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Symmetry properties of finite sums involving generalized Fibonacci numbers

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Abstract. We extend a result of I. J. Good and prove more symmetry properties of sums involving generalized Fibonacci numbers.
Key Words: generalized Fibonacci number, Fibonacci number, Lucas number, symmetry property, Horadam sequence, telescoping summation identity.
2010 MSC: Primary 11B39; Secondary 11B37.

1 Introduction

The generalized Fibonacci numbers G_i, $i \ge 0$, with which we are mainly concerned in this paper, are defined through the second order recurrence relation $G_{i+1} = G_i + G_{i-1}$, where the seeds G_0 and G_1 need to be specified. As particular cases, when $G_0 = 0$ and $G_1 = 1$, we have the Fibonacci numbers, denoted F_i, while when $G_0 = 2$ and $G_1 = 1$, we have the Lucas numbers, L_i.

I. J. Good [4] proved the symmetry property:

$$F_q \sum_{k=1}^{n} \frac{(-1)^k}{G_k G_{k+q}} = F_n \sum_{k=1}^{q} \frac{(-1)^k}{G_k G_{k+n}}, \tag{1}$$

where q and n are non-negative integers, and all the numbers $G_1, G_2, \ldots, G_{n+q}$ are non-zero.

The identity (1) is a particular case (corresponding to setting $p = 1$) of the following result, to be proved in this present paper:

$$F_{pq} \sum_{k=1}^{n} \frac{(-1)^{pk}}{G_{pk} G_{pk+pq}} = F_{pn} \sum_{k=1}^{q} \frac{(-1)^{pk}}{G_{pk} G_{pk+pn}}, \tag{2}$$

where q, p and n are non-negative integers, and all the numbers $G_p, G_{2p}, \ldots, G_{pn+pq}$ are non-zero.

In the limit as n approaches infinity, and specializing to Fibonacci numbers, the identity (2) gives

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^{pk}}{F_{pk} F_{pk+pq}} &= \frac{1}{F_{pq}} \sum_{k=1}^{q} \left\{ \frac{(-1)^{pk}}{F_{pk}} \lim_{n\to\infty} \left(\frac{F_{pn}}{F_{pk+pn}} \right) \right\} \\ &= \frac{1}{F_{pq}} \sum_{k=1}^{q} \frac{(-1)^{pk}}{\phi^{pk} F_{pk}},\end{aligned} \tag{3}$$

where $\phi = (1 + \sqrt{5})/2$ is the golden ratio.

The identity (3) generalizes the result of Bruckman and Good [3, Identity (19)], which corresponds to setting $q = 1$ in (3)).

In sections 3.1 – 3.3 we will prove identity (2) and discover more symmetry properties of sums involving generalized Fibonacci numbers.

Finally, in section 3.4 we shall extend the discussion to Horadam sequences W_i and U_i by proving

$$U_{pq}\sum_{k=1}^{n}\frac{Q^{pk}}{W_{pk}W_{pk+pq}} = U_{pn}\sum_{k=1}^{q}\frac{Q^{pk}}{W_{pk}W_{pk+pn}} \tag{4}$$

and

$$U_{2pq}\sum_{k=1}^{2n}\frac{(\pm Q^{p})^{k}}{W_{pk}W_{pk+2pq}} = U_{2pn}\sum_{k=1}^{2q}\frac{(\pm Q^{p})^{k}}{W_{pk}W_{pk+2pn}}, \tag{5}$$

for integers p, q, Q and n, thereby extending André-Jeannin's result (Theorem 1 of [2]) and further generalizing the identity (2).

2 Required identities

2.1 Telescoping summation identities

The following telescoping summation identities are special cases of the more general identities proved in [1].

Lemma 2.1. *If $f(k)$ is a real sequence and u, v and w are positive integers, then*

$$\sum_{k=1}^{w}[f(uk+uv)-f(uk)] = \sum_{k=1}^{v}[f(uk+uw)-f(uk)].$$

Lemma 2.2. *If $f(k)$ is a real sequence and u, v and w are positive integers such that v is even and w is even, then*

$$\sum_{k=1}^{w}(\pm 1)^{k-1}(f(uk+uv)-f(uk)) = \sum_{k=1}^{v}(\pm 1)^{k-1}(f(uk+uw)-f(uk)).$$

Lemma 2.3. *If $f(k)$ is a real sequence and u, v and w are positive integers such that vw is odd, then*

$$\sum_{k=1}^{w}(-1)^{k-1}(f(uk+uv)+f(uk)) = \sum_{k=1}^{v}(-1)^{k-1}(f(uk+uw)+f(uk)).$$

2.2 Product of a Fibonacci number and a generalized Fibonacci number

Lemma 2.4 (Howard [6, Corollary 3.5])**.** *For integers a, b, c,*

$$F_aG_{2b+a+c} = \begin{cases} F_{a+b}G_{a+b+c} - F_bG_{b+c} & \text{if } a \text{ is even,}\\ F_{a+b}G_{a+b+c} + F_bG_{b+c} & \text{if } a \text{ is odd.}\end{cases}$$

2.3 Product of a Lucas number and a generalized Fibonacci number

Lemma 2.5 (Vajda [7, Formula 10a])**.** *For integers a, b,*

$$L_aG_b = \begin{cases} G_{b+a} + G_{b-a} & \text{if } a \text{ is even,}\\ G_{b+a} - G_{b-a} & \text{if } a \text{ is odd.}\end{cases}$$

2.4 Difference of products of a Fibonacci number and a generalized Fibonacci number

Lemma 2.6 (Vajda [7, Formula 21]). *For integers a, b,*

$$F_b G_a - F_a G_b = (-1)^a G_0 F_{b-a}.$$

3 Main Results: Symmetry properties

3.1 Sums of products of reciprocals

Theorem 3.1. *If n and q are non-negative integers and p is a non-zero integer, then,*

$$F_{pq} \sum_{k=1}^{n} \frac{(-1)^{pk}}{G_{pk} G_{pk+pq}} = F_{pn} \sum_{k=1}^{q} \frac{(-1)^{pk}}{G_{pk} G_{pk+pn}}.$$

Proof. Dividing through the identity in Lemma 2.6 by $G_a G_b$ and setting $b = pk + pq$ and $a = pk$, we have:

$$\frac{F_{pk+pq}}{G_{pk+pq}} - \frac{F_{pk}}{G_{pk}} = (-1)^{pk} \frac{G_0 F_{pq}}{G_{pk} G_{pk+pq}}. \tag{6}$$

Similarly,

$$\frac{F_{pk+pn}}{G_{pk+pn}} - \frac{F_{pk}}{G_{pk}} = (-1)^{pk} \frac{G_0 F_{pn}}{G_{pk} G_{pk+pn}}. \tag{7}$$

We now use the sequence $f(k) = F_k/G_k$ in Lemma 2.1 with $u = p$, $v = q$ and $w = n$, while taking into consideration identities (6) and (7). □

Theorem 3.2. *If n and q are non-negative even integers and p is a non-zero integer, then*

$$F_{pq} \sum_{k=1}^{n} \frac{(\pm 1)^{k(p-1)}}{G_{pk} G_{pk+pq}} = F_{pn} \sum_{k=1}^{q} \frac{(\pm 1)^{k(p-1)}}{G_{pk} G_{pk+pn}}.$$

Proof. We use the sequence $f(k) = F_k/G_k$ in Lemma 2.2 with $u = p$, $v = q$ and $w = n$. □

3.2 First-power sums

Theorem 3.3. *If p, q, n and t are integers such that pqn is odd, then*

$$L_{pq} \sum_{k=1}^{2n} (\pm 1)^{k-1} G_{pk+pq+t} = L_{pn} \sum_{k=1}^{2q} (\pm 1)^{k-1} G_{pk+pn+t}, \tag{8}$$

$$L_{pq} \sum_{k=1}^{n} G_{2pk+pq+t} = L_{pn} \sum_{k=1}^{q} G_{2pk+pn+t}. \tag{9}$$

Proof. Consider the generalized Fibonacci sequence $f(k) = G_{k+t}$. If we choose $u = p$, $v = 2q$ and $w = 2n$, then Lemma 2.2 gives

$$\sum_{k=1}^{2n} (\pm 1)^{k-1} \left(G_{pk+2pq+t} - G_{pk+t} \right) = \sum_{k=1}^{2q} (\pm 1)^{k-1} \left(G_{pk+2pn+t} - G_{pk+t} \right). \tag{10}$$

But from the second identity of Lemma 2.5 we have

$$G_{pk+2pq+t} - G_{pk+t} = L_{pq} G_{pk+pq+t}, \quad pq \text{ odd}, \tag{11}$$

and

$$G_{pk+2pn+t} - G_{pk+t} = L_{pn} G_{pk+pn+t}, \quad pn \text{ odd}. \tag{12}$$

Using (11) and (12) in (10), identity (8) is proved.

The proof of identity (9) is similar, we use the sequence $f(k) = G_{2k+t}$ in Lemma 2.1 with $u = 2p$, $v = q$ and $w = n$.

□

Theorem 3.4. *If p, q, n and t are integers such that pqn is odd or q and n are even, then*

$$F_{pq} \sum_{k=1}^{n} (-1)^{k-1} G_{2pk+pq+t} = F_{pn} \sum_{k=1}^{q} (-1)^{k-1} G_{2pk+pn+t}.$$

Proof. Consider the sequence $f(k) = F_k G_{k+t}$. If we choose $u = p$, $v = q$ and $w = n$, then Lemma 2.3 gives

$$\begin{aligned} &\sum_{k=1}^{n} (-1)^{k-1} \left(F_{pk+pq} G_{pk+pq+t} + F_{pk} G_{pk+t}\right) \\ &\quad = \sum_{k=1}^{q} (-1)^{k-1} \left(F_{pk+pn} G_{pk+pn+t} + F_{pk} G_{pk+t}\right). \end{aligned} \tag{13}$$

From the second identity of Lemma 2.4 we have

$$F_{pk+pq} G_{pk+pq+t} + F_{pk} G_{pk+t} = F_{pq} G_{2pk+pq+t}, \quad pq \text{ odd}, \tag{14}$$

and

$$F_{pk+pn} G_{pk+pn+t} + F_{pk} G_{pk+t} = F_{pn} G_{2pk+pn+t}, \quad pn \text{ odd}. \tag{15}$$

The theorem then follows from using (14) and (15) in (13). If q and n are even then we use $f(k) = F_k G_{k+t}$ with $u = p$, $v = q$ and $w = n$ in Lemma 2.2 together with the first identity of Lemma 2.4. □

Theorem 3.5. *If p, q, n and t are integers such that p is even or q and n are even, then*

$$F_{pq} \sum_{k=1}^{n} G_{2pk+pq+t} = F_{pn} \sum_{k=1}^{q} G_{2pk+pn+t}.$$

Proof. Consider the sequence $f(k) = F_k G_{k+t}$. Lemma 2.1 with $u = p$, $v = q$ and $w = n$ gives

$$\begin{aligned} &\sum_{k=1}^{n} \left(F_{pk+pq} G_{pk+pq+t} - F_{pk} G_{pk+t}\right) \\ &\quad = \sum_{k=1}^{q} \left(F_{pk+pn} G_{pk+pn+t} - F_{pk} G_{pk+t}\right). \end{aligned} \tag{16}$$

From the first identity of Lemma 2.4 we have

$$F_{pk+pq} G_{pk+pq+t} - F_{pk} G_{pk+t} = F_{pq} G_{2pk+pq+t}, \quad pq \text{ even}, \tag{17}$$

and

$$F_{pk+pn} G_{pk+pn+t} - F_{pk} G_{pk+t} = F_{pn} G_{2pk+pn+t}, \quad pn \text{ even}. \tag{18}$$

Using (17) and (18) in (16), Theorem 3.5 is proved. □

Theorem 3.6. *If p, q, n and t are integers such that p is even, then*

$$F_{pq}\sum_{k=1}^{2n}(\pm 1)^{k-1}G_{pk+pq+t} = F_{pn}\sum_{k=1}^{2q}(\pm 1)^{k-1}G_{pk+pn+t}.$$

Proof. Consider the sequence $f(k) = F_k G_{k+t}$. Lemma 2.2 with $u = p$, $v = 2q$ and $w = 2n$ gives

$$\begin{aligned}&\sum_{k=1}^{2n}(\pm 1)^{k-1}\left(F_{pk+2pq}G_{pk+2pq+t} - F_{pk}G_{pk+t}\right)\\ &\quad = \sum_{k=1}^{2q}(\pm 1)^{k-1}\left(F_{pk+2pn}G_{pk+2pn+t} - F_{pk}G_{pk+t}\right).\end{aligned} \tag{19}$$

From identities (17) and (18) we have

$$F_{pk+2pq}G_{pk+2pq+t} - F_{pk}G_{pk+t} = F_{2pq}G_{2pk+2pq+t}, \tag{20}$$

and

$$F_{pk+2pn}G_{pk+2pn+t} - F_{pk}G_{pk+t} = F_{2pn}G_{2pk+2pn+t}. \tag{21}$$

Using (20) and (21) in (19), Theorem 3.6 is proved. □

Theorem 3.7. *If p, q, n and t are integers such that p is even and nq is odd, then*

$$L_{pq}\sum_{k=1}^{n}(-1)^{k-1}G_{2pk+pq+t} = L_{pn}\sum_{k=1}^{q}(-1)^{k-1}G_{2pk+pn+t},$$

Proof. Consider the sequence $f(k) = G_{2k+t}$. If we choose $u = 2p$, $v = q$ and $w = n$, then Lemma 2.3 gives

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^{k-1}\left(G_{2pk+2pq+t} + G_{2pk+t}\right)\\ &\quad = \sum_{k=1}^{q}(-1)^{k-1}\left(G_{2pk+2pn+t} + G_{2pk+t}\right), \quad nq \text{ odd}.\end{aligned} \tag{22}$$

From the first identity in Lemma 2.5, we have

$$G_{2pk+2pq+t} + G_{2pk+t} = L_{pq}G_{2pk+pq+t}, \quad pq \text{ even}, \tag{23}$$

and

$$G_{2pk+2pn+t} + G_{2pk+t} = L_{pn}G_{2pk+pn+t}, \quad pn \text{ even}. \tag{24}$$

Using (23) and (24) in (22), Theorem 3.7 is proved. □

3.3 More sums involving products of reciprocals

Theorem 3.8. *If p, q, n and t are positive integers such that pnq is odd, then*

$$L_{pq}\sum_{k=1}^{2n}\frac{(\pm 1)^{k-1}G_{pk+pq+t}}{G_{pk+t}G_{pk+2pq+t}} = L_{pn}\sum_{k=1}^{2q}\frac{(\pm 1)^{k-1}G_{pk+pn+t}}{G_{pk+t}G_{pk+2pn+t}}, \tag{25}$$

$$L_{pq}\sum_{k=1}^{n}\frac{G_{2pk+pq+t}}{G_{2pk+t}G_{2pk+2pq+t}} = L_{pn}\sum_{k=1}^{q}\frac{G_{2pk+pn+t}}{G_{2pk+t}G_{2pk+2pn+t}}. \tag{26}$$

Proof. Use of $f(k) = 1/G_{k+t}$ in Lemma 2.2 with $u = p$, $v = 2q$ and $w = 2n$, noting the identites (11) and (12) proves identity (25). To prove identity (26), we use $f(k) = 1/G_{2k+t}$ in Lemma 2.1 with $u = p$, $v = q$ and $w = n$, together with the second identity in Lemma 2.5. □

Theorem 3.9. *If p, q, n and t are positive integers such that p is even and nq is odd, then*

$$L_{pq} \sum_{k=1}^{n} \frac{(-1)^{k-1} G_{2pk+pq+t}}{G_{2pk+t} G_{2pk+2pq+t}} = L_{pn} \sum_{k=1}^{q} \frac{(-1)^{k-1} G_{2pk+pn+t}}{G_{2pk+t} G_{2pk+2pn+t}}.$$

Proof. Use $f(k) = 1/G_{2k+t}$ in Lemma 2.3 with $u = p$, $v = q$ and $w = n$, employing the identities (23) and (24). □

Theorem 3.10. *If p, q, n and t are positive integers such that p is even or n and q are even, then*

$$F_{pq} \sum_{k=1}^{n} \frac{G_{2pk+pq+t}}{F_{pk} G_{pk+t} F_{pk+pq} G_{pk+pq+t}} = F_{pn} \sum_{k=1}^{q} \frac{G_{2pk+pn+t}}{F_{pk} G_{pk+t} F_{pk+pn} G_{pk+pn+t}}.$$

Proof. Use $f(k) = 1/(F_k G_{k+t})$ in Lemma 2.1 with $u = p$, $v = q$ and $w = n$, while taking cognisance of the following identities which follow from identities (17) and (18):

$$\frac{1}{F_{pk} G_{pk+t}} - \frac{1}{F_{pk+pq} G_{pk+pq+t}} = \frac{F_{pq} G_{2pk+pq+t}}{F_{pk} G_{pk+t} F_{pk+pq} G_{pk+pq+t}}, \quad pq \text{ even}, \tag{27}$$

and

$$\frac{1}{F_{pk} G_{pk+t}} - \frac{1}{F_{pk+pn} G_{pk+pn+t}} = \frac{F_{pn} G_{2pk+pn+t}}{F_{pk} G_{pk+t} F_{pk+pn} G_{pk+pn+t}}, \quad pn \text{ even}. \tag{28}$$

□

Theorem 3.11. *If p, q, n and t are positive integers such that p is odd or n and q are even, then*

$$F_{pq} \sum_{k=1}^{n} \frac{(-1)^{k-1} G_{2pk+pq+t}}{F_{pk} G_{pk+t} F_{pk+pq} G_{pk+pq+t}} = F_{pn} \sum_{k=1}^{q} \frac{(-1)^{k-1} G_{2pk+pn+t}}{F_{pk} G_{pk+t} F_{pk+pn} G_{pk+pn+t}}.$$

3.4 Symmetry properties of finite sums involving the terms of the Horadam sequence

Some of the above results can be extended to the Horadam sequence [5], $\{W_i\} = \{W_i(a, b; P, Q)\}$ defined by

$$W_0 = a, W_1 = b, W_i = PW_{i-1} - QW_{i-2}, (i > 2), \tag{29}$$

where a, b, P, and Q are integers, with $PQ \neq 0$ and $\Delta = P^2 - 4Q > 0$. We define the sequence $\{U_i\}$ (Lucas sequence of the first kind) by $U_i = W_i(0, 1; P, Q)$ and note also that our sequence $\{G_i\}$ is given by $G_i = W_i(G_0, G_1; 1, -1)$. It is readily established that [5, 2]:

$$W_i = \frac{A\alpha^i - B\beta^i}{\alpha - \beta}, \tag{30}$$

where $\alpha = (P + \sqrt{\Delta})/2$, $\beta = (P - \sqrt{\Delta})/2$, $A = b - \beta a$ and $B = b - \alpha a$.

Theorem 3.12. *If n and q are non-negative integers and p is a non-zero integer, then*

$$U_{pq} \sum_{k=1}^{n} \frac{Q^{pk}}{W_{pk} W_{pk+pq}} = U_{pn} \sum_{k=1}^{q} \frac{Q^{pk}}{W_{pk} W_{pk+pn}}.$$

Note that when $p = 1$, Theorem 3.12 reduces to Theorem 1 of [2].

Proof. Since n and k in identity (4.1) of [2] are arbitrary non-negative integers, we substitute pk for n and pq for k in the identity, obtaining

$$\frac{\beta^{pk}}{W_{pk}} - \frac{\beta^{pk+pq}}{W_{pk+pq}} = \frac{AQ^{pk}U_{pq}}{W_{pk}W_{pk+pq}}. \tag{31}$$

The theorem now follows by choosing $f(k) = \beta^k/W_k$ in Lemma 2.1 with $w = n$, $u = p$ and $v = q$ while making use of (31). □

Theorem 3.13. *If n and q are non-negative <u>even</u> integers and p is a non-zero integer, then*

$$U_{pq}\sum_{k=1}^{n}\frac{(\pm Q^p)^k}{W_{pk}W_{pk+pq}} = U_{pn}\sum_{k=1}^{q}\frac{(\pm Q^p)^k}{W_{pk}W_{pk+pn}}.$$

Proof. The statement of the theorem follows from choosing $f(k) = \beta^k/W_k$ in Lemma 2.2 with $w = n$, $u = p$ and $v = q$, while making use of (31). □

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