal{FT}$. Set $\mathcal{B} := {}^{\perp}(\mathcal{S}^{\dagger_\infty})$. Then $(\mathcal{B}, \mathcal{B}^{\perp})$ is a hereditary and complete cotorsion theory.

Proof. For each $M \in \mathcal{S}$, fix a projective resolution $\mathbf{P}(M)$ of M. Let $\mathcal{L}_M$ be the set consisting of all syzygies in $\mathbf{P}(M)$ (including M itself) and set $\mathcal{L} := \bigcup_{M \in S} \mathcal{L}_M$. Then $\mathcal{L}$ is naturally a set. By Lemma 3.16(4), $\mathcal{L}^{\perp} = \mathcal{S}^{\dagger_\infty}$, and so $\mathcal{L}^{\dagger} = \mathcal{S}^{\dagger_\infty}$.

Split $\mathcal{L}$ into $\mathcal{L} = \mathrm{GV}(R)^* \cup \mathcal{L}_1$, where $\mathcal{L}_1$ is the set of all syzygies of $M \in \mathcal{S}_1$ and all nonnegative syzygies of R/I. Then $\mathcal{L}_1 \subseteq \mathcal{FT}$. By Proposition 4.14, $(\mathcal{A}, \mathcal{A}^{\perp})$ is a hereditary cotorsion theory. □

Theorem 4.17. $(\mathrm{w}\mathcal{P}_w, {\mathrm{w}\mathcal{P}_w}^{\perp})$ is a hereditary and complete cotorsion theory, and so every module has a special weak w-projective precover.

Proof. Let $\mathcal{S}_1$ be the set of all w-countably generated w-projective w-modules and set $\mathcal{S} = \mathrm{GV}(R)^* \cup \mathcal{S}_1$. Since the collection of all countably generated modules is a set, $\mathcal{S}$ is also a set. By Proposition 4.15, $\mathcal{S}^{\dagger_\infty} = \mathcal{P}_w^{\dagger_\infty}$. Thus $\mathrm{w}\mathcal{P}_w = {}^{\perp}(\mathcal{S}^{\dagger_\infty})$. By Theorem 4.16, $(\mathrm{w}\mathcal{P}_w, {\mathrm{w}\mathcal{P}_w}^{\perp})$ is a hereditary and complete cotorsion theory. □

Proposition 4.18. *Let* M *be a* w*-module. Then there exists a special weak* w*-projective precover* $\varphi : P \to M$ *of* M *such that* P *is a* w*-module and* $\mathrm{Ker}(\varphi) \in \mathcal{P}_w^{\dagger_\infty}$.

Proof. Let the notation be as in the proof of Theorem 4.16 and $\mathcal{S}$ be as in Proposition 4.15(2). Then $\mathcal{L}^{\dagger} = \mathcal{S}^{\dagger_\infty} = \mathcal{P}_w^{\dagger_\infty}$. By Lemma 4.12, M has a special weak w-projective precover $0 \to B \to P \to M \to 0$, where $P \in \mathrm{w}\mathcal{P}_w$, $B \in \mathcal{L}^{\dagger} = P_w^{\dagger_\infty}$. Thus B is a strong w-module, and so P is a w-module. □

5 Homology methods of cotorsion theory

5.1 Homology of cotorsion pairs

When a cotorsion theory is given, naturally we can construct the homological dimension of the cotorsion theory. The description of the homological dimension of a general cotorsion theory can be found in the literature [15, 26]. This section presents the homological method for general cotorsion pairs. In [19] the definition of the global cotorsion dimension of a ring is a good example. That is, through a clear understanding of this dimension, we can grasp the homological dimension of all the hereditary and complete cotorsion theories.

Definition 5.1. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion theory and let M and N be R-modules.

(1) We say that M has an **$\mathcal{A}$-resolution with length at most n** if there is an exact sequence

$$0 \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow M \longrightarrow 0, \qquad A_i \in \mathcal{A}.$$

Use $\mathrm{pd}_{\mathcal{A}}(M)$ to represent the minimal length among all (finite) $\mathcal{A}$-resolutions of M. If such a finite exact sequence does not exist, then we say that M has an $\mathcal{A}$-resolution of infinite length, denoted by $\mathrm{pd}_{\mathcal{A}}(M) = \infty$.

(2) Correspondingly, we say that N has a **$\mathcal{B}$-resolution with length at most n** if there is an exact sequence

$$0 \longrightarrow N \longrightarrow B_0 \longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_{n-1} \longrightarrow B_n \longrightarrow 0, \qquad B_i \in \mathcal{B}.$$

Use $\mathrm{id}_{\mathcal{B}}(M)$ to represent the minimal length among all (finite) $\mathcal{B}$-resolutions of M. If such a finite exact sequence does not exist, then we say that N has a $\mathcal{B}$-resolution of infinite length, denoted by $id_{\mathcal{B}}(N) = \infty$.

Example 5.2. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion theory.

(1) $\mathrm{pd}_{\mathcal{A}}(A) = 0$ if and only if $A \in \mathcal{A}$, that is, A is a $\mathcal{B}$-projective module.

(2) $\mathrm{id}_{\mathcal{B}}(B) = 0$ if and only if $B \in \mathcal{B}$, that is, B is an $\mathcal{A}$-injective module.

If $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion theory, then each module has a special $\mathcal{A}$-precover and a special $\mathcal{B}$-preenvelope. Although a special $\mathcal{A}$-precover and a special $\mathcal{B}$-preenvelope of a module are not unique, for any module N, in order to simplify the statement, we still use $\mathcal{A}(N)$ and $\mathcal{B}(N)$ to represent a special $\mathcal{A}$-precover and a special $\mathcal{B}$-preenvelope of N respectively.

Theorem 5.3. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion theory, n be a nonnegative integer, and M be an R-module. Then the following are equivalent:

(1) $\mathrm{pd}_{\mathcal{A}}(M) \leqslant n$.

(2) $\mathrm{Ext}_R^{n+1}(M, B) = 0$ for any $B \in \mathcal{B}$.

(3) $\mathrm{Ext}_R^{k}(M, B) = 0$ for any $B \in \mathcal{B}$ and any $k > n$.

(4) If $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ is an exact sequence, where $P_0, P_1, \ldots, P_{n-1}$ are projective modules, then $P_n \in \mathcal{A}$.

(5) If $0 \to A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to M \to 0$ is an exact sequence, where $A_0, A_1, \ldots, A_{n-1} \in \mathcal{A}$, then $A_n \in \mathcal{A}$.

If $\mathfrak{C}$ is also a complete cotorsion theory, then each of the above conditions is equivalent to:

(6) $\mathrm{pd}_{\mathcal{A}}(\mathcal{B}(M)) \leqslant n$.

Proof. (1)⇒(3) By the hypothesis, for any $k > n$, there exists an exact sequence

$$0 \to A_k \to A_{k-1} \to \cdots \to A_{n+1} \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0,$$

where if $0 \leqslant i \leqslant n$, then $A_i \in \mathcal{A}$, and if $n < i \leqslant k$, then $A_i = 0$. Denote by L_i the i-th syzygy of M in the above-mentioned exact sequence. Then $L_{n-1} = A_n$ and if $n \leqslant i \leqslant k-1$, then $L_i = 0$. For any given $B \in \mathcal{B}$, since $(\mathcal{A}, \mathcal{B})$ is a hereditary cotorsion theory,

$$\mathrm{Ext}_R^k(M, B) \cong \mathrm{Ext}_R^{k-1}(L_0, B) \cong \mathrm{Ext}_R^{k-2}(L_1, B) \cong \cdots \cong \mathrm{Ext}_R^1(L_{k-2}, B).$$

If $k = n+1$, then $\mathrm{Ext}_R^1(L_{n-1}, B) = \mathrm{Ext}_R^1(A_n, B) = 0$. And if $k \geqslant n+2$, then $L_{k-2} = 0$, and so trivially $\mathrm{Ext}_R^1(L_{k-2}, B) = 0$. Therefore, if $k > n$, then $\mathrm{Ext}_R^k(M, B) = 0$.

(3)⇒(2) Obvious.

(2)⇒(5) Let $0 \to A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to M \to 0$ be an exact sequence, where $A_0, A_1, \ldots, A_{n-1} \in \mathcal{A}$. Denote by L_i the i-th $\mathcal{A}$-syzygy. Then $L_{n-1} = A_n$. For any given $B \in \mathcal{B}$,

$$\mathrm{Ext}_R^{n+1}(M, B) \cong \mathrm{Ext}_R^n(L_0, B) \cong \mathrm{Ext}_R^{n-1}(L_1, B) \cong \cdots \cong \mathrm{Ext}_R^1(L_{n-1}, B) = \mathrm{Ext}_R^1(A_n, B).$$

Since $\mathrm{Ext}_R^{n+1}(M, B) = 0$, it follows that $\mathrm{Ext}_R^1(A_n, B) = 0$. Therefore $A_n \in \mathcal{A}$.

(5)⇒(4) Let $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ be an exact sequence, where $P_0, P_1, \ldots, P_{n-1}$ are projective modules. Since $\mathcal{P} \subseteq \mathcal{A}$, we have $P_0, P_1, \ldots, P_{n-1} \in \mathcal{A}$. It follows from (5) that $P_n \in \mathcal{A}$.

(4)⇒(1) For any given R-module M, there is an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, where $P_0, P_1, \ldots, P_{n-1}$ are projective modules. It follows from (4) that $P_n \in \mathcal{A}$. Thus $\mathrm{pd}_{\mathcal{A}}(M) \leqslant n$.

(1)⇔(6) Since $\mathfrak{C}$ is perfect, there is an exact sequence $0 \to M \to \mathcal{B}(M) \to A \to 0$, where $A \in \mathcal{A}$. For any given $B \in \mathcal{B}$, since $\mathfrak{C}$ is hereditary, $\mathrm{Ext}_R^k(A, B) = 0$ for any $k > 0$. Hence there exists a natural isomorphism $\mathrm{Ext}_R^k(M, B) \cong \mathrm{Ext}_R^k(\mathcal{B}(M), B)$. Therefore $\mathrm{pd}_{\mathcal{A}}(M) = \mathrm{pd}_{\mathcal{A}}(\mathcal{B}(M))$. □

Theorem 5.4. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion theory, n be a nonnegative integer, and N be an R-module. Then the following are equivalent:

(1) $\mathrm{id}_{\mathcal{B}}(N) \leqslant n$.

(2) $\mathrm{Ext}_R^{n+1}(A, N) = 0$ for any $A \in \mathcal{A}$.

(3) $\mathrm{Ext}_R^k(A, N) = 0$ for any $A \in \mathcal{A}$ and any $k > n$.

(4) If $0 \to N \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$ is an exact sequence, where $E_0, E_1, \ldots, E_{n-1}$ are injective modules, then $E_n \in \mathcal{B}$.

(5) If $0 \to N \to B_0 \to B_1 \to \cdots \to B_{n-1} \to B_n \to 0$ is an exact sequence, where $B_0, B_1, \ldots, B_{n-1} \in \mathcal{B}$, then $B_n \in \mathcal{B}$.

If $\mathfrak{C}$ is also a complete cotorsion theory, then each of the above conditions is equivalent to:

(6) $\mathrm{id}_{\mathcal{B}}(\mathcal{A}(N)) \leqslant n$.

Proof. By Theorem 5.3, this can be proved dually. □

Definition 5.5. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion theory. Define:

(1) $\mathrm{gld}_{\mathcal{A}}(R) = \sup\{\mathrm{pd}_{\mathcal{A}}(M) \mid M$ is any R-module$\}$, which is called the **global $\mathfrak{C}_{\mathcal{A}}$-dimension** of R.

(2) $\mathrm{gld}_{\mathcal{B}}(R) = \sup\{\mathrm{id}_{\mathcal{B}}(N) \mid N$ is any R-module$\}$, which is called the **global $\mathfrak{C}_{\mathcal{B}}$-dimension** of R.

Theorem 5.6. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion theory and let n be a nonnegative integer. Then the following are equivalent:

(1) $\mathrm{gld}_{\mathcal{A}}(R) \leqslant n$.

(2) $\mathrm{Ext}_R^{n+1}(M, B) = 0$ for any $B \in \mathcal{B}$ and $M \in \mathfrak{M}$.

(3) $\mathrm{Ext}_R^k(M, B) = 0$ for any $B \in \mathcal{B}$, $M \in \mathfrak{M}$, and any $k > n$.

(4) $\mathrm{id}_R B \leqslant n$ for any $B \in \mathcal{B}$.

If $\mathfrak{C}$ is also a complete cotorsion theory, then each of the above conditions is equivalent to:

(5) $\mathrm{Ext}_R^{n+1}(B,B')=0$ for any $B,B'\in\mathcal{B}$.

(6) $\mathrm{Ext}_R^{k}(B,B')=0$ for any $B,B'\in\mathcal{B}$, and any $k>n$.

(7) $\mathrm{pd}_{\mathcal{A}}(B)\leqslant n$ for any $B\in\mathcal{B}$.

Proof. (1)⇒(3) For any given R-module M, by the hypothesis, $\mathrm{pd}_{\mathcal{A}}(M)\leqslant n$. By Theorem 5.3, we know that $\mathrm{Ext}_R^{k}(M,B)=0$ for any given $B\in\mathcal{B}$ and any $k>n$.

(3)⇒(2) This follows by taking $k=n+1$.

(2)⇒(1) For any $B\in\mathcal{B}$ and $M\in\mathfrak{M}$, we have $\mathrm{Ext}_R^{n+1}(M,B)=0$. By Theorem 5.3, $\mathrm{pd}_{\mathcal{A}}(M)\leqslant n$. Therefore $\mathrm{gld}_{\mathcal{A}}(R)\leqslant n$

(2)⇔(4) This follows from [22, Theorem 3.5.10].

(3)⇒(6)⇒(5) Trivial.

(5)⇒(2) Since $(\mathcal{A},\mathcal{B})$ is a complete cotorsion theory, for any module M, there is an exact sequence $0\to M\to B'\to A\to 0$, where $B'\in\mathcal{B}$ and $A\in\mathcal{A}$. For any $B\in\mathcal{B}$, since $(\mathcal{A},\mathcal{B})$ is a hereditary cotorsion theory, there is an exact sequence

$$0=\mathrm{Ext}_R^{n+1}(B',B)\longrightarrow \mathrm{Ext}_R^{n+1}(M,B)\longrightarrow \mathrm{Ext}_R^{n+2}(A,B)=0.$$

Therefore $\mathrm{Ext}_R^{n+1}(M,B)=0$.

(5)⇔(7) This follows from Theorem 5.3. □

Theorem 5.7. Let $\mathfrak{G}=(\mathcal{A},\mathcal{B})$ be a hereditary cotorsion theory and let n be a nonnegative integer. Then the following are equivalent:

(1) $\mathrm{gld}_{\mathcal{B}}(R)\leqslant n$.

(2) $\mathrm{Ext}_R^{n+1}(A,N)=0$ for any $A\in\mathcal{A}$ and $N\in\mathfrak{M}$.

(3) $\mathrm{Ext}_R^{k}(A,N)=0$ for any $A\in\mathcal{A}$, $N\in\mathfrak{M}$, and any $k>n$.

(4) $\mathrm{pd}_R A\leqslant n$ for any $A\in\mathcal{A}$.

If $\mathfrak{G}$ is also a complete cotorsion theory, then each of the above conditions is equivalent to:

(5) $\mathrm{Ext}_R^{n+1}(A,A')=0$ for any $A,A'\in\mathcal{A}$.

(6) $\mathrm{Ext}_R^{n+1}(A,A')=0$ for any $A,A'\in\mathcal{A}$, and any $k>n$.

(7) $\mathrm{id}_{\mathcal{B}}(A)\leqslant n$ for any $A\in\mathcal{A}$.

Proof. By Theorem 5.6, this can be proved dually. □

Corollary 5.8. *Let $\mathfrak{G}=(\mathcal{A},\mathcal{B})$ be a hereditary cotorsion theory. Then:*

(1) $\mathrm{gld}_{\mathcal{A}}(R)=\sup\{\mathrm{id}_R B\mid B\in\mathcal{B}\}$.

(2) $\mathrm{gld}_{\mathcal{B}}(R)=\sup\{\mathrm{pd}_R A\mid A\in\mathcal{A}\}$.

By Corollary 5.8, we can get

$$\mathrm{gl.dim}(R)\overset{\mathrm{def}}{=}\sup\{\mathrm{pd}_R M\mid M\in\mathfrak{M}\}=\sup\{\mathrm{id}_R N\mid N\in\mathfrak{M}\}.$$

Corollary 5.9. *Let $\mathfrak{G}=(\mathcal{A},\mathcal{B})$ be a hereditary complete cotorsion theory. Then:*

(1) $\mathrm{gld}_{\mathcal{A}}(R)=\sup\{\mathrm{pd}_{\mathcal{A}}(B)\mid B\in\mathcal{B}\}$.

(2) $\mathrm{gld}_{\mathcal{B}}(R) = \sup\{\mathrm{id}_{\mathcal{B}}(A) \mid A \in \mathcal{A}\}$.

Example 5.10. (1) For the cotorsion theory $\mathfrak{C} = (\mathcal{P}, \mathfrak{M})$, we have $\mathrm{gld}_{\mathcal{P}}(R) = \mathrm{gl.dim}(R)$ and $\mathrm{gld}_{\mathfrak{M}}(R) = 0$.

(2) For the cotorsion theory $\mathfrak{C} = (\mathfrak{M}, \mathcal{I})$, we have $\mathrm{gld}_{\mathfrak{M}}(R) = 0$ and $\mathrm{gld}_{\mathcal{I}}(R) = \mathrm{gl.dim}(R)$.

Theorem 5.11. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary complete cotorsion theory.

(1) If $\mathrm{gld}_{\mathcal{A}}(R) < \infty$, then

$$\mathrm{gld}_{\mathcal{A}}(R) = \sup\{\mathrm{id}_R N \mid N \in \mathcal{K}\} = \sup\{\mathrm{pd}_{\mathcal{A}}(E) \mid E \in \mathcal{I}\}.$$

(2) If $\mathrm{gld}_{\mathcal{B}}(R) < \infty$, then

$$\mathrm{gld}_{\mathcal{B}}(R) = \sup\{\mathrm{pd}_R M \mid M \in \mathcal{K}\} = \sup\{\mathrm{id}_{\mathcal{B}}(P) \mid P \in \mathcal{P}\}.$$

Proof. (1) Write $\sup\{\mathrm{id}_R N \mid N \in \mathcal{K}\} = n$ and $\mathrm{gld}_{\mathcal{A}}(R) = m$. By Theorem 5.6, $n \leqslant m$. And by Theorem 5.6, for any $B \in \mathcal{B}$, we have $\mathrm{pd}_{\mathcal{A}}(B) \leqslant m$. Thus B has a $\mathfrak{C}$-projective resolution of length at most m

$$0 \to A_m \to A_{m-1} \to \cdots \to A_1 \to A_0 \to B \to 0, \qquad A_i \in \mathcal{A}.$$

Since $\mathfrak{C}$ is a hereditary complete cotorsion theory, we can assume that each syzygy K_i in the just above exact sequence is in $\mathcal{B}$. Since $\mathcal{B}$ is closed under extensions, each $A_i \in \mathcal{K}$. Thus for any module M,

$$\mathrm{Ext}_R^{n+1}(M, B) \cong \mathrm{Ext}_R^{n+2}(M, K_0) \cong \cdots \cong \mathrm{Ext}_R^{n+m+1}(M, A_m) = 0.$$

Hence $m \leqslant n$, and so $n = m$.

Again set $n = \sup\{\mathrm{pd}_{\mathcal{A}}(E) \mid E \in \mathcal{I}\}$. Then $n \leqslant m$. By Theorem 5.6, $\mathrm{id}_R B \leqslant m$ for any $B \in \mathcal{B}$. Thus B has an injective resolution

$$0 \to B \to E_0 \to E_1 \to \cdots \to E_{m-1} \to E_m \to 0, \qquad E_i \in \mathcal{I}.$$

Denote by L_i the i-th cosyzygy of the just above exact sequence. For any $B' \in \mathcal{B}$, since $\sup\{\mathrm{pd}_{\mathcal{A}}(E) \mid E \in \mathcal{I}\} = n$, we have $\mathrm{Ext}_R^k(E_i, B') = 0$ for any $k > n$. Thus

$$\mathrm{Ext}_R^{n+1}(B, B') \cong \mathrm{Ext}_R^{n+2}(L_0, B') \cong \cdots \cong \mathrm{Ext}_R^{n+m+1}(E_m, B) = 0.$$

By Theorem 5.6, $m \leqslant n$. Therefore $n = m$.

(2) By (1), this can be proved dually. □

Theorem 5.12. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion theory and let M be an R-module. Then

(1) $\mathrm{pd}_R M \leqslant \mathrm{gld}_{\mathcal{B}}(R) + \mathrm{pd}_{\mathcal{A}}(M)$.

(2) $\mathrm{id}_R M \leqslant \mathrm{gld}_{\mathcal{A}}(R) + \mathrm{id}_{\mathcal{B}}(M)$.

Proof. (1) We may assume that $m = \mathrm{gld}_{\mathcal{B}}(R) < \infty$ and $n = \mathrm{pd}_{\mathcal{A}}(M) < \infty$. Then M has an $\mathcal{A}$-resolution:

$$0 \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow M \longrightarrow 0, \qquad A_i \in \mathcal{A}.$$

By Theorem 5.7(4), $\mathrm{pd}_R A_i \leqslant m$, $i = 0, 1, \ldots, n$. Let K_i be the i-th syzygy of the above exact sequence. Decompose the exact sequence into n short exact sequences:

$$0 \longrightarrow K_i \longrightarrow A_i \longrightarrow K_{i-1} \longrightarrow 0, \qquad i = 0, 1, \ldots, n-1, \quad K_{-1} = M.$$

Since $K_{n-1} = A_n$, we obtain that $\mathrm{pd}_R K_{n-2} \leqslant m+1$. Successively we obtain that $\mathrm{pd}_R K_{n-3} \leqslant m+2, \cdots$. Finally we get that $\mathrm{pd}_R M \leqslant m+n$.

(2) By (1), this can be proved dually. □

Corollary 5.13. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion theory. Then*

$$\mathrm{gl.dim}(R) \leqslant \mathrm{gld}_{\mathcal{A}}(R) + \mathrm{gld}_{\mathcal{B}}(R).$$

5.2 Homology of Tor-torsion theories

Definition 5.14. Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a hereditary Tor-torsion theory and let M be an R-module.

(1) We say that M has a **weak $\mathcal{A}$-resolution of length at most** n if there exists an exact sequence

$$0 \to A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to M \to 0, \qquad A_i \in \mathcal{A}.$$

Use $\mathrm{fd}_{\mathcal{A}}(M)$ to represent the minimal length among all (finite) $\mathcal{A}$-resolutions of M. If such a finite exact sequence does not exist, then we say that M has an $\mathcal{A}$-resolution of infinite length, denoted by $\mathrm{fd}_{\mathcal{A}}(M) = \infty$.

(2) The definition of a **weak $\mathcal{B}$-resolution of length** $\mathrm{fd}_{\mathcal{B}}(N)$ of a module N is defined correspondingly, needless to say.

Remark 5.1 If $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ is a hereditary Tor-torsion theory, then by Theorem 3.26, $\mathfrak{G}_1 = (\mathcal{A}, \mathcal{A}^{\perp})$ is a hereditary perfect cotorsion theory. In this case, $\mathrm{pd}_{\mathcal{A}}(M) = \mathrm{fd}_{\mathcal{A}}(M)$ for any module M.

Example 5.15. Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a hereditary Tor-torsion theory. Then $\mathrm{fd}_{\mathcal{A}}(A) = 0$ if and only if $A \in \mathcal{A}$. By the same argument, $\mathrm{fd}_{\mathcal{B}}(B) = 0$ if and only if $B \in \mathcal{B}$.

Theorem 5.16. Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a hereditary Tor-torsion theory, n be a nonnegative integer, and M be an R-module. Then the following are equivalent:

(1) $\mathrm{fd}_{\mathcal{A}}(M) \leqslant n$.

(2) $\mathrm{Tor}^R_{n+1}(M, B) = 0$ for any $B \in \mathcal{B}$.

(3) $\mathrm{Tor}^R_k(M, B) = 0$ for any $B \in \mathcal{B}$ and any $k > n$.

(4) If $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ is an exact sequence, where $P_0, P_1, \ldots, P_{n-1}$ are projective modules, then $P_n \in \mathcal{A}$.

(5) If $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ is an exact sequence, where $P_0, P_1, \ldots, P_{n-1}$ are flat modules, then $P_n \in \mathcal{A}$.

(6) If $0 \to A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to M \to 0$ is an exact sequence, where $A_0, A_1, \ldots, A_{n-1} \in \mathcal{A}$, then $A_n \in \mathcal{A}$.

Proof. The proof is similar to that of the corresponding situation of cotorsion theory, and so the proof will be omitted. □

Definition 5.17. Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a hereditary Tor-torsion theory. Define:

(1) $\mathrm{wgld}_{\mathcal{A}}(R) = \sup\{\mathrm{fd}_{\mathcal{A}}(M) \mid M \text{ is any } R\text{-module}\}$, which is called the **global $\mathfrak{G}_{\mathcal{A}}$-flat dimension** of R.

(2) $\mathrm{wgld}_{\mathcal{B}}(R) = \sup\{\mathrm{fd}_{\mathcal{B}}(N) \mid N \text{ is any } R\text{-module}\}$, which is called the **global $\mathfrak{G}_{\mathcal{B}}$-flat dimension** of R.

Theorem 5.18. Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a hereditary Tor-torsion theory and let n be a nonnegative integer. Then the following are equivalent:

(1) $\mathrm{wgld}_{\mathcal{A}}(R) \leqslant n$.

(2) $\mathrm{Tor}^R_{n+1}(M, B) = 0$ for any $B \in \mathcal{B}$, $M \in {}_R\mathfrak{M}$.

(3) $\mathrm{Tor}^R_k(M, B) = 0$ for any $B \in \mathcal{B}$, $M \in {}_R\mathfrak{M}$, and any $k > n$.

(4) $\mathrm{fd}_R B \leqslant n$ for any $B \in \mathcal{B}$.

Proof. The proof is similar to that of Theorem 5.6, and so we omit it. □

Corollary 5.19. *Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a hereditary Tor-torsion theory. Then*

$$\mathrm{wgld}_{\mathcal{A}}(R) = \sup\{\mathrm{fd}_R B \mid B \in \mathcal{B}\}.$$

Example 5.20. Let $\mathfrak{G} = (\mathcal{F}, \mathfrak{M})$. Then $\mathrm{wgld}_{\mathcal{F}}(R) = w.\mathrm{gl.dim}(R)$ and $\mathrm{wgld}_{\mathfrak{M}}(R) = 0$.

Corollary 5.21. *Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a hereditary Tor-torsion theory. Then every module is an $\mathcal{A}$-flat module (i.e., $\mathcal{B} = {}_R\mathfrak{M}$) if and only if $\mathcal{A} = \mathcal{F}$.*

Proposition 5.22. *Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a hereditary Tor-torsion theory and let N be an R-module with $\mathrm{fd}_{\mathcal{B}} N = m > 0$. Then there exists $E \in \mathcal{A} \cap \mathcal{A}^{\perp}$ such that $\mathrm{Tor}_m^R(E, N) \neq 0$.*

Proof. By Theorem 5.16, $\mathrm{Tor}_{m+1}^R(A, N) = 0$ for any module $A \in \mathcal{A}$, and there exists $M \in \mathcal{A}$ such that $\mathrm{Tor}_m^R(M, N) \neq 0$. By Theorem 3.26, $(\mathcal{A}, \mathcal{A}^{\perp})$ is a hereditary perfect cotorsion theory. Thus there is an exact sequence $0 \to M \to E \to A \to 0$, where $E = \mathcal{A}^{\perp}(M) \in \mathcal{A}^{\perp}$ and $A \in \mathcal{A}$. Since $\mathcal{A}$ is closed under extensions, $E \in \mathcal{A}$.

It follows by the exact sequence of $0 = \mathrm{Tor}_{m+1}^R(A, N) \to \mathrm{Tor}_m^R(M, N) \to \mathrm{Tor}_m^R(E, N)$ and the fact that $\mathrm{Tor}_m^R(M, N) \neq 0$ that $\mathrm{Tor}_m^R(E, N) \neq 0$. □

Theorem 5.23. Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a hereditary Tor-torsion theory. If $\mathrm{wgld}_{\mathcal{B}}(R) < \infty$, then

$$\mathrm{wgld}_{\mathcal{B}}(R) = \sup\{\mathrm{fd}_R A \mid A \in \mathcal{A} \cap \mathcal{A}^{\perp}\} = \sup\{\mathrm{fd}_{\mathcal{B}}(E) \mid E \in \mathcal{A}^{\perp}\}$$

Proof. Write $\sup\{\mathrm{fd}_R A \mid A \in \mathcal{A} \cap \mathcal{A}^{\perp}\} = n$ and $\mathrm{wgld}_{\mathcal{B}}(R) = m$. By Theorem 5.18, $n \leqslant m$. By Proposition 5.22, $n \geqslant m$. Therefore $n = m$.

Again set $n = \sup\{\mathrm{fd}_{\mathcal{B}}(E) \mid E \in \mathcal{A}^{\perp}\}$. Then obviously $n \leqslant m$. For any $B \in \mathcal{B}$, by Proposition 5.22, $m \leqslant n$. Thus $n = m$. □

Theorem 5.24. Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a hereditary Tor-torsion theory and let M be an R-module. Then

$$\mathrm{fd}_R M \leqslant \mathrm{wgld}_{\mathcal{B}}(R) + \mathrm{fd}_{\mathcal{A}}(M).$$

Proof. The proof is similar to that of Theorem 5.12. □

Corollary 5.25. *Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a hereditary Tor-torsion theory. Then*

$$w.\mathrm{gl.dim}(R) \leqslant \mathrm{wgld}_{\mathcal{A}}(R) + \mathrm{wgld}_{\mathcal{B}}(R).$$

6 n-cotorsion modules and n-torsion-free modules

Below we always set n to be a nonnegative integer if not specified otherwise. Using a cotorsion theory $(\mathcal{F}_n, \mathcal{C}_n)$ as an example, we introduce homological properties of cotorsion theory which play an important role in characterizations of ring structures.

6.1 n-cotorsion modules and n-torsion-free modules

Definition 6.1. (1) An $\mathcal{F}_n$-injective module is called an n-**cotorsion module**. In particular, a 0-cotorsion module (i.e., $\mathcal{F}$-injective module) is called a cotorsion module. Denote by $\mathcal{C}_n$ the class of n-cotorsion modules and by $\mathcal{C}$ the class of cotorsion modules.

(2) An $\mathcal{F}_n$-flat module is called an n-**torsion-free module**. Denote by $\mathcal{T}_n$ the class of n-torsion-free modules.

(3) A $\mathcal{T}_n$-injective module is called an n-**Warfield cotorsion module**. Denote by $\mathcal{WC}_n$ the class of n-Warfield cotorsion modules.

Example 6.2. The following statements are obvious.

(1) Every module is a 0-torsion-free module and a 0-Warfield cotorsion module is injective.

(2) Let $m \leqslant n$. Since $\mathcal{F}_m \subseteq \mathcal{F}_n$, we have $\mathcal{T}_n \subseteq \mathcal{T}_m$, and $\mathcal{C}_n \subseteq \mathcal{C}_m$, but $\mathcal{WC}_m \subseteq \mathcal{WC}_n$, that is, every n-torsion-free module is m-torsion-free, every n-cotorsion module is an m-cotorsion module, but every m-Warfield cotorsion module is an n-Warfield cotorsion module.

(3) Let R be a domain and $n \geqslant 1$. Then every n-torsion-free module is torsion-free, and every n-cotorsion module is a divisible module.

(4) Let R be a domain. Then a module M is 1-torsion-free if and only if M is torsion-free. In other words, if we denote by $\mathcal{T}$ the class of torsion-free modules, then $\mathcal{T}_1 = \mathcal{T}$.

(5) Since $\mathcal{F} \subseteq \mathcal{T}_n$, we have $\mathcal{WC}_n \subseteq \mathcal{C}$, that is, every n-Warfield cotorsion module is a cotorsion module.

(6) By [22, Theorem 3.4.14], the direct limit of n-torsion-free modules over a directed set is also n-torsion-free.

Remark 6.1 When R is a domain, 1-Warfield cotorsion modules have been called Warfield cotorsion modules [11]. Therefore, following this terminology, we call a 1-Warfield cotorsion module over any ring a Warfield cotorsion module.

Theorem 6.3. Every $\mathcal{F}_n$-pure injective module is n-Warfield cotorsion.

Proof. This follows by taking $\mathcal{L} = \mathcal{F}_n$ in Theorem 1.20. □

Corollary 6.4. *Every pure injective module is a cotorsion module.*

Theorem 6.5. Let D be an R-module.

(1) $D \in \mathcal{T}_n$ if and only if $D^+ \in \mathcal{C}_n$, that is, D is an n-torsion-free module if and only if its character module D^+ is an n-cotorsion module. In particular, M^+ is a cotorsion module for any module M.

(2) $D \in \mathcal{F}_n$ if and only if $D^+ \in \mathcal{WC}_n$, that is, $\mathrm{fd}_R D \leqslant n$ if and only if its character module D^+ is an n-Warfield cotorsion module.

Proof. This follows by taking $\mathcal{L} = \mathcal{F}_n$ and $\mathcal{L} = \mathcal{T}_n$ respectively in Theorem 1.14. □

Theorem 6.6. (1) If C is an m-cotorsion module, then the n-th injective cosyzygy of C is an $(m + n + 1)$-cotorsion module. In particular, the n-th injective cosyzygy of a cotorsion module is an $(n + 1)$-cotorsion module.

(2) If M is an m-torsion-free module, then the n-th flat weak syzygy of M is an $(m+n+1)$-torsion-free module. In particular, the n-th flat weak syzygy of any module is an $(n+1)$-torsion-free module.

Proof. (1) Let $0 \to C \to E_0 \to E_1 \to \cdots \to E_n \to L \to 0$ be an exact sequence, where $E_0, E_1, \ldots, E_n$ are injective modules. For any $X \in \mathcal{F}_{n+m+1}$, there is an exact sequence $0 \to F_{n+m+1} \to F_{m+n} \to \cdots \to F_1 \to F_0 \to X \to 0$, where $F_0, F_1, \ldots, F_{m+n+1}$ are flat modules. Let Y be the $(n-1)$-th syzygy of this flat resolution. Then $\mathrm{fd}_R Y \leqslant m$. Thus $\mathrm{Ext}_R^1(X,L) \cong \mathrm{Ext}_R^{n+2}(X,C) \cong \mathrm{Ext}_R^1(Y,C) = 0$. Therefore C is an $(m+n+1)$-cotorsion module.

(2) This is similar to (1). □

Theorem 6.7. The following are equivalent for an R-module M:

(1) $M \in \mathcal{F}_n$.

(2) $\mathrm{Tor}_1^R(M,D) = 0$ for any $D \in \mathcal{T}_n$.

(3) $\mathrm{Tor}_1^R(M,D) = 0$ for the $(n-1)$-th weak syzygy D of any R-module X.

Proof. (1)⇒(2)⇒(3) Trivial, but we need Theorem 6.6(2).

(3)⇒(1) It follows from the fact that $\mathrm{Tor}_{n+1}^R(M,X) \cong \mathrm{Tor}_1^R(M,D) = 0$ that $\mathrm{fd}_R M \leqslant n$. Therefore $M \in \mathcal{F}_n$. □

Corollary 6.8. *$(\mathcal{F}_n, \mathcal{T}_n)$ is a hereditary Tor-torsion theory.*

Proof. By Theorem 6.7, $\mathcal{T}_n^{\top} = \mathcal{F}_n$. Thus $(\mathcal{F}_n, \mathcal{T}_n)$ is a Tor-torsion theory. The heredity is trivial. □

Theorem 6.9. The following are equivalent for an R-module M:

(1) $M \in \mathcal{F}_n$.

(2) $\mathrm{Ext}_R^1(M,L) = 0$ for any $L \in \mathcal{C}_n$.

(3) $\mathrm{Ext}_R^1(M,L) = 0$ for the $(n-1)$-th injective cosyzygy L of any R-module X.

Proof. The proof is similar to that of Theorem 6.7. Note that (3)⇒(1) needs to use Theorem 6.5(1). □

Corollary 6.10. *$(\mathcal{F}_n, \mathcal{C}_n)$ is a hereditary perfect cotorsion theory.*

Proof. By Theorem 6.9, ${}^{\perp}\mathcal{C}_n = \mathcal{F}_n$. Thus $(\mathcal{F}_n, \mathcal{C}_n)$ is a cotorsion theory. By Theorem 3.26 and Corollary 6.8, $(\mathcal{F}_n, \mathcal{C}_n)$ is a perfect cotorsion theory. □

Theorem 6.11. The following are equivalent for an R-module D:

(1) $D \in \mathcal{T}_n$.

(2) $\mathrm{Ext}_R^1(D,L) = 0$ for any $L \in \mathcal{WC}_n$.

(3) $\mathrm{Ext}_R^1(D,L) = 0$ for any $\mathcal{F}_n$-pure injective module L.

Proof. This follows by taking $\mathcal{L} = \mathcal{F}_n$ in Theorem 1.21. □

Corollary 6.12. *$(\mathcal{T}_n, \mathcal{WC}_n)$ is a hereditary perfect cotorsion theory.*

Proof. By Theorem 6.11, ${}^{\perp}\mathcal{WC}_n = \mathcal{T}_n$. Thus $\mathfrak{G} = (\mathcal{T}_n, \mathcal{WC}_n)$ is a cotorsion theory. By Theorem 3.24, $\mathfrak{G}$ is a hereditary cotorsion theory. Let $\mathcal{L}$ be the class of $\mathcal{F}_n$-pure injective modules. It follows from Theorem 6.11 that ${}^{\perp}\mathcal{L} = \mathcal{T}_n$. Since $\mathcal{T}_n^{\top} = \mathcal{F}_n$, it follows from Theorem 3.26 that $\mathfrak{G} = \mathfrak{K}_{\mathcal{L}}$ is perfect. □

Theorem 6.13. If L is an n-Warfield cotorsion module, then $\mathrm{id}_R L \leqslant n$.

Proof. Let X be any R-module and choose A to be the $(n-1)$-th projective syzygy of X. By Theorem 6.6, $A \in \mathcal{T}_n$. Thus $\mathrm{Ext}_R^{n+1}(X,L) \cong \mathrm{Ext}_R^1(A,L) = 0$. Therefore $\mathrm{id}_R L \leqslant n$. □

The following Theorem 6.14 can be regarded as an application of 1-torsion-free modules.

Theorem 6.14. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence with $\mathrm{fd}_R B \leqslant 1$. Then $\mathrm{fd}_R C \leqslant 1$, and so $\mathrm{fd}_R A \leqslant 1$.

Proof. By Theorem 6.7, the class of $\mathcal{T}_1$-flat modules is exactly $\mathcal{F}_1$. Taking $\mathcal{L} = \mathcal{T}_1$ in Proposition 1.17, we get $\mathrm{fd}_R C \leqslant 1$. □

6.2 n-cotorsion dimension and n-torsion-free dimension

By Corollary 6.10, $\mathfrak{G} = (\mathcal{F}_n, \mathcal{C}_n)$ is a hereditary perfect cotorsion theory. For an R-module N, we can set

$$\mathrm{c}_n\mathrm{d}_R N = \mathrm{id}_{\mathcal{C}_n}(N),$$

which is called the n-**cotorsion dimension** of N. Correspondingly, for a ring R, set

$$\mathrm{gld}_{\mathcal{C}_n}(R) = \{\mathrm{c}_n\mathrm{d}_R N \mid N \in \mathfrak{M}\},$$

which is called the **global** n-**cotorsion dimension** of R. Correspondingly, for a Tor-torsion theory $\mathfrak{G} = (\mathcal{F}_n, \mathcal{T}_n)$ we also define

$$\mathrm{t}_n\mathrm{d}_R N = \mathrm{fd}_{\mathcal{T}_n}(N),$$

which is called the n-**torsion-free dimension** of N. At this time

$$\mathrm{wgld}_{\mathcal{T}_n}(R) = \{\mathrm{t}_n\mathrm{d}_R N \mid N \in \mathfrak{M}\},$$

which is called the **global** n-**torsion-free dimension** of R

Remark 6.2 Since we have already written $\mathcal{C}_0 = \mathcal{C}$, we write $\mathrm{c}_n\mathrm{d}_R N$ for $\mathrm{cd}_R N$, which is the original notation of [19]. Correspondingly, we also write $\mathrm{gld}_{\mathcal{C}}(R)$ for $\mathrm{gld}_{\mathcal{C}_0}(R)$.

Example 6.15. Let R be a ring. Then:

(1) $\mathrm{gld}_{\mathcal{C}_n}(R) \leqslant \mathrm{gld}_{\mathcal{C}_{n+1}}(R)$.

(2) By [22, Theorem 3.10.26], $\mathrm{gld}_{\mathcal{C}_n}(R) \leqslant \mathrm{FPD}(R) \leqslant \mathrm{gl.dim}(R)$.

(3) $\mathrm{wgld}_{\mathcal{T}_n}(R) \leqslant \mathrm{wgld}_{\mathcal{T}_{n+1}}(R) \leqslant w.\mathrm{gl.dim}(R)$.

Theorem 6.16. Let R be a ring. Then:

(1) $\mathrm{gld}_{\mathcal{C}_n}(R) = \sup\{\mathrm{pd}_R M \mid M \in \mathcal{F}_n\} = \sup\{\mathrm{c}_n\mathrm{d}_R M \mid M \in \mathcal{F}_n\}$.

(2) $\mathrm{wgld}_{\mathcal{T}_n}(R) = \sup\{\mathrm{fd}_R M \mid M \in \mathcal{F}_n\}$.

(3) $\mathrm{wgld}_{\mathcal{T}_n}(R) \leqslant \mathrm{gld}_{\mathcal{C}_n}(R)$.

Proof. (1) This follows from Corollary 5.8 and Corollary 5.9.
(2) This follows from Corollary 5.19.
(3) This is trivial. □

Theorem 6.17. For any ring R, $\mathrm{wgld}_{\mathcal{T}_n}(R) \leqslant n$.

Proof. Let M be any R-module. By Theorem 6.6, the $(n-1)$-th weak syzygy of M must be an n-torsion-free module. Hence $\mathrm{t}_n\mathrm{d}_R M \leqslant n$. Therefore $\mathrm{wgld}_{\mathcal{T}_n}(R) \leqslant n$. □

Theorem 6.18. Let R be a ring. Then:

(1) If $\mathrm{gld}_{\mathcal{C}_n}(R) < \infty$, then

$$\mathrm{gld}_{\mathcal{C}_n}(R) = \sup\{\mathrm{pd}_R M \mid M \in \mathcal{F}_n \cap \mathcal{C}_n\} = \sup\{\mathrm{c}_n\mathrm{d}_R M \mid M \in \mathcal{P}\}.$$

(2) $\mathrm{wgld}_{\mathcal{T}_n}(R) = \sup\{\mathrm{fd}_R M \mid M \in \mathcal{F}_n \cap \mathcal{T}_n\}$.

Proof. (1) This follows by applying Theorem 5.11(2).
(2) This follows by applying Theorem 5.23 and Theorem 6.17. □

Theorem 6.19. (**Mao-Ding**) $\mathrm{gl.dim}(R) \leqslant \mathrm{gld}_{\mathcal{C}}\,\mathrm{dim}(R) + w.\mathrm{gl.dim}(R)$.

Proof. For the cotorsion theory $(\mathcal{F}, \mathcal{C})$, this follows by applying Corollary 5.13. □

Proposition 6.20. *If $\mathrm{t}_n\mathrm{d}_R M = m > 0$, then there exists $E \in \mathcal{F}_n \cap \mathcal{C}_n$ such that $\mathrm{Tor}_m^R(M, E) \neq 0$.*

Proof. This follows from Proposition 5.22. □

Proposition 6.21. *(1) Let $0 \to A \to F \to B \to 0$ be an exact sequence, where F is an n-torsion-free module. If $m = \mathrm{t}_n\mathrm{d}_R B > 0$, then $\mathrm{t}_n\mathrm{d}_R A = m - 1$.*

(2) Let $0 \to A \to F \to B \to 0$ be an exact sequence, where F is a flat module. If $\mathrm{t}_n\mathrm{d}_R B \leqslant m$, then $\mathrm{t}_{n+1}\mathrm{d}_R A \leqslant m$. In particular, if B is an n-torsion-free module, then A is an $(n+1)$-torsion-free module.

Proof. (1) This is trivial.
(2) Let $0 \to L \to P \to N \to 0$ be an exact sequence, where P is a flat module and $N \in \mathcal{F}_{n+1}$. Then $L \in \mathcal{F}_n$. Thus $\mathrm{Tor}_{m+1}^R(A, N) \cong \mathrm{Tor}_{m+2}^R(B, N) \cong \mathrm{Tor}_{m+1}^R(B, L) = 0$. Therefore $\mathrm{t}_{n+1}\mathrm{d}_R A \leqslant m$. □

Theorem 6.22. For any ring R,

$$\begin{aligned} \mathrm{wgld}_{\mathcal{T}_n}(R) &= \sup\{\mathrm{t}_n\mathrm{d}_R R/I \mid I \text{ is an ideal of } R\} \\ &= \sup\{\mathrm{t}_n\mathrm{d}_R R/I \mid I \text{ is a finitely generated ideal of } R\}. \end{aligned}$$

Proof. It is sufficient to prove that if for each finitely generated ideal I of R, $\mathrm{t}_n\mathrm{d}_R R/I \leqslant m$, and then we can get $\mathrm{wgld}_{\mathcal{T}_n}(R) \leqslant m$.

Let M be any R-module with $\mathrm{t}_n\mathrm{d}_R M = k$. Then there exists $N \in \mathcal{F}_n$ such that $\mathrm{Tor}_{k+1}^R(M, N) \neq 0$. Write $\mathrm{fd}_R N = s$. Then $k \leqslant s \leqslant n$. Thus there is a finitely generated ideal I such that $\mathrm{Tor}_{s+1}^R(N, R/I) \neq 0$. Hence $m \geqslant \mathrm{t}_n\mathrm{d}_R R/I \geqslant s \geqslant k$. It follows that $\mathrm{wgld}_{\mathcal{T}_n}(R) \leqslant m$. □

6.3 Change of rings theorems for n-torsion-free dimensions

Theorem 6.23. Let $\phi : R \to T$ be a ring homomorphism and let T as an R-module be an n-torsion-free module. Then:

(1) If N is an R-module with $\mathrm{fd}_R N \leqslant n$, then $\mathrm{fd}_T(T \otimes_R N) \leqslant n$.

(2) If L is an n-torsion-free T-module, then L is also an n-torsion-free R-module.

(3) $\mathrm{t}_n\mathrm{d}_R L \leqslant \mathrm{t}_n\mathrm{d}_T L$ for any T-module L.

Proof. (1) Let $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to N \to 0$ be a flat resolution of N. Since T is an n-torsion-free R-module, $\mathrm{Tor}_i^R(T,N) = 0$ for any $i > 0$. Thus

$$0 \to T \otimes_R P_n \to T \otimes_R P_{n-1} \to \cdots \to T \otimes_R P_1 \to T \otimes_R P_0 \to T \otimes_R N \to 0$$

is an exact sequence. Hence $\mathrm{fd}_T(T \otimes_R N) \leqslant n$.

(2) Let $N \in \mathcal{F}_n(R)$ and let $0 \to A \to P \to N \to 0$ be an exact sequence, where P is a free R-module. By (1), $\mathrm{fd}_T(T \otimes_R N) \leqslant n$. Since L is an n-torsion-free T-module, $\mathrm{Tor}_1^T(L, T \otimes_R N) = 0$. From the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & L \otimes_T (T \otimes_R A) & \longrightarrow & L \otimes_T (T \otimes_R P) \\
 & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & \mathrm{Tor}_1^R(L,N) & \longrightarrow & L \otimes_R A & \longrightarrow & L \otimes_R P
\end{array}$$

we have $\mathrm{Tor}_1^R(L,N) = 0$. Therefore L is also an n-torsion-free R-module.

(3) Write $m = \mathrm{t}_n\mathrm{d}_T L$. Then there is an exact sequence $0 \to F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 \to L \to 0$, where $F_0, F_1, \ldots, F_{m-1}, F_m$ are n-torsion-free T-modules. Therefore it follows by (2) that $\mathrm{t}_n\mathrm{d}_R L \leqslant \mathrm{t}_n\mathrm{d}_T L$. □

Theorem 6.24. Let $\phi : R \to T$ be a ring homomorphism and let T as an R-module be a flat module. If B is an n-torsion-free R-module, then $T \otimes_R B$ is an n-torsion-free T-module.

Proof. Let L be a T-module with $\mathrm{fd}_T L \leqslant n$. Since T is a flat R-module, every flat T-module is also a flat R-module. Hence $\mathrm{fd}_R L \leqslant n$. Note that $\mathrm{Tor}_1^T(T \otimes_R B, L) \cong T \otimes_R \mathrm{Tor}_1^R(B,L) = 0$. Therefore $T \otimes_R B$ is an n-torsion-free T-module. □

Theorem 6.25. Let S be a multiplicative subset of R. Then:

(1) If B is an n-torsion-free R-module, then B_S is an n-torsion-free R_S-module.

(2) $\mathrm{wgld}_{\mathcal{T}_n}(R_S) \leqslant \mathrm{wgld}_{\mathcal{T}_n}(R)$.

Proof. (1) This follows from Theorem 6.24.

(2) Write $m = \mathrm{wgld}_{\mathcal{T}_n}(R)$. Let $L \in \mathcal{F}_n(R_S)$. Then $L \in \mathcal{F}_n(R)$. Let A be an R_S-module. Then $\mathrm{t}_n\mathrm{d}_R A \leqslant m$. Thus $\mathrm{Tor}_{m+1}^{R_S}(A,L) = \mathrm{Tor}_{m+1}^{R_S}(A_S, L_S) = (\mathrm{Tor}_{m+1}^R(A,L))_S = 0$. Hence $\mathrm{t}_n\mathrm{d}_{R_S} A \leqslant m$. Therefore $\mathrm{wgld}_{\mathcal{T}_n}(R_S) \leqslant m$. □

Let $u \in R$ and X be an R-module. Write $X^u = \{x \in X \mid ux = 0\}$. Note that X^u is an $R/(u)$-module.

Lemma 6.26. *Let $n > 0$ be an integer, $u \in R$ be a non-zero-divisor nonunit, and $\overline{R} = R/(u)$. Let E be an n-cotorsion R-module. Then:*

(1) E^u is an $(n-1)$-cotorsion $\overline{R}$-module.

(2) If $E \in \mathcal{F}_n(R)$, then $E^u \in \mathcal{F}_{n-1}(\overline{R})$.

Proof. (1) Let B be an $\overline{R}$-module and let A be a submodule of B with $B/A \in \mathcal{F}_{n-1}(\overline{R})$. By [22, Theorem 3.8.15], $\mathrm{fd}_R B/A \leqslant n$. Let $f : A \to E^u$ be a homomorphism. Consider the following diagram

$$\begin{array}{ccccc}
 & & E^u & \longrightarrow & E \\
 & & f \uparrow & \overset{g}{\nwarrow} & \uparrow g \\
0 & \longrightarrow & A & \longrightarrow & B
\end{array}$$

Since E is an n-cotorsion R-module, there is a homomorphism $g : B \to E$ such that the above diagram can be completed to a commutative diagram. Since $ux = 0$ for any $x \in B$, we have $g(ux) = ug(x) = 0$. Thus $\mathrm{Im}(g) \subseteq E^u$. Therefore E^u is an $(n-1)$-cotorsion $\overline{R}$-module.

(2) For any $\overline{R}$-module A, by [22, Theorem 3.8.15], $\mathrm{Tor}_n^{\overline{R}}(A, E^u) \cong \mathrm{Tor}_{n+1}^R(A, E) = 0$. Therefore $\mathrm{fd}_{\overline{R}} E^u \leqslant n-1$. □

Lemma 6.27. *Let $n > 0$ be an integer, $u \in R$ be a non-zero-divisor nonunit, and $\overline{R} = R/(u)$. Let C be an $(n-1)$-cotorsion $\overline{R}$-module with $\mathrm{fd}_{\overline{R}} C \leqslant n-1$. Then there is an n-cotorsion R-module $E \in \mathcal{F}_n(R)$ such that $E^u = C$.*

Proof. Let E be the n-cotorsion envelope of C as an R-module and set $B = E/C$. Then $B \in \mathcal{F}_n(R)$. Since $\mathrm{fd}_{\overline{R}} C \leqslant n-1$, we have $\mathrm{fd}_R C \leqslant n$. Therefore $\mathrm{fd}_R E \leqslant n$.

It follows from $uC = 0$ that $C \subseteq E^u$. Thus we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & E^u & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\
 & & & & {\scriptstyle u}\downarrow & & \downarrow & & \\
 & & & & E & = & E & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

It follows from the heredity of $\mathcal{F}_n$ that $A \in \mathcal{F}_n(R)$. By Lemma 6.26, $\mathrm{fd}_{\overline{R}} E^u \leqslant n-1$. It follows from the exactness of the first row that $\mathrm{fd}_{\overline{R}} A < \infty$. It also follows from [22, Theorem 3.8.15] that $\mathrm{fd}_{\overline{R}} A \leqslant n-1$. Since C is an $(n-1)$-cotorsion module, $\mathrm{Ext}^1_{\overline{R}}(A, C) = 0$. Thus the first row is split. So there exists a submodule A' of E^u such that $A' \cong A$ and $E^u = C \oplus A$. Since E is the n-cotorsion envelope of C, $E/E^u \cong E \in \mathcal{F}_n(R)$. Hence it follows from Theorem 3.8 that $A = 0$, which implies that $C = E^u$. □

Theorem 6.28. Let $n > 0$ be an integer, $u \in R$ be a non-zero-divisor nonunit, and $\overline{R} = R/(u)$.

(1) Let M be a nonzero $\overline{R}$-module. Then $\mathrm{t}_n\mathrm{d}_R M = \mathrm{t}_{n-1}\mathrm{d}_{\overline{R}} M + 1$.

(2) $\mathrm{wgld}_{\mathcal{T}_n}(R) \geqslant \mathrm{wgld}_{\mathcal{T}_{n-1}}(\overline{R}) + 1$.

Proof. (1) Set $m := \mathrm{t}_{n-1}\mathrm{d}_{\overline{R}} M$. By Proposition 6.20, there exists an $(n-1)$-cotorsion $\overline{R}$-module C with $\mathrm{fd}_{\overline{R}} C \leqslant n-1$ such that $\mathrm{Tor}_m^{\overline{R}}(M, C) \neq 0$. By Lemma 6.27, there exists an n-cotorsion R-module E with $\mathrm{fd}_R E \leqslant n$ such that $C = E^u$. By [22, Theorem 3.8.15], $\mathrm{Tor}_{m+1}^R(M, E) \cong \mathrm{Tor}_m^{\overline{R}}(M, C) \neq 0$. Therefore $k := \mathrm{t}_n\mathrm{d}_R M \geqslant m+1$.

If $k > m+1$, then again by Proposition 6.20 and Lemma 6.26, there exists an n-cotorsion R-module E with $\mathrm{fd}_R E \leqslant n$ such that $\mathrm{Tor}_k^R(M, E) \cong \mathrm{Tor}_{k-1}^{\overline{R}}(M, E^u) \neq 0$, which contradicts the fact that $\mathrm{t}_{n-1}\mathrm{d}_{\overline{R}} M = m$. Therefore $k = m+1$.

(2) Let $m = \mathrm{wgld}_{\mathcal{T}_{n-1}}(\overline{R})$. Then there is an $\overline{R}$-module M such that $\mathrm{t}_{n-1}\mathrm{d}_{\overline{R}} M = m$. By (1), $\mathrm{t}_n\mathrm{d}_R M = m+1$. Therefore $\mathrm{wgld}_{\mathcal{T}_n}(R) \geqslant m+1$. □

7 The weak finitistic dimension of a ring

The weak finitistic dimension of the ring is a dimension introduced by Bass in [2] in 1960. Few literature contains the properties of ring structures using weak finitistic dimensions. In this section, we present several methods through torsion theory to characterize ring structures using weak finitistic dimensions. The contents of this section are excerpts from [23].

7.1 The weak finitistic dimension of a ring

Definition 7.1. Let R be a ring. Set

$$\mathrm{FFD}(R) = \sup\{\mathrm{fd}_R M \mid \mathrm{fd}_R M < \infty\},$$

which is called the **weak finitistic dimension** of R.

Remark 7.1 Trivially for any ring R, $\mathrm{fPD}(R) \leqslant \mathrm{FFD}(R) \leqslant \mathrm{FPD}(R)$.

Theorem 7.2. Let S be a multiplicative subset of R. Then $\mathrm{FFD}(R_S) \leqslant \mathrm{FFD}(R)$.

Proof. Without loss of generality, we assume that $m := \mathrm{FFD}(R) < \infty$. Let N be an R_S-module and $\mathrm{fd}_{R_S} N < \infty$. By [22, Corollary 3.8.6], $\mathrm{fd}_R N = \mathrm{fd}_{R_S} N < \infty$. Thus $\mathrm{fd}_{R_S} N \leqslant m$. Therefore $\mathrm{FFD}(R_S) \leqslant m$. □

Theorem 7.3. Let R be a ring. Then

$$\begin{aligned}\mathrm{FFD}(R) &= \sup\{\mathrm{FFD}(R_{\mathfrak{m}}) \mid \mathfrak{m} \in \mathrm{Max}(R)\}\\ &= \sup\{\mathrm{FFD}(R_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\}.\end{aligned}$$

Proof. We only prove the maximal ideal situation. Let m be a nonnegative integer. Suppose that $\mathrm{FFD}(R) \leqslant m$. By Theorem 7.2, $\mathrm{FFD}(R_{\mathfrak{m}}) \leqslant m$ for any maximal ideal $\mathfrak{m}$ of R. Now assume that the hypothesis of the converse is satisfied. Let N be an R-module with $\mathrm{fd}_R N < \infty$. Let $0 \to F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ be an exact sequence, where $F_0, F_1, \ldots, F_{m-1}$ are flat modules. Since $\mathrm{fd}_{R_{\mathfrak{m}}} N_{\mathfrak{m}} < \infty$ for any $\mathfrak{m} \in \mathrm{Max}(R)$ and the sequence

$$0 \to (F_m)_{\mathfrak{m}} \to (F_{m-1})_{\mathfrak{m}} \to \cdots \to (F_1)_{\mathfrak{m}} \to (F_0)_{\mathfrak{m}} \to N_{\mathfrak{m}} \to 0$$

is exact, it follows from the given condition that $(F_m)_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module. Thus F_m is a flat R-module. Therefore $\mathrm{fd}_R M \leqslant m$. □

Proposition 7.4. *The following statements are equivalent for a ring R.*

(1) $\mathrm{FFD}(R) \leqslant n$.

(2) $\mathcal{F}_m = \mathcal{F}_n$ *for any integer* $m > n$.

(3) *There exists an integer* $m > n$ *such that* $\mathcal{F}_m = \mathcal{F}_n$.

(4) $\mathcal{C}_m = \mathcal{C}_n$ *for any integer* $m > n$.

(5) *There exists an integer* $m > n$ *such that* $\mathcal{C}_m = \mathcal{C}_n$.

(6) $\mathcal{T}_m = \mathcal{T}_n$ *for any integer* $m > n$.

(7) *There exists an integer* $m > n$ *such that* $\mathcal{T}_m = \mathcal{T}_n$.

(8) $\mathcal{WC}_m = \mathcal{WC}_n$ *for any integer* $m > n$.

(9) *There exists an integer* $m > n$ *such that* $\mathcal{WC}_m = \mathcal{WC}_n$.

Proof. Exercise. □

Theorem 7.5. Let $m < n$. Then the following statements are equivalent for a ring R.

(1) $\mathrm{FFD}(R) \leqslant m$.

(2) If W is an n-Warfield cotorsion module, then $\mathrm{id}_R W \leqslant m$.

(3) If U is an $\mathcal{F}_n$-pure injective module, then $\mathrm{id}_R U \leqslant m$.

(4) $\mathrm{wgld}_{\mathcal{T}_n}(R) \leqslant m$.

(5) $\mathrm{wgld}_{\mathcal{T}_{m+1}}(R) \leqslant m$.

Proof. (1)⇒(2) By Proposition 7.4, $W \in \mathcal{WC}_m$. By Theorem 6.13, $\mathrm{id}_R W \leqslant m$.

(2)⇒(4) This follows immediately from Corollary 5.8.

(4)⇒(5) Since $n \geqslant m+1$, this follows from the fact that every n-torsion-free module is $(m+1)$-torsion-free.

(5)⇒(1) Let M be an R-module and let $k := \mathrm{fd}_R M < \infty$. Assume on the contrary that $k > m$. Then there exists an R-module X such that $\mathrm{fd}_R X = m+1$. Let N be any R-module. By the hypothesis, $\mathrm{t}_{m+1}\mathrm{d}_R N \leqslant m < k$. Thus $\mathrm{Tor}^R_{m+1}(X,N) = 0$, a contradiction. Therefore $\mathrm{FFD}(R) \leqslant m$.

(2)⇒(3) This follows from Theorem 6.3.

(3)⇒(4) Let N be any R-module and let D be the $(m-1)$-th weak syzygy of N. Then for any $\mathcal{F}_n$-pure injective module U, by the hypothesis, $\mathrm{Ext}^1_R(D,U) \cong \mathrm{Ext}^{m+1}_R(N,U) = 0$. By Theorem 6.11, $D \in \mathcal{T}_m$. Therefore $\mathrm{wgld}_{\mathcal{T}_n}(R) \leqslant m$. □

Corollary 7.6. *The following statements are equivalent for a ring R.*

(1) $\mathrm{FFD}(R) = 0$.

(2) Every R-module is n-torsion-free for any $n \geqslant 1$.

(3) Every cotorsion module is an n-cotorsion module for any $n \geqslant 1$.

(4) $\mathrm{wgld}_{\mathcal{T}_n}\dim(R) = 0$ *for any* $n \geqslant 1$.

(5) Every R-module is 1-torsion-free.

(6) $\mathrm{wgld}_{\mathcal{T}_1}(R) = 0$.

Proof. (1)⇔(2)⇔(3)⇔(5) follows from Proposition 7.4, while (1)⇔(4)⇔(6) follows from Theorem 7.5. □

For the finitistic dimension of a ring, we have the following corresponding characterization.

Theorem 7.7. Let $m < n$. Then the following statements are equivalent for a ring R.

(1) $\mathrm{FPD}(R) \leqslant m$.

(2) $\mathrm{gld}_{\mathcal{C}_n}(R) \leqslant m$.

(3) $\mathrm{gld}_{\mathcal{C}_{m+1}}(R) \leqslant m$.

Proof. (1)⇒(2) Let $N \in \mathcal{F}_n$. By [22, Theorem 3.10.26], $\mathrm{pd}_R N < \infty$. By the hypothesis, $\mathrm{pd}_R N \leqslant m$. By Theorem 6.16, $\mathrm{gld}_{\mathcal{C}_n}(R) \leqslant m$.

(2)⇒(3) This is trivial.

(3)⇒(1) Let M be an R-module and set $k := \mathrm{pd}_R M < \infty$. Assume on the contrary that $k > m$. Then without loss of generality, we may assume that $k = m+1$. Thus $M \in \mathcal{F}_{m+1}$. By Theorem 6.16, $\mathrm{pd}_R M \leqslant m$, a contradiction. Thus $\mathrm{FPD}(R) \leqslant m$. □

Theorem 7.8. The following statements are equivalent for a ring R.

(1) $\mathrm{FPD}(R) = 0$.

(2) Every R-module is an n-cotorsion module for any $n \geqslant 1$.

(3) $\mathrm{gld}_{\mathcal{C}_n}(R) = 0$ for any $n \geqslant 1$.

(4) Every R-module is a cotorsion module.

(5) R is a perfect ring.

(6) $\mathrm{gld}_{\mathcal{C}}(R) = 0$.

Proof. (1)⇒(2)⇒(3) This follows from Theorem 7.7.
(3)⇒(6)⇒(4) This is trivial.
(4)⇒(5) By the hypothesis, $\mathcal{C} = \mathfrak{M}$, and so $\mathcal{F} = \mathcal{P}$. By [22, Theorem 3.10.22], R is a perfect ring.
(5)⇒(1). This follows from [22, Theorem 3.10.25]. □

Theorem 7.9. The following statements are equivalent for a ring R.

(1) $w.\mathrm{gl.dim}(R) \leqslant n$.

(2) Every n-cotorsion module is injective.

(3) Every n-torsion-free module is flat.

Proof. (1)⇒(2) Let L be an n-cotorsion module and let M be any R-module. By the hypothesis, $\mathrm{fd}_R M \leqslant n$. Thus $\mathrm{Ext}^1_R(M, L) = 0$. Therefore L is injective.
(2)⇒(1) Since $\mathcal{C}_n = \mathcal{I}$, we have $\mathcal{F}_n = {}^{\perp}\mathcal{C}_n = {}^{\perp}\mathcal{I} = \mathfrak{M}$. Thus $w.\mathrm{gl.dim}(R) \leqslant n$.
(1)⇔(3) The proof is similar to that of (1)⇔(2). □

7.2 Integral domains with weak finitistic dimension 1

Theorem 7.10. The following statements are equivalent for an integral domain R.

(1) $\mathrm{FFD}(R) \leqslant 1$.

(2) $\mathrm{FFD}(R/(u)) = 0$ for any nonzero nonunit $u \in R$.

(3) $\mathrm{wgld}_{\mathcal{T}_1}(R/(u)) = 0$ for any nonzero nonunit $u \in R$..

(4) Every torsion-free R-module is 2-torsion-free.

(5) If A is a torsion-free R-module with $\mathrm{fd}_R A < \infty$, then A is flat.

(6) If A is a torsion-free R-module with $\mathrm{fd}_R A \leqslant 1$, then A is flat.

(7) $\mathcal{F}_2 = \mathcal{F}_1$.

(8) Every submodule of a flat R-module is 2-torsion-free.

(9) Every ideal of R is 2-torsion-free.

(10) $\mathrm{wgld}_{\mathcal{T}_2}(R) \leqslant 1$.

(11) $\mathrm{wgld}_{\mathcal{T}_n}(R) \leqslant 1$ for any $n \geqslant 2$.

Proof. For a nonzero nonunit $u \in R$, write $\overline{R} = R/(u)$.

(1)⇒(2) Let B be a nonzero $\overline{R}$-module with $\mathrm{fd}_{\overline{R}}B < \infty$. By [22, Theorem 3.8.15], $\mathrm{fd}_R B = \mathrm{fd}_{\overline{R}}B + 1 \leqslant 1$. Thus $\mathrm{fd}_{\overline{R}}B = 0$. Therefore $\mathrm{FFD}(\overline{R}) = 0$.

(2)⇒(3) This follows from Corollary 7.6.

(3)⇒(10) Let I be a nonzero proper ideal of R and set $M = R/I$. Take $u \in I$ with $u \neq 0$. Thus $uM = 0$, and so M is an $\overline{R}$-module. By the hypothesis, $\mathrm{t_1d}_{\overline{R}}M = 0$. By Theorem 6.28, $\mathrm{t_2d}_R M = 1$. By Theorem 6.22, $\mathrm{wgld}_{\mathcal{T}_2}(R) \leqslant 1$.

(10)⇒(11)⇒(1) This follows from Theorem 7.5.

(10)⇒(4) Let M be a torsion-free module. Then M can be embedded in a flat module $F = K \otimes_R M$. By the hypothesis, $\mathrm{t_2d}_R F/M \leqslant 1$. Therefore M is 2-torsion-free.

(4)⇒(8)⇒(9) This is trivial.

(9)⇒(10) This follows from Theorem 6.22.

(1)⇒(5) Since A is a torsion-free module, there exists an exact sequence $0 \to A \to F \to C \to 0$, where F is a flat module. Thus $\mathrm{fd}_R C < \infty$. By the hypothesis, $\mathrm{fd}_R C \leqslant 1$. Therefore A is flat.

(5)⇒(6) This is trivial.

(6)⇒(7) This follows from direct verification.

(7)⇒(1) Let N be an R-module and set $k := \mathrm{fd}_R N < \infty$. Assume on the contrary that $k \geqslant 2$. Then there exists a module B such that $\mathrm{fd}_R B = 2$. Thus $B \in \mathcal{F}_2 = \mathcal{F}_1$, which yields immediately that $\mathrm{fd}_R B \leqslant 1$, a contradiction. Therefore $\mathrm{FFD}(R) \leqslant 1$. □

Proposition 7.11. *Let $R \subseteq T$ be an extension of domains.*

1. *T is a 2-torsion-free R-module if and only if $T \otimes_R N$ is a torsion-free T-module for any torsion-free R-module N with $\mathrm{fd}_R N \leqslant 1$.*
2. *If $\mathrm{FFD}(R) \leqslant 1$, then T is a 2-torsion-free R-module.*
3. *Let S be a multiplicative set of R such that $R_S \subseteq T$. If T is a 2-torsion-free R-module, then T is a 2-torsion-free R_S-module.*

Proof. (1) Assume that T is a 2-torsion-free R-module. Since N is a torsion-free R-module, there exists an exact sequence $0 \to N \to P \to C \to 0$, where P is a flat R-module. Thus $\mathrm{fd}_R C \leqslant 2$. Hence $\mathrm{Tor}_1^R(T, C) = 0$. Thus $0 \to T \otimes_R N \to T \otimes_R P \to T \otimes_R C \to 0$ is an exact sequence, and so $T \otimes_R N$ is a torsion-free T-module.

Assume that the opposite is true. Let C be an R-module with $\mathrm{fd}_R C \leqslant 2$. Take an exact sequence $0 \to N \to P \to C \to 0$, where P is a flat R-module. Then we have an exact sequence: $0 \to \mathrm{Tor}_1^R(T, C) \to T \otimes_R N \to T \otimes_R P \to T \otimes_R C \to 0$. By the hypothesis $T \otimes_R N$ is a torsion-free T-module. Since T is included in the quotient field of R, $T \otimes_R N$ is also a torsion-free R-module. By [22, Exercise 3.7], $\mathrm{Tor}_1^R(T, C) = 0$. Therefore T is a 2-torsion-free R-module.

(2) This follows immediately from Theorem 7.10.

(3) Let N be an R_S-module with $\mathrm{fd}_{R_S} N \leqslant 2$. By [22, Corollary 3.8.6], $\mathrm{fd}_R N \leqslant 2$, and so $\mathrm{Tor}_1^R(T, N) = 0$. Note that $T_S = T$ and $N_S = N$. By [22, Corollary 3.4.12], $\mathrm{Tor}_1^{R_S}(T, N) \cong \mathrm{Tor}_1^R(T, N)_S = \mathrm{Tor}_1^R(T, N) = 0$. Therefore T is a 2-torsion-free R_S-module. □

Theorem 7.12. Let $(R, \mathfrak{m})$ be a local ring with $\mathrm{fPD}(R) = 0$. Then $R/\mathfrak{m}$ is a 1-torsion-free module.

Proof. Let B be an R-module with $\mathrm{fd}_R B \leqslant 1$. Then there exists an exact sequence $0 \to A \to P \to B \to 0$, where P, A are flat R-modules. By [22, Theorem 3.10.9], $\mathfrak{m}A = \mathfrak{m}P \cap A$. By [22, Exercise 3.46], $\mathrm{Tor}_1^R(R/\mathfrak{m}, B) = 0$. Therefore $R/\mathfrak{m}$ is 1-torsion-free. □

7.3 The weak finitistic dimension in Cartesian square

For a Cartesian square $(RDTF, M)$, we always assume that $\pi : T \to F := T/M$ is the natural homomorphism.

Proposition 7.13. *Let $(RDTF, M)$ be a Cartesian square, where D, F are fields. Let N be a T-module. Then:*

(1) There exist a D-module I and an F-isomorphism $h : F \otimes_D I \to F \otimes_T N$.

(2) Set $B := (I, N, h)$. Then $D \otimes_R B \cong I$ and $T \otimes_R B \cong N$.

Proof. (1) Since F is a field, $F \otimes_T N$ is an F-vector space. Write s as the dimension of this vector space. (It can be infinite dimensional.) Take a D-vector space I of dimension s. Then there exists an F-isomorphism $h : F \otimes_D I \to F \otimes_T N$.

(2) This follows from [22, Proposition 8.1.8 and Theorem 8.1.9]. □

Theorem 7.14. Let $(RDTF, M)$ be a Cartesian square, where D is a field and T as an R-module is $(n+1)$-torsion-free. If $\mathrm{FFD}(T) \leqslant n$, then $\mathrm{FFD}(R) \leqslant n$.

Proof. In order to prove that $\mathrm{FFD}(R) \leqslant n$, it suffices that $\mathrm{fd}_R N \leqslant n+1$ implies that $\mathrm{fd}_R N \leqslant n$. Let N be an R-module with $\mathrm{fd}_R N \leqslant n+1$. Let $0 \to B \to P_{n-1} \to \cdots \to P_1 \to P_0 \to N \to 0$ be an exact sequence, where $P_0, P_1, \ldots, P_{n-1}$ are flat R-modules. Since T is an $(n+1)$-torsion-free R-module, $\mathrm{Tor}_i^R(T, N) = 0$ for each $i \geqslant 1$. Thus

$$0 \to T \otimes_R B \to T \otimes_R P_{n-1} \to \cdots \to T \otimes_R P_1 \to T \otimes_R P_0 \to T \otimes_R N \to 0$$

is an exact sequence. By Theorem 6.23, $\mathrm{fd}_T(T \otimes_R N) \leqslant n+1$. Since $\mathrm{FFD}(T) \leqslant n$, it follows that $T \otimes_R B$ is a flat T-module. Since D is a field, $D \otimes_R B = B/MB$ is naturally a flat D-module. By [22, Theorem 8.2.1], B is a flat R-module. Therefore $\mathrm{FFD}(R) \leqslant n$. □

Theorem 7.15. Let $(RDTF, M)$ be a Cartesian square, where D is a field and T is a domain. If $\mathrm{FFD}(R) \leqslant 1$ and M is a flat R-module, then T is a flat R-module.

Proof. By [23, Lemma 3.3], $\mathrm{fd}_R T \leqslant 1$. By Theorem 7.10, T is a flat R-module. □

Theorem 7.16. Let $(RDTF, M)$ be a Cartesian square, where D, F are fields. Then $\mathrm{fPD}(R) = 0$ if and only if $\mathrm{fPD}(T) = 0$.

Proof. Assume that $\mathrm{fPD}(R) = 0$. Let A be a finitely generated proper ideal of T. Then there exists a finitely generated proper ideal I of R such that $A = IT$. By [22, Theorem 3.10.11], there exists a nonzero $a \in R$ such that $aI = 0$. Thus $aA = 0$. By [12, Theorem 3.3.16] (or [22, Theorem 3.10.8 and Theorem 3.10.11]), $\mathrm{fPD}(T) = 0$.

Conversely, assume that $\mathrm{fPD}(T) = 0$. Let I be a finitely generated proper ideal of R and set $A = IT$. Assume by way of contradiction that $A = T$. By [22, Proposition 8.3.2], $M \subset I$. Since D is a field, $I = R$, a contradiction. Thus $A \neq T$. Hence there exists $t \in T$ such that $tA = 0$, and so $tI = 0$. If $t \notin R$, then $t \notin M$. Take a quasi-inverse t' of t such that $a := t't \in R$ and $a \neq 0$. In this case, $aI = t'tI = 0$. By [12, Theorem 3.3.16], $\mathrm{fPD}(R) = 0$. □

Lemma 7.17. *Let $(RDTF, M)$ be a Cartesian square, where D is a field. Let B be an R-module and set $N = T \otimes_R B$. If $\mathrm{Tor}_1^T(F, N) = 0$ and $\mathrm{fd}_T N \leqslant 1$, then $\mathrm{fd}_R B \leqslant 1$.*

Proof. Let $0 \to A \to P \to B \to 0$ be an exact sequence, where P is a flat R-module. Thus we have the following exact sequence:

$$0 \to A_1 \to D \otimes_R P \to D \otimes_R B \to 0 \quad \text{and} \quad 0 \to A_2 \to T \otimes_R P \to T \otimes_R B \to 0,$$

where $A_1 = \mathrm{Ker}(D \otimes_R P \to D \otimes_R B)$ and $A_2 = \mathrm{Ker}(T \otimes_R P \to T \otimes_R B)$. By [22, Proposition 8.1.10], we can set $P = (D \otimes_R P, T \otimes_R P, \delta_P)$. Since D is a field, A_1 is a flat D-module. Since $\mathrm{fd}_T N \leqslant 1$, A_2 is a flat T-module. Since F is naturally a flat D-module, $0 \to F \otimes_D A_1 \to F \otimes_R P \to F \otimes_R B \to 0$ is an exact sequence. Since $\mathrm{Tor}_1^T(F, N) = 0$, it follows that $0 \to F \otimes_T A_2 \to F \otimes_R P \to F \otimes_R B \to 0$ is also an exact sequence. Therefore we have the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F \otimes_D A_1 & \longrightarrow & F \otimes_R P & \longrightarrow & F \otimes_R B & \longrightarrow & 0 \\
 & & {\scriptstyle h_1}\downarrow & & \| & & \| & & \\
0 & \longrightarrow & F \otimes_T A_2 & \longrightarrow & F \otimes_R P & \longrightarrow & F \otimes_R B & \longrightarrow & 0
\end{array}$$

where h_1 is the induced homomorphism from the right square. By [22, Theorem 1.9.9], $h_1 : F \otimes_D A_1 \to F \otimes_T A_2$ is an isomorphism.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & A_1 \oplus A_2 & \longrightarrow & F \otimes_T A_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & (D \otimes_R P) \oplus (T \otimes_R P) & \longrightarrow & F \otimes_R P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & (D \otimes_R B) \oplus (T \otimes_R B) & \longrightarrow & F \otimes_R B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where three columns and two bottom rows are exact sequences. By [22, Theorem 1.9.12], the first row is also exact. It follows that $A \cong (A_1, A_2, h_1)$. By [22, Theorem 8.2.2], A is a flat R-module. Thus $\mathrm{fd}_R B \leqslant 1$. □

Theorem 7.18. Let $(RDTF, M)$ be a Cartesian square, where D, F are fields. Then $\mathrm{FFD}(R) = 0$ if and only if $\mathrm{FFD}(T) = 0$.

Proof. Suppose that $\mathrm{FFD}(R) = 0$. Let N be a T-module with $\mathrm{fd}_T N \leqslant 1$. By Proposition 7.13, there exist a D-module I and an F-isomorphism $h : F \otimes_D I \to F \otimes_T N$ such that $B = (I, N, h)$ and $T \otimes_R B \cong N$.

First assume that T is a local ring, so that R is a local ring. By Theorem 7.16, $\mathrm{fPD}(T) = 0$. By Theorem 7.12, F is a 1-torsion-free T-module, and so $\mathrm{Tor}_1^T(F, N) = 0$. By Lemma 7.17, $\mathrm{fd}_R B \leqslant 1$. Since $\mathrm{FFD}(R) = 0$, B is a flat R-module. Thus $N \cong T \otimes_R B$ is a flat T-module. Therefore $\mathrm{FFD}(T) = 0$.

Now consider the general case. Let Q be a maximal ideal of T and set $P := Q \cap R$. If $Q \neq M$, then [22, Proposition 8.3.1] and Theorem 7.3, $\mathrm{FFD}(T_Q) = \mathrm{FFD}(R_P) = 0$. If $M = Q$ and set $S := R \setminus M$, then $(R_S D T_S F, M T_S)$ is a Cartesian square. By [22, Lemma 8.3.8], $T_S = T_M$. It is proved by the local situation that $\mathrm{FFD}(T_M) = 0$. By Theorem 7.3, $\mathrm{FFD}(T) = 0$.

Conversely, assume that $\mathrm{FFD}(T) = 0$. By Theorem 7.16, $\mathrm{fPD}(R) = 0$. First assume that T is a local ring. By Theorem 7.12, D is a 1-torsion-free R-module. Since D is a field, F is a direct sum of copies of D. Thus F is also a 1-torsion-free R-module. From the exact sequence $0 \to M \to T \to F \to 0$ and the fact that M is a 1-torsion-free R-module, it follows that T is a 1-torsion-free R-module. By Theorem 7.14, $\mathrm{FFD}(R) = 0$.

Now consider the general case. Let P be a maximal ideal of R and set $Q = Q(P)$ as [22, Theorem 8.3.17]. If $P \neq M$, then by [22, Theorem 8.3.17], Q is a maximal ideal of T and $Q \neq M$. At this time, $\mathrm{FFD}(R_P) = \mathrm{FFD}(T_Q) = 0$. If $P = M$, then $(R_S D T_S F, MT_S)$ is a Cartesian square. It has been proved by the above that $\mathrm{FFD}(R_P) = 0$. Again by Theorem 7.3, $\mathrm{FFD}(R) = 0$. □

Proposition 7.19. *Let $(RDTF, M)$ be a Milnor square. If $\mathrm{FFD}(R) \leqslant 1$, then $\mathrm{FFD}(D) \leqslant 1$.*

Proof. The situation where D is a field is obvious. Now suppose that D is not a field. First assume that (T, M) is a local ring. Let $u \in D$ be a nonzero nonunit. Take $x \in R$ such that $\pi(x) = u$. Then $x \in R$, $x \notin M$, and x is not a unit. By [22, Theorem 8.3.6], $M \subset xR$. In this case, $R/(x) \cong D/(u)$. By Theorem 7.10, $\mathrm{FFD}(D/(u)) = 0$. Again by Theorem 7.10, $\mathrm{FFD}(D) \leqslant 1$.

Now consider the general situation. Let $\mathfrak{p}$ be a maximal ideal of D. Then there is a maximal ideal P of R such that $\pi(P) = \mathfrak{p}$. Set $S_1 = R \setminus P$ and $S_2 = T \setminus M$. Then $D_{\mathfrak{p}} = D_{S_1}$. By [22, Lemma 8.3.8], $T_{S_1} = T_{S_2}$ is a local ring. Since $(R_{S_1} D_{S_1} T_{S_1} F, MT_{S_1})$ is a Milnor square, it follows from the above that $\mathrm{FFD}(D_{\mathfrak{p}}) \leqslant 1$. By Theorem 7.3, $\mathrm{FFD}(D) \leqslant 1$. □

Theorem 7.20. Let $(RDTF, M)$ be a Milnor square of type I. Then $\mathrm{FFD}(R) \leqslant 1$ if and only if $\mathrm{FFD}(D) \leqslant 1$ and $\mathrm{FFD}(T) \leqslant 1$.

Proof. Assume that $\mathrm{FFD}(R) \leqslant 1$. By [22, Theorem 8.3.10], $T = R_S$ is a flat R-module, where $S = R \setminus M$. By Theorem 7.2, $\mathrm{FFD}(T) \leqslant 1$. By Proposition 7.19, $\mathrm{FFD}(D) \leqslant 1$.

Conversely, let A be an R-module with $\mathrm{fd}_R A \leqslant 2$. Let $0 \to B \to P \to A \to 0$ be an exact sequence, where P is a flat R-module. Since T is a flat R-module and $\mathrm{FFD}(T) \leqslant 1$, $T \otimes_R B$ is a flat T-module, and by Theorem 7.10 and Theorem 6.23(2), M is both a 2-torsion-free T-module and a 2-torsion-free R-module. Thus $\mathrm{Tor}_1^R(M, A) \cong \mathrm{Tor}_2^R(D, A) \cong \mathrm{Tor}_1^R(D, B) = 0$. Let $0 \to B_1 \to P_1 \to B \to 0$ be an exact sequence, where P_1 is a flat R-module. Since $\mathrm{fd}_R B \leqslant 1$, B_1 is a flat module. Note that $0 \to B_1/MB_1 \to P_1/MP_1 \to B/MB \to 0$ is a D-module exact sequence. Thus $\mathrm{fd}_D(B/MB) \leqslant 1$. It follows from [22, Proposition 8.2.8] that B/MB is a torsion-free D-module. Since $\mathrm{FFD}(D) \leqslant 1$, applying Theorem 7.10, we get that B/MB is a flat D-module. It follows from [22, Theorem 8.2.1] that B is a flat R-module. So $\mathrm{fd}_R A \leqslant 1$, and thus $\mathrm{FFD}(R) \leqslant 1$. □

Theorem 7.21. Let $(RDTF, M)$ be a Milnor square of type II.

(1) Assume that $\mathrm{t}_2\mathrm{d}_T F \leqslant 1$. If $\mathrm{FFD}(R) \leqslant 1$, then $\mathrm{FFD}(T) \leqslant 1$.

(2) Assume that T is a 2-torsion-free R-module. If $\mathrm{FFD}(D) \leqslant 1$ and $\mathrm{FFD}(T) \leqslant 1$, then $\mathrm{FFD}(R) \leqslant 1$.

Proof. (1) First assume that D is a field. Let N be a torsion-free T-module with $\mathrm{fd}_T N \leqslant 1$. By Proposition 7.13, we can let $N = T \otimes_R B$, where B is the pullback of N and a certain D-module I. By the hypothesis, $\mathrm{t}_2\mathrm{d}_T F \leqslant 1$, and so $\mathrm{Tor}_1^T(F, N) = 0$. By Lemma 7.17, $\mathrm{fd}_R B \leqslant 1$. Since B is a torsion-free module, B is a flat module. Hence N is a flat T-module. Therefore $\mathrm{FFD}(T) \leqslant 1$.

Now consider the general case. Let L be the quotient field of D. Split the original Milnor square into the following two Milnor squares:

$$\begin{array}{ccccc} R & \longrightarrow & T_1 & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & L & \longrightarrow & F \end{array}$$

Since L is the quotient field of D, we have $\mathrm{FFD}(T_1) \leqslant 1$. Since L is a field, $\mathrm{FFD}(T) \leqslant 1$.

(2) First assume that D is a field. By the hypothesis, T is a 2-torsion-free R-module. By applying Theorem 7.14, we know that $\mathrm{FFD}(R) \leqslant 1$.

Now consider the general case. Let L be the quotient field of D. Split the original Milnor square into the following two Milnor squares:

$$\begin{array}{ccccc} R & \longrightarrow & T_2 & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & L & \longrightarrow & F \end{array}$$

By Proposition 7.11, T is a 2-torsion-free T_2-module. Thus $\mathrm{FFD}(T_2) \leqslant 1$. Since L is the quotient field of D, we get that $\mathrm{FFD}(R) \leqslant 1$. □

Lemma 7.22. *Let $(RDTF, M)$ be a Milnor square, where D is a field. Then:*

(1) There exists an exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow C \longrightarrow 0,$$

where P is a finitely generated free R-module, C is a finitely generated torsion-free R-module. Thus M is a 2-torsion-free R-module.

(2) If M is a flat ideal of T with $M^2 = M$, then M is also a flat R-module.

Proof. (1) By [22, Theorem 8.5.5], there exists a finitely generated fractional ideal I of R such that $M = I^{-1}$. Take an exact sequence $0 \to A \xrightarrow{f} Q \to I \to 0$, where Q is a finitely generated projective R-module. Thus we have an exact sequence $0 \to I^* \to Q^* \xrightarrow{f^*} A^*$. Set $P = Q^*$ and $C = \mathrm{Im}(f^*)$. Now the proof follows immediately from $I^* \cong I^{-1} = M$.

(2) Since $M^2 = M$, we have $F \otimes_T M = 0$. Computation gets the pullback $(0, M, 0) \cong M$. By [22, Theorem 8.2.2], M is a flat R-module. □

Theorem 7.23. Let $(RDTF, M)$ be a Milnor square, where D is a field and T is a local ring.

(1) If T as an R-module is not a 2-torsion-free module, then M is a flat T-module and $M = M^2$.

(2) If $\mathrm{FFD}(T) \leqslant 1$ and M is a flat ideal of T with $M = M^2$, then T as an R-module is not a 2-torsion-free module.

Proof. (1) Assume that T as an R-module is not a 2-torsion-free module. By Proposition 7.11(1), there exists a torsion-free R-module B such that $\mathrm{fd}_R B \leqslant 1$ and $T \otimes_R B$ is not a torsion-free T-module. Since T is a 1-torsion-free R-module, it follows from Theorem 6.23 that $\mathrm{fd}_T(T \otimes_R B) \leqslant 1$. Take an exact sequence $0 \to B \to P \to N \to 0$, where P is a flat R-module. Let $\theta : T \otimes_R B \to T \otimes_R P$ be a natural homomorphism and write $I = \mathrm{Im}(\theta)$ and $L = \mathrm{Ker}(\theta)$. Then I is a submodule of a flat T-module $T \otimes_R P$, and thus a torsion-free T-module. By Lemma 7.22(1), $\mathrm{t}_2\mathrm{d}_R D \leqslant 1$, and so $\mathrm{Tor}_1^R(D, B) = 0$. By [22, Exercise 3.46], $M \otimes_R B \cong MB$. Note that $MB \subseteq B \subseteq I$. Thus we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & T \otimes_R B & \longrightarrow & I & \longrightarrow & 0 \\ & & \| & & \downarrow & {\scriptstyle h} \swarrow & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & L & \longrightarrow & F \otimes_R B & \xrightarrow{g} & I/MB & \longrightarrow & 0 \end{array}$$

Thus L is an F-vector space. Since the bottom row in the above diagram is an exact sequence of F-vector spaces, it is split. Hence we have a homomorphism $h : I \to F \otimes_R B$ such that $hg = \beta$. By [22, Exercise 1.60], the top row is split. Thus $T \otimes_R B \cong L \oplus I$. Hence $\mathrm{fd}_T I \leqslant 1$ and $\mathrm{fd}_T L \leqslant 1$.

Since $T \otimes_R B$ is not a torsion-free T-module, it follows that $L \neq 0$. Since L is an F-vector space, L is the direct sum of some copies of F. From this we have $\mathrm{fd}_T F = \mathrm{fd}_T L \leqslant 1$, so that M is a flat T-module.

If $M \neq M^2$, by [22, Theorem 2.5.22] M is a principal ideal, and so $M = Tu$. Thus $T \cong M$. By Lemma 7.22(1), T as an R-module is 2-torsion-free, which contradicts the known conditions. Thus $M = M^2$.

(2) Assume on the contrary that T as an R-module is a 2-torsion-free module. By Lemma 7.22(2), M is a flat ideal of R. By Theorem 7.21, $\mathrm{FFD}(R) \leqslant 1$. By Theorem 7.15, T as an R-module is flat. By [22, Theorem 8.3.10], F is the quotient field of D, a contradiction. Therefore T as an R-module is not 2-torsion-free. □

7.4 Weak global dimension and weak finitistic dimension of pseudo-valuation rings

Let R be a pseudo-valuation ring. By [22, Corollary 11.8.9], there is a Milnor square $(RDTF, M)$, where (T, M) is a valuation ring and D is a field.

Lemma 7.24. *Let (R, M) be a pseudo-valuation ring and A be a finitely generated proper ideal of R. Then there is an exact sequence*

$$0 \longrightarrow M^{n-1} \longrightarrow R^n \longrightarrow A \longrightarrow 0,$$

where n is an appropriate positive integer.

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be a minimal generating set of A. The situation of $n = 1$ is obvious, and so we may assume that $n > 1$. Let $\varphi : R^n \to A$ be a projective cover. Then $\mathrm{Ker}(\varphi) \subseteq M^n$. Let $z = (m_1, m_2, \ldots, m_n) \in \mathrm{Ker}(\varphi)$. Define $g(z) = (m_2, m_3, \ldots, m_n)$. If $g(x) = 0$, then $m_2 = \cdots = m_n = 0$. Thus $m_1 x_1 = 0$. Since A is a torsion-free module, it follows that $m_1 = 0$. Thus g is a monomorphism.

Again let $m_2, \ldots, m_n \in M$. Set $I = Rx_2 + \cdots + Rx_n$ and $J = Rx_1$. By minimality of a generating set, $J \nsubseteq I$. By [22, Theorem 11.8.6], $IM \subseteq JM = Mx_1$. Thus there exists $m_1 \in M$ such that $m_1 x_1 + m_2 x_2 + \cdots + m_n x_n = 0$, and so $(m_1, m_2, \ldots, m_n) \in \mathrm{Ker}(\varphi)$. Thus g is an epimorphism. Therefore $\mathrm{Ker}(\varphi) \cong M^{(n-1)}$. □

Proposition 7.25. *Let (R, M) be a pseudo-valuation ring, but not a valuation ring. Then:*

(1) $M = M^2$ if and only if M is a flat ideal of R.

(2) If $M \neq M^2$, then there exists an exact sequence

$$0 \longrightarrow M^{(L_1)} \longrightarrow R^{(L)} \longrightarrow M \longrightarrow 0,$$

where L, L_1 are appropriate index sets.

Proof. (1) Assume that $M = M^2$. By Lemma 7.22(2), M is a flat ideal of R. Conversely, assume that M is a flat ideal of R. If $M \neq M^2$, then by [22, Theorem 2.5.22], M is a principal ideal, which contradicts [22, Theorem 8.3.3 (2)]. Thus $M = M^2$.

(2) Let $X = \{x_i \mid i \in L\} \subseteq R$ such that $\{\overline{x_i}\}$ is an R/M-basis of T/M. We may assume that $x_1 = 1$. By [22, Theorem 2.5.22], $M = Tu$, where $u \in M \setminus M^2$. Define $\varphi : R^{(L)} \to M$ by $\varphi(e_i) = x_i u$, where $\{e_i \mid i \in L\}$ is the standard basis of $R^{(L)}$. Thus $\mathrm{Im}(\varphi) = \sum\limits_i Rx_i u = \sum\limits_i Rx_i u + Mu = (\sum\limits_i Rx_i + M)u = Tu = M$, and so φ is an epimorphism. It is easy to see that $\mathrm{Ker}(\varphi) \subseteq M^{(L)}$. Set $L_1 = L - \{1\}$ and $I = \sum\limits_{j \neq 1} Rx_j u$, $J = Ru$. Then $J \nsubseteq I$. Similarly to the proof of Lemma 7.24, we have $\mathrm{Ker}(\varphi) \cong M^{(L_1)}$. □

Let (R, M) be a pseudo-valuation ring, but not a valuation ring. In [8], Dobbs proved that if $M = M^2$ holds, then $w.\mathrm{gl.dim}(R) = 2$; when $M \neq M^2$, one has $w.\mathrm{gl.dim}(R) = +\infty$. We now provide a more precise form of Dobbs' theorem.

Theorem 7.26. Let (R, M) be a pseudo-valuation ring, but not a valuation ring, then $\mathrm{FFD}(R) \leqslant 2$. More specifically:

(1) If $M = M^2$, then $w.\mathrm{gl.dim}(R) = 2$.

(2) If $M \neq M^2$, then $\mathrm{FFD}(R) = 1$, at this time $w.\mathrm{gl.dim}(R) = +\infty$.

Proof. (1) Since $M = M^2$, by Proposition 7.25 M is a flat ideal of R. Let A be a finitely generated proper ideal of R. By Lemma 7.24, $\mathrm{fd}_R R/A \leqslant 2$, and so $w.\mathrm{gl.dim}(R) \leqslant 2$. Since R is not a valuation ring, $w.\mathrm{gl.dim}(R) = 2$.

(2) Since M is a flat ideal of T, by applying Theorem 7.23, T as an R-module is 2-torsion-free. It follows by Theorem 7.21 that $\mathrm{FFD}(R) \leqslant 1$. Assume on the contrary that $\mathrm{fd}_R M = n < +\infty$. Then it follows by Proposition 7.25 that $n > 0$. By Proposition 7.25 and [22, Theorem 3.6.6], we get that $n = n + 1$, a contradiction. Thus $\mathrm{fd}_R M = +\infty$, and so $w.\mathrm{gl.dim}(R) = +\infty$. □

8 Matlis cotorsion modules and Matlis domains

This section uses the Matlis domain as an example to show some methods of cotorsion theory for describing the structure of the ring. The Matlis domain is the domain class to which Matlis is primarily concerned [20]. For the domain R, K is always used below to indicate the quotient field of R.

8.1 h-divisible modules and Matlis cotorsion modules

Definition 8.1. Let R be a domain and let D be an R-module.

(1) D is called an **h-divisible module** if it is a factor module of an injective module (i.e., 0-th cosyzygy module). Denote by $\mathcal{D}_h$ the class of h-divisible modules.

(2) D is called a **reduced module** (resp., an **h-reduced module**) if it does not contain any nonzero divisible (resp., h-divisible) submodule.

Theorem 8.2. Let R be a domain and let D be an R-module. Then the following are equivalent:

(1) D is h-reduced.

(2) $0 \to \mathrm{Hom}_R(K/R, D) \to \mathrm{Hom}_R(K, D) \to \mathrm{Hom}_R(R, D) \to 0$ is an exact sequence.

(3) D is a factor module of a torsion-free divisible module (i.e., a K-vector space).

(4) For any $x \in D$, there exists a homomorphism $g : K \to D$ such that $g(1) = x$.

Proof. (1)⇒(2) Since D is h-reduced, there exists an epimorphism $E \to D$, where E is an injective module. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_R(K,E) & \longrightarrow & \mathrm{Hom}_R(K,D) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_R(R,E) & \longrightarrow & \mathrm{Hom}_R(R,D) \end{array}$$

where all mappings are natural homomorphisms. Since E is injective, the left vertical map is an epimorphism. Since R is a free module, the second row is an epimorphism. Therefore the right vertical map is an epimorphism.

(2)⇒(3) Note that $\mathrm{Hom}_R(K, D)$ is trivially a K-vector space. Now this follows from the fact that $D \cong \mathrm{Hom}_R(R, D)$.

(3)⇒(1) By [22, Theorem 2.4.7], every torsion-free divisible module is injective, and so D is h-divisible.

(2)⇔(4) This is trivial. □

Example 8.3. Let R be a domain.

(1) Every factor module of an h-divisible module is h-divisible and any direct product of h-divisible modules is h-divisible.

(2) Since every factor module of a divisible module is divisible, every h-divisible module is divisible. Denote by $\mathcal{D}$ the class of divisible modules. Then $\mathcal{D}_h \subseteq \mathcal{D}$.

(3) By Zorn's lemma and Exercise 16, any module has a maximal divisible (resp., h-divisible) submodule. Denote by $d(X)$ (resp., $d_h(X)$) a **maximal divisible submodule** (resp., **maximal h-divisible submodule**) of a module X. Trivially a module M is reduced (resp., h-reduced) if and only if $\operatorname{Hom}_R(D,M)=0$ for any divisible (resp., h-divisible) module D.

(4) Any submodule of a reduced module (resp., an h-reduced module) is naturally reduced (resp., h-reduced).

Proposition 8.4. *Let R be a domain and let M be an R-module. Then $\operatorname{pd}_R M \leqslant 1$ if and only if $\operatorname{Ext}^1_R(M,D)=0$ for any h-divisible module D. In other words, $\mathcal{P}_1 = {}^{\perp}\mathcal{D}_h$.*

Proof. Note that for a ring R and a positive integer n, $\operatorname{pd}_R M \leqslant n$ if and only if $\operatorname{Ext}^1_R(M,X)=0$ for any $(n-1)$-th cosyzygy module X. Now the assertion follows by taking $n=1$. □

Proposition 8.5. *Let R be a domain and let M be an R-module. Then M is h-reduced if and only if $\operatorname{Hom}_R(K,M)=0$. In particular, if R is not a field, then R is an h-reduced module; more generally every free module is h-reduced.*

Proof. Assume that $\operatorname{Hom}_R(K,M)\neq 0$. Then there is a nonzero homomorphism $f: K \to M$. Thus $f(K)$ is a nonzero h-divisible submodule of M. Hence M is not h-reduced.

Conversely, assume that M is not h-reduced. Then M has a nonzero h-divisible submodule D. By Theorem 8.2, there exists a homomorphism $g: K \to D$ such that $g(1)\neq 0$. Let $\lambda: D \to M$ be an embedding. Then $\lambda g: K \to M$ is a nonzero. Thus $\operatorname{Hom}_R(K,M)\neq 0$. □

Proposition 8.6. *Let R be a domain and let D be an h-divisible module. Then the total torsion submodule $\operatorname{tor}(D)$ is a direct summand of D. In particular, $\operatorname{tor}(E)$ of an injective module E is a direct summand of E, and so is an injective module.*

Proof. Let E be a K-vector space and let $g: E \to D$ be an epimorphism. Set $T := \operatorname{tor}(D)$ and $N := D/T$. Then N is a torsion-free divisible module, and so a K-vector space. Let $\{x_i\} \subseteq D$ such that $\{\overline{x_i}\}$ is a K-basis of N. For any i, choose $e_i \in E$ such that $g(e_i)=x_i$. Then it is clear that $\{e_i\}$ in E is K-linearly independent. Let M be a K-vector space in E generated by $\{e_i\}$. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{h} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle g} & & \| & & \\
0 & \longrightarrow & T & \longrightarrow & D & \xrightarrow{\pi} & N & \longrightarrow & 0
\end{array}$$

where π is a natural homomorphism. By Exercise 9, h is a K-homomorphism, and so A is also a K-vector space. For $k \in K$ and $x \in M$, define $kg(x)=g(kx)$. Since $\operatorname{Ker}(g)\subseteq A$, it follows that $g(M)$ has been made into a K-vector space, which is a torsion-free module as an R-module. Thus $T\cap g(M)=0$. It follows by direct verification that $D=T+g(M)$. Therefore $D=T\oplus g(M)$. □

Definition 8.7. Let R be a domain and let M, W be R-modules.

(1) W is called a **Matlis cotorsion module** if $\operatorname{Ext}^1_R(K,W)=0$. Denote by $\mathcal{MC}$ the class of Matlis cotorsion modules.

(2) W is called a **Lee cotorsion module** if $\mathrm{Ext}_R^1(K/R, W) = 0$. Denote by $\mathcal{LC}$ the class of Lee cotorsion modules.

(3) M is called a **strongly flat module** if $M \in {}^{\perp}\mathcal{MC}$. Denote by $\mathcal{SF}$ the class of strongly flat modules.

(4) Write $\mathcal{SF}_1 = {}^{\perp}\mathcal{LC}$.

Proposition 8.8. *Let R be a domain. Then:*

(1) $\mathcal{C} \subseteq \mathcal{MC}$ and $\mathcal{C}_1 \subseteq \mathcal{LC}$. In other words, every cotorsion module is Matlis cotorsion and every 1-cotorsion module is Lee cotorsion.

(2) $\mathcal{LC} \subseteq \mathcal{MC} \cap \mathcal{D}_h$. That is, every Lee cotorsion module is both h-divisible and Matlis cotorsion.

(3) $\mathcal{P} \subseteq \mathcal{SF} \subseteq \mathcal{F}$ and $\mathcal{P}_1 \subseteq \mathcal{SF}_1 \subseteq \mathcal{F}_1$.

(4) $\mathcal{SF} \subseteq \mathcal{SF}_1$.

(5) Let G be an R-module. If there exists $u \in R \setminus \{0\}$ such that $uG = 0$, then $G \in \mathcal{MC}$.

(6) The quotient field K of R is trivially a strongly flat R-module.

Proof. Exercise. □

Theorem 8.9. Let R be a domain. Then $(\mathcal{SF}, \mathcal{MC})$ and $(\mathcal{SF}_1, \mathcal{LC})$ are all complete cotorsion theories.

Proof. This follows by taking $\mathcal{S} = \{K\}$ and $\mathcal{S} = \{K/R\}$ respectively in Theorem 3.11. □

Lemma 8.10. *Let R be a domain.*

(1) Let $0 \to A \to B \to C \to 0$ be an exact sequence. If $B \in \mathcal{MC}$ and C is h-reduced, then $A \in \mathcal{MC}$.

(2) Let A be a torsion module. Then $\mathrm{Hom}_R(A, X)$ is an h-reduced Matlis cotorsion module for any R-module X. In particular, $\mathrm{Hom}_R(K/R, X)$ is an h-reduced Matlis cotorsion module.

(3) Let D be an h-divisible module. Then there exists an exact sequence $0 \to M \to E \to D \to 0$, where E is a K-vector space and M is an h-reduced Matlis cotorsion module.

(4) Let M be a torsion-free module. Then $\mathrm{Ext}_R^1(K/R, M)$ is always an h-reduced Matlis cotorsion module..

Proof. (1) Since C is h-reduced, $\mathrm{Hom}_R(K, C) = 0$. Thus there exists an exact sequence $0 \to \mathrm{Ext}_R^1(K, A) \to \mathrm{Ext}_R^1(K, B) = 0$. Therefore $\mathrm{Ext}_R^1(K, A) = 0$, and so $A \in \mathcal{MC}$.

(2) Since A is a torsion module, by [22, Theorem 2.2.16] $\mathrm{Hom}_R(K, \mathrm{Hom}_R(A, X)) = 0$. By Proposition 8.5, $\mathrm{Hom}_R(A, X)$ is an h-reduced module.

Let $0 \to X \to E \to Y \to 0$ be an exact sequence, where E is an injective module. Then there exists an exact sequence $0 \to \mathrm{Hom}_R(A, X) \to \mathrm{Hom}_R(A, E) \to \mathrm{Hom}_R(A, Y)$. By [22, Theorem 3.4.11], $\mathrm{Ext}_R^1(K, \mathrm{Hom}_R(A, E)) \cong \mathrm{Hom}_R(\mathrm{Tor}_1^R(K, A), E) = 0$. Thus $\mathrm{Hom}_R(A, E)$ is a Matlis cotorsion module. Since $\mathrm{Hom}_R(A, Y)$ is an h-reduced module, it follows by (1) that $\mathrm{Hom}_R(A, X)$ is a Matlis cotorsion module.

(3) This follows from Theorem 8.2 by taking $E = \mathrm{Hom}_R(K, D)$ and $M = \mathrm{Hom}_R(K/R, D)$.

(4) By [22, Theorem 3.6.12] there exists an exact sequence $0 \to M \to K \otimes_R M \to (K/R) \otimes_R M \to 0$. Since M and $K \otimes_R M$ are torsion-free, $K \otimes_R M$ is an injective R-module. Hence there is an exact sequence:

$$0 \to \mathrm{Hom}_R(K/R, (K/R) \otimes_R M) \to \mathrm{Ext}_R^1(K/R, M) \to \mathrm{Ext}_R^1(K/R, K \otimes_R M) = 0.$$

Therefore $\mathrm{Hom}_R(K/R, (K/R) \otimes_R M) \cong \mathrm{Ext}_R^1(K/R, M)$. Now the assertion follows by (2). □

Lemma 8.11. *Let R be a domain and let M be an h-reduced module.*

(1) $M \in \mathcal{MC}$ if and only if $M \cong \mathrm{Ext}^1_R(K/R, M)$.

(2) If M is a torsion-free module, then there exists an exact sequence

$$0 \to M \to C \to E \to 0,$$

where E is a K-vector space and C is an h-reduced Matlis cotorsion module.

Proof. (1) Since M is h-reduced, $\mathrm{Hom}_R(K, M) = 0$, and hence there exists an exact sequence

$$0 \to M = \mathrm{Hom}_R(R, M) \to \mathrm{Ext}^1_R(K/R, M) \to \mathrm{Ext}^1_R(K, M) \to 0.$$

Thus $M \in \mathcal{MC}$ if and only if $M \cong \mathrm{Ext}^1_R(K/R, M)$.

(2) This follows immediately by taking $C = \mathrm{Ext}^1_R(K/R, M)$ and $E = \mathrm{Ext}^1_R(K, M)$ in the exact sequence in (1) and by applying Lemma 8.10. □

Theorem 8.12. Let R be a domain and let M be an R-module. Then the following are equivalent:

(1) M is a strongly flat module.

(2) M is a direct summand of a certain module N, where N fits into an exact sequence $0 \to F \to N \to E \to 0$, where F is a free module and E is a K-vector space.

(3) $M \in \mathcal{F}$ and $\mathrm{Ext}^1_R(M, C) = 0$ for any $C \in \mathcal{F} \cap \mathcal{MC}$.

(4) $M \in \mathcal{F}$ and $\mathrm{Ext}^1_R(M, C) = 0$ for any reduced Matlis cotorsion module C.

Proof. (1)⇒(2) Let $0 \to H \to F \to M \to 0$ be an exact sequence, where F is a free module. Since F is a torsion-free h-reduced module, H is also torsion-free h-reduced. By Lemma 8.11(2), there exist a Matlis cotorsion module C and an embedding map $i : H \to C$ such that $E := \mathrm{Coker}(i)$ is a K-vector space. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & C & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where the left square is a pushout diagram. Note that $\mathrm{Coker}(g) \cong E$. Since C is a Matlis cotorsion module, the second row is split. Thus M is a direct summand of N.

(2)⇒(1) Since F and E are strongly flat modules, N is strongly flat, and so M is strongly flat.

(1)⇒(4) Trivial.

(4)⇒(3) Since C is a flat module, C is torsion-free. Let D be a maximal divisible submodule of C. Then D is injective, and hence we have a direct sum decomposition $C = D \oplus C_1$. Thus C_1 is a reduced Matlis cotorsion module. By the hypothesis, $\mathrm{Ext}^1_R(M, C) = \mathrm{Ext}^1_R(M, C_1) = 0$.

(3)⇒(1) Let C be a Matlis cotorsion module. Since $(\mathcal{F}, \mathcal{C})$ is a perfect cotorsion theory, there is an exact sequence $0 \to X \to G \to C \to 0$, where G is a flat module and X is a cotorsion module. Thus $G \in \mathcal{F} \cap \mathcal{MC}$. By the hypothesis, $\mathrm{Ext}^1_R(M, G) = 0$. By the exact sequence $0 = \mathrm{Ext}^1_R(M, G) \to \mathrm{Ext}^1_R(M, C) \to \mathrm{Ext}^2_R(M, X)$ and $\mathrm{Ext}^2_R(M, X) = 0$, it follows that $\mathrm{Ext}^1_R(M, C) = 0$. Therefore M is a strongly flat module. □

Theorem 8.13. Let R be a domain, P be a projective module, and F be a strongly flat submodule of P. Then F is a projective module.

Proof. Let D be an h-divisible module. By Lemma 8.10, there is an exact sequence $0 \to M \to E \to D \to 0$, where E is a K-vector space and $M \in \mathcal{MC}$. Thus $\mathrm{Ext}_R^1(P/F, D) \cong \mathrm{Ext}_R^2(P/F, M) \cong \mathrm{Ext}_R^1(F, M) = 0$. By Proposition 8.4, $\mathrm{pd}_R P/F \leqslant 1$, and so F is projective. □

Theorem 8.14. Let R be a domain, S be a multiplicative subset of R, and C be a Matlis cotorsion R-module. Then:

(1) $C^S := \mathrm{Hom}_R(R_S, C)$ is a Matlis cotorsion R_S-module.

(2) If C is also an h-reduced module, then C^S is an h-reduced Matlis cotorsion R_S-module and $C^S \cong \mathrm{Ext}_R^1(K/R_S, C)$.

Proof. (1) Let $\xi : 0 \to C \to E \to N \to 0$, where E is an injective R-module. Then we have an exact sequence:

$$0 \to \mathrm{Hom}_R(R_S, C) \xrightarrow{f} \mathrm{Hom}_R(R_S, E) \to H \to 0,$$

where $H := \mathrm{Coker}(f)$. Thus we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_{R_S}(K, \mathrm{Hom}_R(R_S, E)) & \longrightarrow & \mathrm{Hom}_{R_S}(K, H) & \longrightarrow & \mathrm{Ext}^1_{R_S}(K, \mathrm{Hom}_R(R_S, C)) & \longrightarrow & 0 \\
\downarrow{\cong} & & \downarrow{\sigma} & & \downarrow{\tau} & & \\
\mathrm{Hom}_R(K \otimes_{R_S} R_S, E) & \longrightarrow & \mathrm{Hom}_R(K \otimes_{R_S} R_S, N) & \longrightarrow & \mathrm{Ext}^1_R(K \otimes_{R_S} R_S, C) & \longrightarrow & 0
\end{array}$$

Since $H \subseteq \mathrm{Hom}_R(R_S, N)$, σ is a monomorphism, and so τ is a monomorphism. Since C is a Matlis cotorsion module, $\mathrm{Ext}^1_{R_S}(K, \mathrm{Hom}_R(R_S, C)) = 0$, and thus $\mathrm{Hom}_R(R_S, C)$ is a Matlis cotorsion R_S-module.

(2) Since $\mathrm{Hom}_{R_S}(K, \mathrm{Hom}_R(R_S, C)) \cong \mathrm{Hom}_R(K \otimes_{R_S} R_S, C) = 0$, C^S is a reduced R_S-module. From the exact sequence $0 \to R_S \to K \to K/R_S \to 0$, we have an exact sequence:

$$0 = \mathrm{Hom}_R(K, C) \to \mathrm{Hom}_R(R_S, C) \to \mathrm{Ext}^1_R(K/R_S, C) \to \mathrm{Ext}^1_R(K, C) = 0.$$

Now the last assertion follows. □

Theorem 8.15. Let R be a domain, S be a multiplicative subset of R, and M be a strongly flat R-module. Then M_S is a strongly flat R_S-module.

Proof. By Theorem 8.12, M is a direct summand of a module N, where N fits in an exact sequence $0 \to F \to N \to E \to 0$ with F a free module and E a K-vector space. Thus M_S is a direct summand of N_S and $0 \to F_S \to N_S \to E \to 0$ is an exact sequence. It follows by Theorem 8.12 that M_S is a strongly flat R_S-module. □

8.2 Characterizations of Matlis domains

Definition 8.16. A domain R is called a **Matlis domain** if $\mathrm{pd}_R K \leqslant 1$, equivalently, $\mathrm{pd}_R K/R \leqslant 1$.

Trivially every Dedekind domain is a Matlis domain.

Theorem 8.17. The following are equivalent for a domain R:

(1) R is a Matlis domain.

(2) $\mathcal{D}_h = \mathcal{LC}$, that is, every h-divisible module is a Lee cotorsion module.

(3) Every factor module of a Lee cotorsion module is a Lee cotorsion module.

(4) $\mathcal{P}_1 = \mathcal{SF}_1$.

(5) $(\mathcal{SF}, \mathcal{MC})$ is a hereditary cotorsion theory.

(6) The projective dimension of any strong flat module is at most 1.

Proof. (1)⇒(2) Let D be an h-divisible module. By Proposition 8.4, $\mathrm{Ext}_R^1(K/R, D) = 0$. Thus D is a Lee cotorsion module.

(2)⇒(3) Obviously, because $\mathcal{LC} \subseteq \mathcal{D}_h$, every factor module of a Lee cotorsion module is h-divisible.

(3)⇒(1) Let D be any h-divisible module. By the hypothesis, D is a Lee cotorsion module, and so $\mathrm{Ext}_R^1(K/R, D) = 0$. By Proposition 8.4, $\mathrm{pd}_R K/R \leqslant 1$, that is, R is a Matlis domain.

(2)⇒(4) By the hypothesis, $\mathcal{D}_h = \mathcal{LC}$. By Proposition 8.4, $\mathcal{P}_1 = {}^{\perp}\mathcal{D}_h = {}^{\perp}\mathcal{LC} = \mathcal{SF}_1$.

(4)⇒(1) This follows from the fact that $K/R \in \mathcal{SF}_1 = \mathcal{P}_1$.

(1)⇒(5) Let $0 \to A \to B \to C \to 0$ be an exact sequence, where A, B are Matlis cotorsion modules. Then by an exact sequence $0 = \mathrm{Ext}_R^1(K, B) \to \mathrm{Ext}_R^1(K, C) \to \mathrm{Ext}_R^2(K, A) = 0$ it follws that $\mathrm{Ext}_R^1(K, C) = 0$. Thus C is a Matlis cotorsion module. By Theorem 3.5 and Theorem 8.9, $(\mathcal{SF}, \mathcal{MC})$ is a hereditary cotorsion theory.

(5)⇒(1) Let D be an h-divisible module. By Lemma 8.10, there exists an exact sequence $0 \to M \to E \to D \to 0$, where E is a K-vector space and M is an h-reduced Matlis cotorsion module. By the hypothesis, D is a Matlis cotorsion module, and so $\mathrm{Ext}_R^1(K, D) = 0$. By Proposition 8.4, $\mathrm{pd}_R K \leqslant 1$, that is, R is a Matlis domain.

(1)⇒(6) Let M be a strongly flat module and $0 \to N \to P \to M \to 0$ be an exact sequence, where P is a projective module. By Theorem 8.17, $(\mathcal{SF}, \mathcal{MC})$ is a hereditary cotorsion theory. By Theorem 3.5, N is a strongly flat module. By Theorem 8.13, N is a projective module, and so $\mathrm{pd}_R M \leqslant 1$.

(6)⇒(1) This is obvious because K is a strongly flat module. □

Let R be a Matlis domain. By Theorem 8.17, $\mathfrak{G} = (\mathcal{SF}, \mathcal{MC})$ is a hereditary cotorsion theory. From Section 5 one can define the **strongly flat dimension** of a module M to be ∞, or the shortest length of strongly flat resolutions of M, denoted by $\mathrm{Sfd}_R M$. By Theorem 5.3, $\mathrm{fd}_R M \leqslant \mathrm{Sfd}_R M \leqslant \mathrm{pd}_R M$.

Proposition 8.18. *Let R be a Matlis domain and M be a torsion-free R-module. Then the following are equivalent:*

(1) M is a strongly flat R-module.

(2) $\mathrm{pd}_R((K/R) \otimes_R M) \leqslant 1$.

(3) $\mathrm{Sfd}_R((K/R) \otimes_R M) \leqslant 1$.

Proof. (1)⇒(2) By tensoring K/R with the exact sequence of Theorem 8.12(2), we have the following exact sequence

$$0 \longrightarrow (K/R) \otimes_R F \longrightarrow (K/R) \otimes_R N \longrightarrow (K/R) \otimes_R E \longrightarrow 0.$$

Since $(K/R) \otimes_R F$ and $(K/R) \otimes_R E$ are direct sums of some copies of K/R respectively,

$$\mathrm{pd}_R((K/R) \otimes_R N) \leqslant \max\{\mathrm{pd}_R((K/R) \otimes_R F), \mathrm{pd}_R((K/R) \otimes_R E)\} \leqslant 1.$$

Thus $\mathrm{pd}_R((K/R) \otimes_R N) \leqslant 1$. Therefore $\mathrm{pd}_R((K/R) \otimes_R M) \leqslant 1$.

(2)⇒(3) This is trivial.

(3)⇒(1) Consider the exact sequence $0 \to R \to K \to K/R \to 0$. Since M is a torsion-free module, it follows by [22, Exercise 3.6] that

$$0 \longrightarrow M \longrightarrow K \otimes_R M \longrightarrow (K/R) \otimes_R M \longrightarrow 0$$

is an exact sequence. Since $\mathrm{Sfd}_R((K/R) \otimes_R M) \leqslant 1$, it follows that M is a strongly flat module. □

Lemma 8.19. *Let $a_1,\dots,a_n,\dots$ be a sequence of nonzero elements of R and let F be a free R-module with its basis $x_0,x_1,\dots,x_n,\dots$. Set*

$$y_n = x_n - a_{n+1}x_{n+1}, \qquad n = 0,1,2,\dots.$$

Let $n \geqslant 1$ and $y = r_0x_0 + r_1x_1 + \cdots + r_nx_n \in F$. Then

$$y = \sum_{k=0}^{n-1}(\sum_{j=0}^{k} r_ja_{j+1}\cdots a_k)y_k + (r_n + r_{n-1}a_n + \cdots + r_1a_2\cdots a_n + r_0a_1\cdots a_n)x_n.$$

where if $j = k$, regard $a_{j+1}\cdots a_k = 1$.

Proof. If $n = 1$, then $y = r_0x_0 + r_1x_1 = r_0y_0 + (r_1 + r_0a_1)x_1$. Thus the assertion is true. Assume that $n > 1$ and set $z = r_0x_0 + r_1x_1 + \cdots + r_{n-1}x_{n-1}$. By the induction hypothesis,

$$z = \sum_{k=0}^{n-2}(\sum_{j=0}^{k} r_ja_{j+1}\cdots a_k)y_k + (\sum_{j=0}^{n-1} r_ja_{j+1}\cdots a_{n-1})x_{n-1}.$$

Therefore

$$\begin{aligned} y &= z + r_nx_n \\ &= \sum_{k=0}^{n-2}(\sum_{j=0}^{k} r_ja_{j+1}\cdots a_k)y_k + (\sum_{j=0}^{n-1} r_ja_{j+1}\cdots a_{n-1})y_{n-1} + a_n(\sum_{j=0}^{n-1} r_ja_{j+1}\cdots a_k)x_n + r_nx_n \\ &= \sum_{k=0}^{n-1}(\sum_{j=0}^{k} r_ja_{j+1}\cdots a_k)y_k + (\sum_{j=0}^{n} r_ja_{j+1}\cdots a_n)x_n. \end{aligned}$$ □

Lemma 8.20. *Let R be a domain, $a_1,\dots,a_n,\dots$ be a sequence of nonzero elements of R, and A be an R-submodule of K generated by*

$$\{1, \frac{1}{a_1}, \dots, \frac{1}{a_1\cdots a_n}, \cdots\}.$$

Then $\mathrm{pd}_R A \leqslant 1$. In particular, if $u \in R \setminus \{0\}$, then $\mathrm{pd}_R R_u \leqslant 1$.

Proof. Let F be a free module with its basis $x_0,x_1,\dots,x_n,\dots$ and let $\phi : F \to A$ be an epimorphism such that $\phi(x_0) = 1$ and $\phi(x_n) = \frac{1}{a_1\cdots a_n}$, $n \geqslant 1$. Set $P = \mathrm{Ker}(\phi)$ and $y_n := x_n - a_{n+1}x_{n+1}$, $n \geqslant 0$. Below we prove that P is a submodule of F generated by $y_0,y_1,y_2,\dots,y_n,\dots$. Then by [22, Theorem 3.10.19(1)], P is a free module, and so $\mathrm{pd}_R A \leqslant 1$.

By a direct verification, one has $\phi(y_0) = 1 - a_1\frac{1}{a_1} = 0$, and if $n \geqslant 1$, then $\phi(y_n) = \frac{1}{a_1\cdots a_n} - a_{n+1}\frac{1}{a_1\cdots a_na_{n+1}} = 0$. Thus $y_n \in P$. On the other hand, for any $y \in P$ we can write $y = r_0x_0 + r_1x_1 + \cdots + r_nx_n$. Since $\phi(y) = 0$, we have $r_0a_1\cdots a_n + r_1a_2\cdots a_n + \cdots + r_{n-1}a_n + r_n = 0$. If $n = 0$, then $y = r_0x_0$. Since $\phi(y) = r_0 = 0$, we have $y = 0$. Let $n \geqslant 1$. By Lemma 8.19, $y = r_0y_0 + \sum_{k=1}^{n-1}(r_k + r_{k-1}a_k + \cdots + r_0a_1\cdots a_k)y_k$.

The last assertion follows from the fact that $1, \frac{1}{u}, \dots, \frac{1}{u^n}, \dots$ is as an R-module a generating system of R_u. □

Theorem 8.21. Let R be a domain. If K as an R-module is countably generated, then R is a Matlis domain. In particular, every umbrella ring is a Matlis domain.

Proof. Suppose that K is generated by $\{x_n\}_{n=1}^{\infty}$. Write $x_n = \frac{b_n}{a_n}$, $n \geqslant 1$. Note that $\{1, \frac{1}{a_1}, \frac{1}{a_1a_2}, \dots, \frac{1}{a_1\cdots a_n}, \dots\}$ is also a generating system of K. By Lemma 8.20, $\mathrm{pd}_R K \leqslant 1$. □

Lemma 8.22. *Let R be a domain, S be a multiplicative set of R, and T be a multiplicative subset of S. If $\mathrm{pd}_R R_S \leqslant 1$ and $\mathrm{pd}_R R_S/R_T \leqslant 1$, then we have:*

(1) R_T/sR_T is a projective R/sR-module for any $s \in S$.

(2) Let $\mathfrak{m} \in \mathrm{Max}(R)$. If $T \cap \mathfrak{m} = \varnothing$, then $(R_T)_\mathfrak{m} = R_\mathfrak{m}$. If $T \cap \mathfrak{m} \neq \varnothing$, then $(R_T)_\mathfrak{m} = (R_S)_\mathfrak{m}$.

(3) R_T/R is a direct summand of R_S/R.

Proof. (1) Obviously $\mathrm{pd}_R R_T \leqslant 1$. Write $\overline{R} = R/sR$. Since $\mathrm{pd}_R R_S \leqslant 1$ and R_T is a torsion-free module, $\mathrm{pd}_{\overline{R}} R_T/sR_T \leqslant 1$. Since $sR_S = R_S$, we have $R_S/R_T \cong sR_S/sR_T = R_S/sR_T$. From the exact sequence $0 \to R_T/sR_T \to R_S/sR_T \to R_S/R_T \to 0$ we know that $\mathrm{pd}_R R_T/sR_T \leqslant 1$. It follows by [22, Theorem 3.8.13] that $\mathrm{pd}_{\overline{R}} R_T/sR_T = 0$.

(2) If $T \cap \mathfrak{m} = \varnothing$, then obviously $(R_T)_\mathfrak{m} = R_\mathfrak{m}$, which is true for any domain. If $T \cap \mathfrak{m} \neq \varnothing$, then over the ring $R_\mathfrak{m}$, by (1), $(R_T)_\mathfrak{m}/s(R_T)_\mathfrak{m}$ is a free $R_\mathfrak{m}/sR_\mathfrak{m}$-module. Since $t(R_T)_\mathfrak{m} = (R_T)_\mathfrak{m}$ if $t \in T \cap \mathfrak{m}$, it follows that $(R_T)_\mathfrak{m}/s(R_T)_\mathfrak{m}$ is a t-divisible module. Since t is not a unit of $R_\mathfrak{m}$, but $(R_T)_\mathfrak{m}/s(R_T)_\mathfrak{m}$ is a free $R_\mathfrak{m}/sR_\mathfrak{m}$-module, there can only be $(R_T)_\mathfrak{m}/s(R_T)_\mathfrak{m} = 0$, that is, $(R_T)_\mathfrak{m}$ is a divisible $R_\mathfrak{m}$-submodule of K. Therefore $(R_T)_\mathfrak{m} = K$.

(3) Let $\mathfrak{m}$ be a maximal ideal of R. If $T \cap \mathfrak{m} = \varnothing$, then $(R_T)_\mathfrak{m} = R_\mathfrak{m}$. Thus $(R_T/R)_\mathfrak{m} = R_\mathfrak{m}/R_\mathfrak{m} = 0$. If $T \cap \mathfrak{m} \neq \varnothing$, then $(R_T)_\mathfrak{m} = K$, and so $(R_T/R)_\mathfrak{m} = K/R_\mathfrak{m}$. Therefore $(R_T/R)_\mathfrak{m}$ is a divisible $R_\mathfrak{m}$-module. Note that for a domain R, an R-module X is divisible if and only if $X_\mathfrak{m}$ is a divisible $R_\mathfrak{m}$-module for any $\mathfrak{m} \in \mathrm{Max}(R)$. Then R_T/R is a divisible module. Thus $R_T = sR_T + R$ for any $s \in R \setminus \{0\}$. Hence $R/(sR_T \cap R) \cong R_T/sR_T$ is a cyclic projective R/sR-module. Therefore the natural homomorphism $h : R/sR \to R/(sR_T \cap R)$ is a split epimorphism, and it is easy to get $\mathrm{Ker}(h) = (sR_T \cap R)/sR \cong (R_T \cap s^{-1}R)/R$. Thus

$$R/sR \cong s^{-1}R/R = (R_T \cap s^{-1}R)/R \oplus C, \qquad \text{where} \quad C \cong R_T/sR_T. \tag{8.1}$$

Write $\Gamma_1 = \{\mathfrak{m} \in \mathrm{Max}(R) \mid T \cap \mathfrak{m} = \varnothing\}$ and $\Gamma_2 = \{\mathfrak{m} \in \mathrm{Max}(R) \mid T \cap \mathfrak{m} \neq \varnothing\}$. And set

$$A = \bigcap_{\mathfrak{m} \in \Gamma_1} R_\mathfrak{m}, \qquad B = \bigcap_{\mathfrak{m} \in \Gamma_2} R_\mathfrak{m}.$$

Then $A \cap B = \bigcap_{\mathfrak{m} \in \mathrm{Max}(R)} R_\mathfrak{m} = R$. Thus $A/R + B/R = A/R \oplus B/R$. Since $R_T \subseteq A$, it follows that $R_T/R + B/R = R_T/R \oplus B/R$.

In the direct sum decomposition of (8.1), the first term $(R_T \cap s^{-1}R)/R \subseteq R_T/R$. Now we come to prove $C \subseteq B/R$.

If $\mathfrak{m} \in \Gamma_1$, then by (2), $C_\mathfrak{m} \subseteq (R_T/R)_\mathfrak{m} \oplus (B/R)_\mathfrak{m} = (B/R)_\mathfrak{m}$. If $\mathfrak{m} \in \Gamma_2$, then

$$C_\mathfrak{m} \cong (R_T/sR_T)_\mathfrak{m} = (R_T)_\mathfrak{m}/s(R_T)_\mathfrak{m} = K/K = 0 \subseteq (B/R)_\mathfrak{m}.$$

Therefore $C \subseteq B/R$.

Based on the above, it has been proved that $s^{-1}R/R \subseteq R_T/R \oplus B/R$, and so $s^{-1} + R \in R_T/R \oplus B/R$. From the arbitrariness of s, we get $R_S/R = R_T/R \oplus B/R$. □

Lemma 8.23. *Let R be a Matlis domain and $u \in R$ be a nonzero nonunit element. Then there is a multiplicative set T of countable elements of R such that $u \in T$ and R_T/R is a direct summand of K/R, and so R_T/R is an h-divisible module.*

Proof. Let $\mathcal{T}_1$ be the set of submodules of the form R_T/R of K/R. Then $\mathcal{T}_1$ is a **weak tight system** of K/R, that is, the set of submodules that satisfy the following conditions (a), (b):

(a) $0, K/R \in \mathcal{T}_1$ and $\mathcal{T}_1$ is closed under unions of chains;

(b) If $A \in \mathcal{T}_1$ and X is a countable subset of K/R, then there exists $B \in \mathcal{T}_1$ such that $A, X \subseteq B$ and B/A is countably generated.

By Lemma 3.19, K/R has a weak tight system $\mathcal{T}$. Obviously a tight system is a weak tight system, and the intersection of a tight system and a weak tight system is a tight system. Set $\mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}$. Then $\mathcal{T}_2$ is a tight system. Thus if $R_T/R \in \mathcal{T}_2$, then $\mathrm{pd}_R K/R_T \leqslant 1$. By Lemma 8.22, each $R_T/R \in \mathcal{T}_2$ is a direct summand of K/R. Now the assertion follows immediately by taking $A = 0$ and $X = \{u\}$ in Lemma 3.19(1)(c). □

Theorem 8.24. The following are equivalent for a domain R:

(1) R is a Matlis domain.

(2) $\mathcal{D} = \mathcal{D}_h$, that is, every divisible module is h-divisible.

(3) $\mathcal{D} = \mathcal{LC}$, that is every divisible module is a Lee cotorsion module.

(4) $(\mathcal{SF}_1, \mathcal{D})$ is a cotorsion theory.

(5) $(\mathcal{SF}_1, \mathcal{D}_h)$ is a cotorsion theory.

Proof. (1)⇒(2) Let D be a divisible module. First assume that D is a torsion module. For any $x \in D$ with $x \neq 0$, there exists $u \in R$ with $u \neq 0$ such that $ux = 0$. Set $x_0 = x$. By Lemma 8.23, there exists a multiplicative set T with countable elements of R such that $u \in T$ and R_u/R is an h-divisible module.

Write $T = \{1, s_0 = u, s_n \mid n \geqslant 1\}$. For $n \geqslant 1$, recursively take $x_n \in D$ satisfying $s_n x_n = x_{n-1}$. Note that the elements in R_T can be expressed as $\frac{r}{s_n} = \frac{r s_0 s_1 \cdots s_{n-1}}{s_0 s_1 \cdots s_n}$. Define $f : R_T \to D$ by $f(\frac{a}{s_0 s_1 \cdots s_n}) = a x_n$, $a \in R$, $n \geqslant 0$. It is easy to see that f is a well-defined homomorphism. Since $f(1) = u f(\frac{1}{u}) = ux = 0$, we have $R \subseteq \mathrm{Ker}(f)$. By [22, Theorem 1.2.18], f induces a homomorphism $g : R_u/R \to D$ such that $g(\frac{a}{s_0 s_1 \cdots s_n} + R) = a x_n$. So any element in D is contained in an h-divisible submodule, and thus D is an h-divisible module.

Now consider the general situation. Set $D_0 = \mathrm{tor}(D)$. Then $0 \to D_0 \to D \to D/D_0 \to 0$ is an exact sequence. Since D/D_0 is a torsion-free divisible module. Therefore it is a K-vector space, that is, the direct sum of some copies of K. Obviously D_0 is also a divisible module. From the above proof, D_0 is an h-divisible module. Since $\mathrm{pd}_R K \leqslant 1$, by Proposition 8.4, $\mathrm{Ext}^1_R(D/D_0, D_0) = 0$. Therefore the exact sequence is split, so that D is also an h-divisible module.

(2)⇒(1) By Proposition 8.4, it suffices to prove that $\mathrm{Ext}^1_R(K, D) = 0$ for any h-divisible modulus D. By Proposition 8.6, we may assume that D is a torsion module. Let $\xi : 0 \to D \to G \xrightarrow{g} K \to 0$ be an exact sequence. For $x \in G$ and $s \in R \setminus \{0\}$, since K is divisible, there exists $y \in G$ such that $g(x) = sg(y)$. Thus $x - sy \in D$. So there exists $z \in D$ such that $x - sy = sz$. Thus $x = s(y + z)$, that is, G is a divisible module. By the hypothesis G is an h-divisible module. Obviously $D = \mathrm{tor}(G)$. By Proposition 8.6, ξ is a split exact sequence. It follows by [22, Theorem 3.3.5] that $\mathrm{Ext}^1_R(K, D) = 0$.

(2)⇒(3) This follows immediately from Theorem 8.17.

(3)⇒(4) By the hypothesis, $\mathcal{D} = \mathcal{LC}$. Then $(\mathcal{SF}_1, \mathcal{LC})$ is a cotorsion theory.

(4)⇒(5) By the hypothesis, $\mathcal{D} = \mathcal{LC} = \mathcal{SF}_1^{\perp}$. Since $\mathcal{LC} \subseteq \mathcal{D}_h \subseteq \mathcal{D}$, it follows that $\mathcal{D}_h = \mathcal{LC}$.

(5)⇒(1) By the hypothesis, $\mathcal{SF}_1 = {}^{\perp}\mathcal{D}_h = \mathcal{P}_1$. Now the assertion follows immediately from Theorem 8.17. □

Proposition 8.25. *Let R be a Matlis domain and N be an R-module. Then $N/d(N)$ is a reduced module.*

Proof. By Proposition 8.4 and Theorem 8.24, $\mathrm{Ext}^1_R(K, d(N)) = 0$. Thus we have the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(K, d(N)) \longrightarrow \mathrm{Hom}_R(K, N) \longrightarrow \mathrm{Hom}_R(K, N/d(N)) \longrightarrow 0.$$

Let $f \in \mathrm{Hom}_R(K,N)$. Then $f(K)$ is a divisible submodule of N, and so $f(K) \subseteq d(N)$. Thus $\mathrm{Hom}_R(K,d(N)) = \mathrm{Hom}_R(K,N)$. Therefore $\mathrm{Hom}_R(K,N/d(N)) = 0$, that is, $N/d(N)$ is reduced. □

Theorem 8.26. (**Kaplansky**) Let $(R,\mathfrak{m})$ be a local domain. Then R is a Matlis domain if and only if K as an R-module is a countably generated module.

Proof. Suppose that R is a Matlis domain. Take any $u \in \mathfrak{m}$ with $u \neq 0$. By Lemma 8.22(3), $(R_u)_\mathfrak{m} = R_u = K$. Thus K is a countably generated R-module. The converse follows immediately from Theorem 8.21. □

Definition 8.27. Let R be a domain. The R is called an ***h*-local ring** if R satisfies the following two conditions:

(1) For any nonzero element $a \in R$, a is contained in only a finite number of maximal ideals;

(2) Each nonzero prime ideal is only contained in one maximal ideal.

Obviously every local integral domain is an h-local ring. In addition, an integral domain with Krull dimension 1 that satisfies the above condition (1) is also an h-local ring.

Theorem 8.28. Let R be a h-local ring. Then:

(1) If $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are two distinct maximal ideals of R, then $R_{\mathfrak{m}_1} \otimes_R R_{\mathfrak{m}_2} = K$.

(2) Let $\mathfrak{m}$ be a maximal ideal of R and let B be a torsion R-module. Then the natural homomorphism $\theta : B \to B_\mathfrak{m}$ is an epimorphism.

(3) Let $\mathfrak{m}$ be a maximal ideal of R, B be a torsion $R_\mathfrak{m}$-module, and A be an R-submodule of B. Then A is also an $R_\mathfrak{m}$-module.

(4) Let $\mathfrak{m}$ be a maximal ideal of R and let B be a countably generated torsion $R_\mathfrak{m}$-module, Then B as an R-module is also countably generated.

(5) For any R-torsion module B, $B \cong \bigoplus_{\mathfrak{m}\in \mathrm{Max}(R)} B_\mathfrak{m}$. In particular,

$$K/R \cong \bigoplus_{\mathfrak{m}\in \mathrm{Max}(R)} (K/R_\mathfrak{m}).$$

Proof. (1) Set $S_1 = R \setminus \mathfrak{m}_1$, $S_2 = R \setminus \mathfrak{m}_2$, $S = S_1S_2$, and $S_0 = R \setminus \{0\}$. Then $K = R_{S_0}$. Trivially $R_{\mathfrak{m}_1} \otimes_R R_{\mathfrak{m}_2} = R_S \subseteq K$. If $R_S \neq K$, then $S \neq S_0$. By [22, Theorem 1.4.7], there exists a nonzero prime ideal $\mathfrak{p}$ satisfying the condition that $\mathfrak{p} \cap S = \varnothing$. Since $S_1, S_2 \subseteq S$, we have $\mathfrak{p} \subseteq \mathfrak{m}_1 \cap \mathfrak{m}_2$. Since R is assumed to be an h-local ring, there is no such $\mathfrak{p}$. So $S = S_0$, and thus $R_S = K$.

(2) From the exact sequence $0 \to R \to R_\mathfrak{m} \to R_\mathfrak{m}/R \to 0$, we have an exact sequence $B \to B_\mathfrak{m} \to (R_\mathfrak{m}/R) \otimes_R B \to 0$. Set $X = (R_\mathfrak{m}/R) \otimes_R B$ and let $\mathfrak{m}'$ be any maximal ideal of R. Note that for a multiplicative subset consisting of non-zero-divisors of a ring R, $R_S \otimes_R (R_S/R) = 0$. Thus if $\mathfrak{m}' = \mathfrak{m}$, then $X_\mathfrak{m} = 0$. If $\mathfrak{m}' \neq \mathfrak{m}$, then it follows by (1) that $R_\mathfrak{m} \otimes_R R_{\mathfrak{m}'} \otimes_R B = 0$. From the exact sequence

$$R_{\mathfrak{m}'} \otimes_R B \to R_{\mathfrak{m}'} \otimes_R R_\mathfrak{m} \otimes_R B \to R_{\mathfrak{m}'} \otimes_R (R_\mathfrak{m}/R) \otimes_R B \to 0,$$

it follows that $X_{\mathfrak{m}'} = 0$. Thus $X = 0$. Therefore θ is an epimorphism.

(3) From the commutative diagram below, we can see that the natural homomorphism $\theta : A \to A_\mathfrak{m}$ is an isomorphism:

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & {\scriptstyle\theta}\downarrow & & \| \\ 0 & \longrightarrow & A_\mathfrak{m} & \longrightarrow & B_\mathfrak{m} \end{array}$$

(4) Let $\{x_i\}_{i=1}^{\infty}$ be a generating system of B as an $R_{\mathfrak{m}}$-module. Set $A = \sum_{i=1}^{\infty} Rx_i$. Then A is an R-submodule of B. It follows by (3) that $A = A_{\mathfrak{m}} = B$. Therefore B as an R-module is also countably generated.

(5) For any $x \in B$, since B is a torsion module, there exists $u \in R \setminus \{0\}$ such that $ux = 0$. Since u is contained in only finitely many maximal ideals of R, it follows that if u is not contained in these maximal ideals, then $\frac{x}{1} = 0$. Define $\phi : B \to \bigoplus_{\mathfrak{m}\in\mathrm{Max}(R)} B_{\mathfrak{m}}$ by $\phi(x) = [\frac{x}{1}]$. Then ϕ is a well-defined R-homomorphism. For any $\mathfrak{m} \in \mathrm{Max}(R)$,

$$R_{\mathfrak{m}} \otimes_R (\bigoplus_{\mathfrak{m}\in\mathrm{Max}(R)} B_{\mathfrak{m}}) = (R_{\mathfrak{m}} \otimes_R B_{\mathfrak{m}}) \oplus (\bigoplus_{\mathfrak{m}'\neq\mathfrak{m}} (R_{\mathfrak{m}} \otimes_R R_{\mathfrak{m}'}) \otimes_{R_{\mathfrak{m}'}} B_{\mathfrak{m}'}) = B_{\mathfrak{m}}$$

Therefore ϕ is an isomorphism. □

Theorem 8.29. Let R be a domain with $\dim(R) \leqslant 1$. If R is of finite character, i.e., satisfying Definition 8.27(1), then we have:

(1) Let $u \in R$ be a nonzero element, $\mathfrak{m}$ be a maximal ideal of R, and $u \in \mathfrak{m}$. Then $K/R_{\mathfrak{m}} = (R_u/R)_{\mathfrak{m}}$.

(2) Let $u \in R$ be a nonzero nonunit element. Then $R_u/R \cong \bigoplus_{i=1}^{k}(K/R_{\mathfrak{m}_i})$, where $\mathfrak{m}_1, \dots, \mathfrak{m}_k$ are all the maximal ideals of R containing u.

(3) R is a Matlis domain.

Proof. (1) Since $(R_u)_{\mathfrak{m}} = (R_{\mathfrak{m}})_u$, we may assume that $(R, \mathfrak{m})$ is a local ring. Let $x \in R$ with $x \neq 0$. If x is a unit, then $u \in Rx$. If x is not a unit, then, since $\sqrt{Rx} = \mathfrak{m}$, there exists n such that $u^n \in Rx$, and so $x^{-1} \in Ru^{-n} \subseteq R_u$. It follows that $K = R_u$.

(2) If $u \notin \mathfrak{m}$, then $(R_u/R)_{\mathfrak{m}} = 0$. Now the assertion follows by taking $B = R_u/R$ in Theorem 8.28(5) and applying (1).

(3) Let $\mathfrak{m}$ be a maximal ideal of R. Take any $u \in \mathfrak{m} \setminus \{0\}$. By (2) and Lemma 8.20, $K/R_{\mathfrak{m}}$ is a direct summand of R_u/R, and so $\mathrm{pd}_R K/R_{\mathfrak{m}} \leqslant 1$. It follows from the second isomorphism in Theorem 8.28(5) that $\mathrm{pd}_R K/R \leqslant 1$. □

9 Almost perfect domains

Similarly to the Matlis domains discussed in the previous section, this section uses almost perfect domains as an example to show some methods of cotorsion theory for describing the structure of the ring. For a study of almost perfect domains, please refer to [1, 3, 4, 5, 10, 21]. To this end, the so-called 1-perfect domain will be described first.

9.1 Characterizations of 1-perfect domains

Definition 9.1. Let $n \geqslant 0$ be an integer. A ring R is called an n-**perfect ring** if $\mathrm{gld}_{\mathcal{C}}(R) \leqslant n$.

By Theorem 7.8, the 0-perfect ring is exactly the perfect ring. In this subsection, we will characterize 1-perfect domains.

Theorem 9.2. Let R be a 1-perfect domain. Then R is a Matlis domain.

Proof. By Theorem 6.16, $\mathrm{pd}_R K \leqslant \mathrm{gld}_{\mathcal{C}}(R) \leqslant 1$. □

Theorem 9.3. The following statements are equivalent for an integral domain R.

(1) R is a 1-perfect domain.

(2) $\mathcal{D} \subseteq \mathcal{C}$, that is, every divisible module is a cotorsion module.

(3) $\mathcal{D}_h \subseteq \mathcal{C}$, that is, every h-divisible module is a cotorsion module.

(4) Every factor module of a cotorsion module is a cotorsion module.

(5) Every pure submodule of a projective module is a projective module.

(6) The projective dimension of a flat module is at most 1.

Proof. (5) $\Leftrightarrow$ (6) This is trivial.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) The proofs of these equivalences are similar to those of Theorem 8.24.

(1) $\Rightarrow$ (5) Let A be a pure submodule of a projective module P. By Proposition 1.17, P/A is a flat module. By Theorem 6.16, $\mathrm{pd}_R(P/A) \leqslant 1$. Therefore A is a projective module.

(5) $\Rightarrow$ (1) Let F be a flat module and let $0 \to A \to P \to F \to 0$ be an exact sequence, where P is a projective module. Thus A is a pure submodule of P. By hypothesis, A is a projective module. Thus $\mathrm{pd}_R(F) \leqslant 1$. By Theorem 6.16, R is a 1-perfect domain. □

Theorem 9.4. Let $(RDTF, M)$ be a Milnor square. Then R is a 1-perfect domain if and only if both D and T are 1-perfect domains.

Proof. Assume that both D and T are 1-perfect domains. Let A be a flat R-module and let $0 \to B \to P \to A \to 0$ be an exact sequence, where P is a projective R-module. Then $T \otimes_R A$ is a flat T-module and $D \otimes_R A$ is a flat D-module. Since $\mathrm{Tor}_1^R(T, A) = 0$ and $\mathrm{Tor}_1^R(R/M, A) = 0$, it follows that both $0 \to T \otimes_R B \to T \otimes_R P \to T \otimes_R A \to 0$ and $0 \to D \otimes_R B \to D \otimes_R P \to D \otimes_R A \to 0$ are exact sequences. Since both D and T are 1-perfect domains, $T \otimes_R B$ is a flat T-module and $D \otimes_R B$ is a flat D-module. By Theorem [22, Theorem 8.2.3], B is a projective R-module. It follows that $\mathrm{pd}_R(A) \leqslant 1$, and hence R is a 1-perfect domain.

Conversely, assume that R is a 1-perfect domain. Let Q be a flat T-module. Since F is a field, $F \otimes_T Q$ is a free F-module. Therefore there exist a free D-module P and an isomorphism $h : F \otimes_D P \to F \otimes_T Q$. Construct a pullback $A = (P, Q, h)$ over h. By [22, Theorem 8.2.2], A is a flat R-module, and $D \otimes_R A \cong P$ and $T \otimes_R A \cong Q$. Since R is a 1-perfect domain, $\mathrm{pd}_R(M) \leqslant 1$. Let $0 \to A_1 \to A_0 \to M \to 0$ be an exact sequence, where A_0 and A_1 are projective R-modules. Since $\mathrm{Tor}_1^R(T, M) = 0$, it follows that $0 \to T \otimes_R A_1 \to T \otimes_R A_0 \to Q \to 0$ is an exact sequence. Thus $\mathrm{pd}_T(Q) \leqslant 1$. Therefore T is a 1-perfect domain. Similarly we can prove that D is a 1-perfect domain. □

Theorem 9.5. Let $(RDTF, M)$ be a Milnor square. Then R is a Matlis domain if and only if T is a Matlis domain.

Proof. Assume that T is a Matlis domain. Let $0 \to B \to P \to K \to 0$ be an exact sequence, where P is a projective R-module. Note that R and T have the common quotient field K, and D as an R-module is a torsion module. Thus $D \otimes_R K = 0$ is a projective D-module. Similarly to the proof of Theorem 9.4 it can be proved that B is a projective R-module. Thus $\mathrm{pd}_R(K) \leqslant 1$. Therefore R is a Matlis domain.

Conversely, assume that R is a Matlis domain. Again let $0 \to B \to P \to K \to 0$ be an exact sequence, where P is a projective R-module. Then B is a projective R-module. Since $\mathrm{Tor}_1^R(T, K) = 0$, we have $0 \to T \otimes_R B \to T \otimes_R P \to T \otimes_R K = K \to 0$ is an exact sequence. It follows that $\mathrm{pd}_T(K) \leqslant 1$, and so T is a Matlis domain. □

9.2 Characterizations of almost perfect domains

Definition 9.6. A domain R is called an **almost perfect domain** if any nontrivial factor ring of R is a perfect ring.

Theorem 9.7. Let R be an almost perfect domain. Then:

(1) $\dim(R) \leqslant 1$, that is, every nonzero prime ideal is maximal.

(2) R has finite character, that is, every nonzero $x \in R$ is contained in only finitely many maximal ideals.

(3) If I is a nonzero ideal of R, then R/I contains a simple submodule.

(4) R is a Matlis domain.

Proof. (1) Let $\mathfrak{p}$ be a nonzero prime ideal of R. Then $R/\mathfrak{p}$ is a perfect domain. Thus $R/\mathfrak{p}$ is a field. Therefore $\mathfrak{p}$ is a maximal ideal of R.

(2) Since $R/(x)$ is a perfect ring, it follows by [22, Theorem 3.10.22] that $R/(x)$ is a direct product of a finite number of local rings, and hence is a semilocal ring. Therefore x is contained in only finitely many maximal ideals.

(3) Since R/I is a perfect ring, this follows from [22, Theorem 3.10.22].

(4) This follows from Theorem 8.29. □

Theorem 9.8. The following are equivalent for a domain R:

(1) R is an almost perfect domain.

(2) $\mathrm{FPD}(R/(u)) = 0$ for any nonzero nonunit $u \in R$, in other words, $R/(u)$ is a perfect ring.

(3) Every divisible module is an n-cotorsion module for any $n \geqslant 2$.

(4) Every h-divisible module is an n-cotorsion module for any $n \geqslant 2$.

(5) Every factor module of an n-cotorsion module is an n-cotorsion module for any $n \geqslant 2$.

(6) $\mathrm{gld}_{\mathcal{C}_n}(R) \leqslant 1$ for any $n \geqslant 2$.

(7) $\mathrm{gld}_{\mathcal{C}_2}(R) \leqslant 1$.

(8) $\mathrm{FPD}(R) \leqslant 1$.

(9) $\mathcal{P}_n = \mathcal{P}_1$ for any $n \geqslant 1$.

(10) R is of finite character, and $R_{\mathfrak{m}}$ is an almost perfect domain for any maximal ideal $\mathfrak{m}$ of R.

(11) $\mathrm{gld}_{\mathcal{C}_1}(R) \leqslant 1$

(12) $\mathcal{D} \subseteq \mathcal{C}_1$, that is, every divisible module is 1-cotorsion.

(13) $\mathcal{D}_h \subseteq \mathcal{C}_1$, that is, every h-divisible module is 1-cotorsion.

(14) Every factor module of a 1-cotorsion module is 1-cotorsion.

(15) $\mathcal{P}_1 = \mathcal{F}_1$.

(16) $\mathcal{MC} = \mathcal{C}$, that is, every Matlis cotorsion module is a cotorsion module.

(17) (**Bazzoni-Salce**) $\mathcal{F} = \mathcal{SF}$, that is, every flat module is a strongly flat module.

Proof. We first prove that one step from (1) to (10) are equivalent, and one step from (11) to (17) are equivalent. Then we prove that the two steps are equivalent.

The proof of (3)⇔(4)⇔(5)⇔(6) is similar to that of Theorem 8.24, while for (6)⇔(7)⇔(8), see Theorem 7.7.

(1)⇒(2) Trivial.

(2)⇒(1) Let I be a nonzero proper ideal of R. Take $u \in I$ with $u \neq 0$. By the hypothesis, $R/(u)$ is a perfect domain. Since R/I is a factor ring of $R/(u)$, it follows from [22, Corollary 3.10.23] that R/I is a perfect ring. Therefore R is an almost perfect domain.

(8)⇒(2) Let A be any nonzero $\overline{R} := R/(u)$-module with $\mathrm{pd}_{\overline{R}}A < \infty$. By [22, Theorem 3.8.13], $\mathrm{pd}_R A = \mathrm{pd}_{\overline{R}}A + 1 \leqslant 1$. Thus $\mathrm{pd}_{\overline{R}}A = 0$. Therefore $\mathrm{FPD}(\overline{R}) = 0$.

(1)&(2)⇒(8) By Theorem 9.7, R is a Matlis domain. Let M be an R-module with $\mathrm{pd}_R M < \infty$. We may assume that $\mathrm{pd}_R M \leqslant 2$. To prove that $\mathrm{pd}_R M \leqslant 1$, we will prove that for any R-module C, $\mathrm{Ext}^2_R(M,C) = 0$.

Let $0 \to B \to F \to M \to 0$ be an exact sequence, where F is a projective R-module. Thus $\mathrm{pd}_R B \leqslant 1$. Taking any exact sequence $0 \to A \to P \to B \to 0$, where P is a projective module, we know that A is also a projective module. Note that B is a torsion-free module. Then there exists an exact sequence $0 \to A/uA \to P/uP \to B/uB \to 0$. By the hypothesis, $\mathrm{FPD}(R/uR) = 0$. Hence B/uB is a projective R/uR-module. By [22, Theorem 3.8.13], $\mathrm{pd}_R B/uB \leqslant 1$. Thus $\mathrm{Ext}^2_R(B/uB,C) = 0$. Since $0 \to B \xrightarrow{u} B \to B/uB \to 0$ is an exact sequence, there is an exact sequence

$$\mathrm{Ext}^1_R(B/uB,C) \to \mathrm{Ext}^1_R(B,C) \xrightarrow{u} \mathrm{Ext}^1_R(B,C) \to \mathrm{Ext}^2_R(B/uB,C) = 0.$$

Thus $\mathrm{Ext}^1_R(B,C) = u\mathrm{Ext}^1_R(B,C)$, that is, $\mathrm{Ext}^1_R(B,C)$ is a u-divisible module. Since u is arbitrary, $\mathrm{Ext}^1_R(B,C)$ is a divisible module. Hence $\mathrm{Ext}^2_R(M,C)$ is a divisible module. By Theorem 8.24, $\mathrm{Ext}^2_R(M,C)$ is a Lee cotorsion module. Thus $\mathrm{Ext}^1_R(K/R,\mathrm{Ext}^2_R(M,C)) = 0$.

For any free module $F = \bigoplus R$, we have natural isomorphisms

$$\mathrm{Hom}_R(F,\mathrm{Ext}^2_R(M,C)) \cong \prod \mathrm{Ext}^2_R(M,C) \cong \mathrm{Ext}^2_R(F \otimes_R M,C).$$

Since a K-vector space $K \otimes_R M$ is isomorphic to a direct sum of copies of K, we have $\mathrm{pd}_R(K \otimes_R M) = \mathrm{pd}_R K \leqslant 1$. Therefore $\mathrm{Ext}^2_R(K \otimes_R M,C) = 0$. Let $0 \to F_1 \to F_0 \to K \to 0$ be an exact sequence, where F_0, F_1 are free R-modules. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & & 0 = \mathrm{Ext}^2_R(K \otimes_R M,C) & \longrightarrow & \mathrm{Ext}^2_R(F_0 \otimes_R M,C) & \longrightarrow & \mathrm{Ext}^2_R(F_1 \otimes_R M,C) \\
 & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & \mathrm{Hom}_R(K,\mathrm{Ext}^2_R(M,C)) & \longrightarrow & \mathrm{Hom}_R(F_0,\mathrm{Ext}^2_R(M,C)) & \longrightarrow & \mathrm{Hom}_R(F_1,\mathrm{Ext}^2_R(M,C))
\end{array}$$

Thus $\mathrm{Hom}_R(K,\mathrm{Ext}^2_R(M,C)) = 0$. It follows from the exact sequence

$$0 = \mathrm{Hom}_R(K,\mathrm{Ext}^2_R(M,C)) \to \mathrm{Hom}_R(R,\mathrm{Ext}^2_R(M,C)) \to \mathrm{Ext}^1_R(K/R,\mathrm{Ext}^2_R(M,C)) = 0,$$

that $\mathrm{Ext}^2_R(M,C) = 0$.

(8)⇒(9) By the hypothesis, $\mathcal{P}_n \subseteq \mathcal{P}_1$ for any $n > 1$, and thus $\mathcal{P}_n = \mathcal{P}_1$.

(9)⇒(8) This is trivial.

(1)⇒(10) By Theorem 9.7, R has finite character.

Let $\mathfrak{m}$ be a maximal ideal of R and let I be a nonzero proper ideal of R. Then R/I is a perfect ring. By [22, Theorem 3.10.22], $R_\mathfrak{m}/I_\mathfrak{m} = (R/I)_\mathfrak{m}$ is a perfect ring. Therefore $R_\mathfrak{m}$ is an almost perfect domain.

(10)⇒(2) Let $\mathfrak{m} \in \mathrm{Max}(R)$. Since $R_\mathfrak{m}$ is an almost perfect domain, $\dim(R_\mathfrak{m}) \leqslant 1$. Thus $\dim(R) \leqslant 1$. Let $u \in R \setminus \{0\}$. By the hypothesis, there are only finitely many maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ containing u. By Theorem 8.28(5), $R/(u) \cong \bigoplus_{i=1}^{s} R_\mathfrak{m}/R_\mathfrak{m}u$. Therefore $R/(u)$ is a perfect ring.

The proof of (11)⇔(12)⇔(13)⇔(14) is similar to that of the above corresponding cases, while (16)⇔(17) is trivial.

(11)⇒(15) Let $M \in \mathcal{F}_1$. By Theorem 6.16, $\mathrm{pd}_R M \leqslant \mathrm{gld}_{\mathcal{C}_1}(R) \leqslant 1$. Therefore $M \in \mathcal{P}_1$.

(15)⇒(12) Since $K \in \mathcal{F}_1 = \mathcal{P}_1$, R is a Matlis domain. By Theorem 8.24, $\mathcal{D} = \mathcal{D}_h$. Let $D \in \mathcal{D}$. Then $\mathrm{Ext}_R^1(M, D) = 0$ for any $M \in \mathcal{F}_1 = \mathcal{P}_1$. Therefore $D \in \mathcal{C}_1$.

(15)⇒(6) Let M be a Matlis cotorsion module and let F be a flat module. Set $Q = K \otimes_R F$. Then Q is a flat module, and $A := Q/F \in \mathcal{F}_1 = \mathcal{P}_1$. Thus it follows from the exact sequence $0 = \mathrm{Ext}_R^1(Q, M) \to \mathrm{Ext}_R^1(F, M) \to \mathrm{Ext}_R^2(A, M) = 0$ that $\mathrm{Ext}_R^1(F, M) = 0$. Therefore M is a cotorsion module.

(17)⇒(12) Let $M \in \mathcal{F}_1$. Then there exists an exact sequence $0 \to F \to P \to M \to 0$, where P is a projective module. Thus F is a flat module. By the hypothesis, F is a strongly flat module. By Theorem 8.13, F is a projective module. Therefore $\mathrm{pd}_R M \leqslant 1$.

(7)⇒(11) This follows immediately from Example 6.15.

(15)⇒(2) Let u be a nonzero nonunit of R and write $\overline{R} = R/(u)$. Let A be a flat $\overline{R}$-module. By [22, Theorem 3.8.15], $\mathrm{fd}_R A \leqslant 1$. By hypothesis, $\mathrm{pd}_R A \leqslant 1$. Thus there exists an exact sequence $0 \to Q \to F \to A \to 0$, where Q and F are projective R-modules. Since $uA = 0$, we have $uP \subseteq Q$. Hence we have an $\overline{R}$-module exact sequence $0 \to B \to F/uF \to A \to 0$, where $B = Q/uF$, and exact sequence $0 \to A \to Q/uQ \to B \to 0$. Thus $0 \to A \oplus B \to Q/uQ \oplus P/uP \to A \oplus B \to 0$ is an $\overline{R}$-module exact sequence. Trivially B is a flat $\overline{R}$-module. By [6, Theorem 2.5], $A \oplus B$, and hence A is a projective $\overline{R}$-module. Hence $\overline{R}$ is a perfect ring, that is, $\mathrm{FPD}(\overline{R}) = 0$. □

Corollary 9.9. *If R is an almost perfect domain, then R is a 1-perfect domain.*

Theorem 9.10. Let R be a coherent domain. Then R is an almost perfect domain if and only if R is a Noetherian domain with $\dim(R) \leqslant 1$.

Proof. Assume that R is an almost perfect domain. For any nonzero nonunit u of R, we have $R/(u)$ is a coherent domain, and is a perfect ring. By [22, Theorem 4.1.10], R is an Artinian ring. By [22, Theorem 4.3.20], R is a Noetherian ring. By Theorem 9.7, $\dim(R) \leqslant 1$. The converse follows from [22, Theorem 4.3.21] and Theorem 9.8. □

Theorem 9.11. Let $(RDTF, M)$ be a Milnor square. Then R is an almost perfect domain if and only if D is a field and T is an almost perfect domain.

Proof. Assume that D is a field and T is an almost perfect domain. Let A be a flat submodule of a projective R-module P and write $B = P/A$. Thus $\mathrm{fd}_R B \leqslant 1$ and $0 \to A \to P \to B \to 0$ is an exact sequence. Since T is a torsion-free module, $\mathrm{Tor}_1^R(T, B) = 0$. Thus $0 \to T \otimes_R A \to T \otimes_R P \to T \otimes_R B \to 0$ is an exact sequence. Hence $T \otimes_R A$ is a flat submodule of a projective T-module $T \otimes_R P$. Since $\mathrm{FPD}(T) \leqslant 1$, it follows that $T \otimes_R A$ is a projective T-module. Since D is a field, $D \otimes_R A$ is trivially a projective D-module. By [22, Theorem 8.2.3], A is a projective R-module. Therefore $\mathrm{FPD}(R) \leqslant 1$.

Conversely, assume that $\mathrm{FPD}(R) \leqslant 1$. Since $D = R/M$ is a proper factor ring of R, it follows that D is a perfect domain, and so is a field. To prove that T is an almost perfect domain, it is sufficient to show that for $u \in T$, $u \neq 0$, T/uT is a perfect domain. Take $a \in M$, $a \neq 0$. Since $auT \subseteq aT$, the natural homomorphism $T/auT \to T/uT$ is an epimorphism, and so we prove that T/auT is a perfect domain. Thus we may assume that $u \in M$. In this case, $uT \subseteq M \subseteq R$, and the commutative diagram

$$\begin{array}{ccc} R/uT & \longrightarrow & T/uT \\ \downarrow & & \downarrow \\ D & \longrightarrow & F \end{array}$$

is a Cartesian square.

Let Q be a flat T/uT-module. Since F is a field, $F \otimes_{T/uT} Q$ is a free F-module. Thus there exist a free D-module P and an F-isomorphism $h : F \otimes_D P \to F \otimes_{T/uT} Q$. Construct a pullback $A = (P,Q,h)$. By [22, Theorem 8.2.2], A is a flat R/uT-module and $(T/uT) \otimes_{R/uT} A \cong Q$. Since R/uT is a perfect ring, A is a projective R/uT-module. So Q is a projective T/uT-module. Therefore T/uT is a perfect ring. □

9.3 Characterizations of Prüfer domains and G-Dedekind domains

The Prüfer domain is a natural extension of the Dedekind domain. Although there are many ways to characterize Prüfer domains, we now look at how to use cotorsion theories to characterize Prüfer domains.

Denote by $\mathcal{FPI}$ the class of all FP-injective modules.

Theorem 9.12. The following are equivalent for a domain R:

(1) R is a Prüfer domain.

(2) $\mathcal{C} = \mathcal{WC}$, that is, each cotorsion module is Warfield cotorsion.

(3) $\mathcal{D} = \mathcal{FPI}$, that is, every divisible module is FP-injective.

(4) $\mathcal{D}_h \subseteq \mathcal{FPI}$, that is, every h-divisible module is FP-injective.

(5) Every factor module of an FP-injective module is FP-injective.

(6) $\mathcal{LC} \subseteq \mathcal{FPI}$, that is, every Lee cotorsion module is FP-injective.

(7) $\mathcal{C}_1 \subseteq \mathcal{FPI}$, that is, every 1-cotorsion module is FP-injective.

(8) $\mathcal{C}_1 = \mathcal{I}$, that is, every 1-cotorsion module is injective.

Proof. (1)⇒(2) By [22, Theorem 3.7.13], $\mathcal{F} = \mathcal{T}$. Thus $\mathcal{WC} = \mathcal{C}$.

(2)⇒(1) By the hypothesis, $\mathcal{WC} = \mathcal{C}$. Thus $\mathcal{T} = {}^{\perp}\mathcal{WC} = {}^{\perp}\mathcal{C} = \mathcal{F}$, that is, each torsion-free module is flat. Therefore it follows from [22, Theorem 3.7.13] that R is a Prüfer domain.

(1)⇒(3) Let $I = (a_1,\ldots,a_k)$ be a finitely generated ideal of R. Without loss of generality, we assume that $a_i \neq 0$. By the hypothesis, I is invertible. Thus there exist $x_1,\ldots,x_n \in I^{-1}$ such that $a_1x_1+\cdots+a_kx_k = 1$. Let D be a divisible module and let $f : I \to D$ be a homomorphism. Then there exists $y_i \in D$ such that $a_iy_i = f(a_i)$, $i = 1,\ldots,k$. Since $a_ix_i \in R$, we have $y = \sum_{i=1}^{k} a_ix_iy_i \in D$. Then for any $b \in I$, we have $bx_i \in R$, and so

$$f(b) = f(\sum_{i=1}^{k} ba_ix_i) = \sum_{i=1}^{k} bx_if(a_i) = \sum_{i=1}^{k} ba_ix_iy_i = b\sum_{i=1}^{k} a_ix_iy_i = by.$$

Define $g : R \to D$ by $g(r) = ry, y \in R$. Then for any $b \in I$, $g(b) = bg(1) = by = f(b)$. Thus $\mathrm{Hom}_R(R,D) \to \mathrm{Hom}_R(I,D)$ is an epimorphism. Therefore it follows by Exercise 24 that D is an FP-injective module.

(3)⇒(4) ⇒ (6) ⇒(7) and (5)⇒(4) These are trivial.

(7)⇒(1) Let A be a torsion-free module. By Theorem 6.5, A^+ is a 1-cotorsion module. By the hypothesis, A^+ is an FP-injective module. Let M be a finitely presented module. Then $\mathrm{Tor}_1^R(M,A)^+ \cong \mathrm{Ext}_R^1(M,A^+) = 0$. Thus $\mathrm{Tor}_1^R(M,A) = 0$. Therefore A is a flat module. Hence it follows by [22, Theorem 3.7.13] that R is a Prüfer domain.

(3)⇒(5) This follows from the fact that every FP-injective module over a domain is divisible.

(1)⇔(8) This follows from Theorem 7.9. □

Since the G-Dedekind domain is a generalization of the Dedekind domain, one can use divisibility to characterize the G-Dedekind domain.

Theorem 9.13. Let R be a G-Dedekind domain. Then R is an almost perfect domain, and so a Matlis domain.

Proof. This follows immediately from [22, Theorem 11.4.8] and Theorem 9.8. □

Theorem 9.14. The following are equivalent for a domain R:

(1) R is a G-Dedekind domain.

(2) Every divisible module is a G-injective module.

(3) Every h-divisible module is a G-injective module.

Proof. (1) ⇒ (2) Let D be a divisible module. Since R is a Matlis domain, D is an h-divisible module. Thus there exists an exact sequence $0 \to X \to E \to D \to 0$, where E is an injective module. By [22, Corollary 11.4.5], D is a G-injective module.

(2) ⇒ (3) This is trivial.

(3) ⇒ (1) For any module X, take an exact sequence $0 \to X \to E \to D \to 0$, where E is an injective module. By hypothesis, D is a G-injective module. Again by [22, Corollary 11.4.5], R is a G-Dedekind domain. □

Example 9.15. By Theorem 8.29, every completely integrally closed valuation domain (valuation domain with Krull dimension 1) is a Matlis domain. By [22, Theorem 8.6.2(2)] and Theorem 9.5, there exists a Matlis valuation domain with any Krull dimension.

Example 9.16. (1) An almost perfect domain is not necessarily a Dedekind domain. For example, set $R = \mathbb{Q} + X\mathbb{R}[[X]]$. Note that $\mathbb{R}[[X]]$ is a DVR. By Theorem 9.11, R is an almost perfect domain. By [22, Theorem 8.5.17], R is not a coherent domain. Naturally R is not a Dedekind domain.

(2) A 1-perfect domain is not necessarily an almost perfect domain. For example, set $R = \mathbb{Z} + X\mathbb{R}[[X]]$. By Theorem 9.4, R is a 1-perfect domain. By Theorem 9.11, R is not an almost perfect domain.

(3) A Matlis domain is not necessarily a 1-perfect domain. For example, let D be a valuation domain with $\mathrm{gl.dim}(D) = 3$. Let F be the quotient field of D and set $R = D + XF[X]$. By Exercise 26, D is not a 1-perfect domain. By Theorem 9.5, R is a Matlis domain. By Theorem 9.4, R is not a 1-perfect domain.

10 Exercise

1. For any module M, the canonical homomorphism

$$\rho : M \to M^{++} = \mathrm{Hom}(\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$

is a pure monomorphism.

2. Let $\xi : 0 \to A \to B \to C \to 0$ be an exact sequence, where C is a finitely presented module. Then ξ is pure if and only if ξ is split.

3. Let M be an R-module and let I be an index set. Then $M^{(I)}$ is a pure submodule of M^I.

4. Let R be a PID and let A be a submodule of a module B. Then A is a pure submodule of B if and only if $A \cap rB = rA$ for any nonzero $r \in R$.

5. Let A and B be submodules of N such that $A \subseteq B$.

(1) If B is a pure submodule of N, then B/A is a pure submodule of N/A.

(2) If A is a pure submodule of N and B/A is a pure submodule of N/A, then B is a pure submodule of N.

6. If E is a pure injective module, then $\mathrm{Hom}_R(M,E)$ is a pure injective module for any R-module M.

7. Let $\mathcal{L}$ be a class of modules. Then:
(1) $\mathcal{L} \subseteq {}^{\perp}(\mathcal{L}^{\perp}) \cap ({}^{\perp}\mathcal{L})^{\perp}$, $({}^{\perp}(\mathcal{L}^{\perp}))^{\perp} = \mathcal{L}^{\perp}$, and ${}^{\perp}(({}^{\perp}\mathcal{L})^{\perp}) = {}^{\perp}\mathcal{L}$.
(2) $\mathcal{L} \subseteq (\mathcal{L}^{\top})^{\top}$ and $((\mathcal{L}^{\top})^{\top})^{\top} = \mathcal{L}^{\top}$.

8. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence and let E be a pure injective module. Then

$$0 \longrightarrow \mathrm{Hom}_R(C,E) \longrightarrow \mathrm{Hom}_R(B,E) \longrightarrow \mathrm{Hom}_R(A,E) \longrightarrow 0$$

is a split exact sequence.

9. Let R be a domain with quotient field K, let A,B be K-modules, and let $f : A \to B$ be an R-homomorphism. Then f is also a K-homomorphism, and so $\mathrm{Hom}_R(A,B) = \mathrm{Hom}_K(A,B)$.

10. If L is an FP-injective and pure injective module, then L is injective.

11. (1) Let $\{M_i, \varphi_{ij}\}$ be a direct system over a directed set Γ. Then $0 \to N \to \bigoplus_i M_i \to \varinjlim M_i \to 0$ is pure exact.

(2) Let $\mathcal{L}$ be a class of modules which is closed under direct sums and pure quotient modules (i.e., if $0 \to B \to C \to 0$ is a pure exact and $B \in \mathcal{L}$, then $C \in \mathcal{L}$). Then $\mathcal{L}$ is closed under direct limits.

12. Let $(\mathcal{A},\mathcal{B})$ be a Tor-torsion theory and let $0 \to L \to Q \to N \to 0$ be a pure exact sequence. If $M \in \mathcal{A}$, then $L,N \in \mathcal{A}$.

13. Let R be an integral domain. Then:

(1) Any direct sum of h-divisible modules is again h-divisible.

(2) If R is not a field, then R is an h-reduced module.

14. An R-module A is called a **strongly copure flat module** if $\mathrm{Tor}_i^R(A,E) = 0$ for any injective R-module E and any $i \geq 1$. Then every module has a strongly copure flat cover.

15. An R-module A is called a **strongly copure injective module** if $\mathrm{Ext}_R^i(A,E) = 0$ for any injective R-module E and any $i \geq 1$. Then:

(1) Every module has a strongly copure injective special preenvelope.

(2) If R is an n-Gorenstein ring, then every module has a G-injective special preenvelope.

16. Let R be a domain and let M be an R-module. If D_1, D_2 are divisible (resp., h-divisible) submodules of M, then $D_1 + D_2$ is also a divisible (resp., an h-divisible) submodule of M.

17. Let R be a domain and let D be a pure injective R-module. Then the following are equivalent.
(1) D is a Lee cotorsion module.
(2) D is h-divisible.
(3) D is divisible.

18. Let L be a field and let $R = L[X_1,\dots,X_k,\dots]$ be a polynomial ring in countably infinite indeterminates, and $M = R/(X_1,\dots,X_n,\dots)$. Then:
(1) $\mathrm{fd}_R M = \infty$.
(2) Let $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ be a flat resolution of M and let K_n be its n-th weak syzygy. Then K_n is an $(n+1)$-torsion-free module, but not an $(n+2)$-torsion-free module.
(3) The above K_n^+ is an $(n+1)$-cotorsion module, but not an $(n+2)$-cotorsion module.

19. The following are equivalent for a module L.

(1) L is FP-injective.

(2) If L is a submodule of an R-module B, then L is a pure submodule of B (thus in many literature, an FP-injective module is also called an **absolutely pure module**).

(3) L is a pure submodule of $E(L)$.

20. A ring R is called an **IF ring** if every injective R-module is flat. Then:

(1) Every QF ring is IF.

(2) Every IF ring R is coherent and $\mathrm{FFD}(R) = 0$.

(3) If R is a perfect ring which is also an IF ring, then R is a QF ring.

21. Let n be nonnegative integer. Then $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is a hereditary complete cotorsion theory.

22. Let R be a domain. Then:

(1) Any direct sum of h-divisible modules is again h-divisible, and so any direct limit of h-divisible modules over a directed set is also h-divisible.

(2) Let $\{D_i\}$ be a chain of a family of h-divisible submodules of a module M. Then $\bigcup D_i$ is also an h-divisible submodule of M.

23. Let M be an R-module and set $T := \mathrm{tor}_{\mathrm{GV}}(M)$. If $\varphi : F \to M/T$ is a weak w-projective cover of M/T. Consider a pullback diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \xrightarrow{\lambda} & F_1 & \xrightarrow{\beta} & F & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & T & \longrightarrow & M & \xrightarrow{\pi} & M/T & \longrightarrow & 0. \end{array}$$

Then $\alpha : F_1 \to M$ is a weak w-projective cover of M.

24. Let R be a coherent domain. Then an R-module E is an FP-injective module if and only if $\mathrm{Ext}_R^1(R/I, E) = 0$ for any finitely generated ideal I of R, equivalently any homomorphism $f : I \to E$ can be extended to R for any finitely generated ideal I of R

25. Let R be a domain and let M be an R-module. Then:

(1) M is a torsion h-divisible module if and only if $M \cong K/R \otimes_R \mathrm{Hom}_R(K/R, M)$.

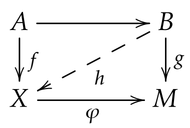
(2) M is a Warfield cotorsion module if and only if M is a Matlis cotorsion module and $\mathrm{id}_R M \leqslant 1$.

(3) M is a Lee cotorsion module if and only if M is an h-divisible Matlis cotorsion module.

26. Let R be a Prüfer domain. Then R is a 1-perfect domain if and only if $\mathrm{gl.dim}(R) \leqslant 2$.

27. Let $\phi : R \to T$ be a ring homomorphism and let L be a cotorsion T-module. Then L is also a cotorsion R-module.

28. Let $\mathcal{L}$ be a class of modules which is closed under extensions. Consider the following commutative diagram:



where A is a submodule of B and φ is an $\mathcal{L}$-cover of M. If $C := B/A \in \mathcal{L}$, then there exists a homomorphism $h : B \to X$ such that $h\varphi = g$.

29. The following are equivalent for a domain R:

(1) Every h-divisible module is a Warfield cotorsion module.

(2) Every factor module of a Warfield cotorsion module is a Warfield cotorsion module.

(3) The projective dimension of any torsion-free module is at most 1.

(4) R is a Matlis domain with $\text{gl.dim}(R) \leqslant 2$.

(5) Every divisible module is a Warfield cotorsion module.

30. Let R be a domain and set $H := \text{Hom}_R(K/R, K/R)$. Then:

(1) H is a commutative ring and H as an R-module is a torsion-free module.

(2) There exists an exact sequence $0 \to R \to H \to \text{Ext}_R^1(K, R) \to 0$.

(3) If each proper submodule of K/R is h-reduced, then H is a domain.

(4) If R is a valuation domain, then H is also a valuation domain.

31. Let M be an R-module and A be a submodule of M. Then there exists a continuous ascending chain of pure submodules of M:

$$A = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\alpha \subset M_{\alpha+1} \subset \cdots \subset M_\tau = M$$

such that each $M_{\alpha+1}/M_\alpha$ is countably generated.

32. Let $\mathcal{A}$ be a class of modules which is closed under w-isomorphisms, M be an R-module, and $\varphi : P \to M$ be an $\mathcal{A}$-cover. Then:

(1) If M is a GV-torsion-free module, then so is P.

(2) If φ is a special $\mathcal{A}$-cove and M is a w-module, then P is a w-module.

33. Let F be an R-module. Then F is a w-flat module if and only if $\text{Ext}_R^1(F, N) = 0$ for any $N \in \mathcal{F}_w^\perp \cap \mathcal{W}$.

34. Let $\mathcal{T}_{\text{GV}}$ denote the class of GV-torsion modules and let $M \in {\mathcal{T}_{\text{GV}}}^\perp$. Then:

(1) If M is a GV-torsion module and N is a submodule with $N \in {\mathcal{T}_{\text{GV}}}^\perp$, then N is a direct summand of M.

(2) $\text{tor}_{\text{GV}}(M) \in {\mathcal{T}_{\text{GV}}}^\perp$. Especially $\text{tor}_{\text{GV}}(E) \in {\mathcal{T}_{\text{GV}}}^\perp$ for any injective module E.

(3) Set $T := \text{tor}_{\text{GV}}(E(M))$. Then $T \subseteq M$, and thus $\text{tor}_{\text{GV}}(M) = \text{tor}_{\text{GV}}(E(M))$.

(4) If M is a GV-torsion module, then there exists an injective module E such that $M \cong \text{tor}_{\text{GV}}(E)$.

35. Let M be an R-module and set $E := E(M)$. Then the following are equivalent.

(1) $M \in {\mathcal{T}_{\text{GV}}}^\perp$

(2) $\text{tor}_{\text{GV}}(E) \subseteq M$ and $\text{Ext}_R^1(R/J, M) = 0$ for any $J \in \text{GV}(R)$.

(3) If $Jx \subseteq M$, where $J \in \text{GV}(R)$ and $x \in E$, then $x \in M$.

(4) E/M is a GV-torsion-free module.

(5) If J is an ideal of R with $J_w = R$, then $\text{Ext}_R^1(R/J, M) = 0$.

36. Let M be an R-module. Then $M \in {}^\perp\mathcal{W}$ if and only if $M \cong P \oplus T$, where P is a projective module and T is a GV-torsion module.

37. Let $D := \mathbb{R}[[X, Y]]$ and set $R := D[Z]/(Z^2 - XY)$. Then:

(1) R is a 2-Gorenstein ring.

(2) R is an integrally closed domain, and thus a Krull domain.

(3) $\text{gl.dim}(R) \neq 2$, and thus $\text{gl.dim}(R) = \infty$.

38. Let M be a GV-torsion-free R-module. Then M has a weak w-projective cover if and only if M_w has a weak w-projective cover. If B is a weak w-projective cover, then B is a GV-torsion-free R-module.

39. Let M be an R-module with $\text{w.}w\text{-pd}_R(M) = n > 0$. Then there exists $P \in \text{w}\mathcal{P}_w \cap \mathcal{P}_w^{\dagger_\infty}$ such that $\text{Ext}_R^n(M, P) \neq 0$.

40. Let $\xi : 0 \to A \to B \to C \to 0$ be a w-exact sequence and let E be an injective w-module. Then $\mathrm{Hom}_R(\xi, E)$ is an exct sequence.

41. Let $\xi : 0 \to A \to B \to C \to 0$ be a w-exact sequence. Then ξ is called a w-**pure exact sequence** if $M \otimes_R \xi$ is a w-exact sequence for any module M. Let $\xi : 0 \to A \to B \to C \to 0$ be a w-pure exact sequence, M an R-module, E be an injective w-module. Then $\mathrm{Hom}_R(\xi, \mathrm{Hom}_R(M, E))$ is an exact sequence.

Let $\mathcal{A}$ be a class of modules. Define

$$w\text{-}\mathcal{A}^{\top} = \{D \in \mathfrak{M} \mid \mathrm{Tor}_1^R(A, D) \text{ is a GV-torsion module for any } A \in \mathcal{A}\}.$$

Let $\mathcal{A}, \mathcal{B}$ be classes of modules. Then $(\mathcal{A}, \mathcal{B})$ is called a w-**Tor-torsion theory** if $w\text{-}\mathcal{A}^{\top} = \mathcal{B}$ and $w\text{-}\mathcal{B}^{\top} = \mathcal{A}$.

42. Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a hereditary w-Tor-torsion theory. Then

(1) Set $\mathcal{L} = \{\mathrm{Hom}_R(B, E) \mid B \in \mathcal{B}$ and E is an injective w-module$\}$. Then $\mathcal{L} \subseteq \mathcal{A}^{\perp}$.

(2) ${}^{\perp}\mathcal{L} = \mathcal{A}$, and so ${}^{\perp}(\mathcal{A}^{\perp}) = \mathcal{A}$.

(3) $(\mathcal{A}, \mathcal{A}^{\perp})$ is a perfect cotorsion theory.

(4) $\mathcal{A}$ is closed under direct limits.

43. Let $\mathfrak{G} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion theory and $\mathcal{L}$ be a class of modules cogenerated by a w-module, where $N \in \mathcal{L}$ satisfies: If ξ is a w-pure exact sequence, then $\mathrm{Hom}_R(\xi, N)$ is an exact sequence. Then

(1) $\mathcal{A}$ is closed under direct limits.

(2) $\mathfrak{G}$ is a perfect cotorsion theory.

Here we correct the errors in the authors' book [22]. The authors would like to thank all readers for pointing out the error.

- (Jesse Elliott) There are some errors in [22, Section 5.7] on valuation methods in rings with zero-divisors. Unfortunately [22, Theorem 5.7.4(2)] is wrong. $R_{[\mathfrak{p}]}$ doesn't have to be a pseudo-local ring. In fact, $[\mathfrak{p}]_{[\mathfrak{p}]}$ doesn't have to be a maximal ideal. See [14, Example 7, p. 182]. The problem is that, if R isn't Marot, then R can have several maximal ideals containing the same regular elements, so just because $[\mathfrak{p}]_{[\mathfrak{p}]}$ contains all of the regular nonunits of $R_{[\mathfrak{p}]}$ doesn't mean that $[\mathfrak{p}]_{[\mathfrak{p}]}$ is the unique regular maximal ideal of $R_{[\mathfrak{p}]}$ (unless R is Marot). Consequently, [22, Proposition 5.7.13(1)] is also incorrect. This means that the proof of [22, Theorem 5.7.21] should also be fixed, and there may also be other proofs that need to be fixed.

  Also, the relation $>$ defined on [22, p. 315, line 1] isn't transitive in general and is therefore not an equivalence relation. If we define $v(x) \leq v(y)$ iff $xz \in \mathfrak{p}$ implies $yz \in \mathfrak{p}$ for all $z \in K$, then $\leq$ is a partial ordering, but not necessarily a total ordering.
- p. 318: line -11, replace "subring" with "overring".
- p. 346: line 7, replace "Coker(f)" with "Coker(g)".
- p. 372: line 14, replace "$F \to M \to C \to 0$" with "$F \to M \to 0$".
- p. 430: line 3, add "Let R be an H-domain." after "(2)".
- p. 444: line 10, replace "over R" with "over T".
- p. 447: line 8, replace "of$T$" with "of T".
- p. 470: line -1, replace "$F = P/R_P$" with "$F = R_P/PR_P$".
- p. 620: line 9, replace "$\mathrm{Im}(P_0 \to P_1)$" with "$\mathrm{Im}(P_0 \to P_{-1})$".
- p. 632: line -14, replace "this L_i" with "this L_i'".
- pp. 685–691: During the editorial process of the publisher, the first letter of the person's name in the titles of articles in the **References** was erroneously changed to lowercase.

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