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# A note on weakly proregular sequence

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**Abstract.** In this paper, we give an elementary proof of the result given by Schenzel that there are functorial isomorphisms between local cohomology groups and Čech cohomology groups, by using weakly proregular sequences. The existence of these isomorphisms was well-known for Noetherian rings, but he generalised it to non-Noetherian rings and gave necessary and sufficient conditions for cohomology to coincide. In [6], he used notions of derived category theory in his proof, but we do not use them in this paper. We give a proof within Abelian category theory.
**Key Words**: Weakly proregular sequence, local cohomology, Čech cohomology.
**2010 MSC**: Primary 13D03; Secondary 13D45.

## 1 Introduction

In this note, we assume all rings are commutative with the identity element. Let $A$ be a ring and $I$ an ideal of $A$. The functor $\Gamma_I$ is defined by;

$$\Gamma_I(M) := \{x \in M \mid I^n x = 0 \text{ for some } n \geq 0\}$$

for an $A$-module $M$. Then, the local cohomology functors $H_I^i(-)$ are defined as the right derived functors of $\Gamma_I(-)$. In other words, let $J^\bullet$ be an injective resolution of the $A$-module $M$, then $H_I^i(M) \cong H^i(\Gamma_I(J^\bullet))$. In Noetherian cases, the local cohomology can be written by using the Čech cohomology. Let $\underline{a} = a_1, \ldots, a_r$ be a sequence of elements of $A$, and $I = (a_1, \ldots, a_r)$. $\check{H}^i(\underline{a}, M)$ denote the Čech cohomology group (see Definition 3.1). It is well-known that there are isomorphisms;

$$H_I^i(M) \cong \check{H}^i(\underline{a}, M) \qquad (*)$$

for any $A$-module $M$ if $A$ is a Noetherian ring and $I = (a_1, \ldots, a_r)$, see for example [1, Theorem 3.5.6.].

This result was generalised by [6]. For an arbitrary ring $A$ and sequence $\underline{a} = a_1, \ldots, a_r$ with $I = (a_1, \ldots, a_r)$, he showed that formula $(*)$ is true for any $A$-module $M$ if and only if $\underline{a}$ is a weakly proregular sequence. Let $H_i(\underline{a})$ be the Koszul homology group of the sequence $\underline{a}$. $\underline{a} = a_1, \ldots, a_r$ is called a weakly proregular sequence if for any $1 \leq i \leq r$ and for each $n > 0$ there is an $m \geq n$ such that the natural map;

$$H_i(\underline{a}^m) \to H_i(\underline{a}^n)$$

is the zero map, where $\underline{a}^n$ is the sequence defined by $a_1^n, \ldots, a_r^n$.

The goal of this note is to explain the following result without using notions of derived category theory.

**Theorem 1.1** (Theorem 5.3)**.** Let $A$ be a ring, $\underline{a} = a_1, \ldots, a_r$ a sequence of elements of $A$ and $I = (a_1, \ldots, a_r)$. $\underline{a}$ is a weakly proregular sequence if and only if for any $i$ and $A$-module $M$, $H_I^i(M) \cong \check{H}^i(\underline{a}, M)$ functorially on $M$.

In section 2, we summarise the $\delta$-functors introduced by [3]. It is also mentioned in [4, Chap.3.1] without proof. And in section 3 we resume the definition of Čech cohomology in commutative algebra. In section 4, we present the theory of weakly proregular sequences, following [6, Sect.2]. As for the weakly proregular sequence and Koszul homology, [5, Sect. 4] also obtain the results using derived categories with a different approach from this note. Finally, we prove Theorem 1.1 in section 5 within Abelian category theory.

## 2 $\delta$-functors

In this section, we outline the $\delta$-functors according to [3].

**Definition 2.1.** Let $\mathscr{A}, \mathscr{B}$ be Abelian categories, and suppose $\mathscr{A}$ has enough injectives. Let $F\colon \mathscr{A} \to \mathscr{B}$ be an additive left exact functor. $I^\bullet$ denotes an injective resolution of $A \in \mathscr{A}$. The functor;

$$R^iF\colon \mathscr{A} \to \mathscr{B}; A \mapsto H^i(F(I^\bullet))$$

is called the **right derived functor** of $F$.

Note that derived functors are independent up to natural transformation of the choice of an injective resolution. The following is a characteristic property of the derived functor.

**Proposition 2.2.** *Let $\mathscr{A}, \mathscr{B}$ be Abelian categories, and suppose $\mathscr{A}$ has enough injectives. Let $F\colon \mathscr{A} \to \mathscr{B}$ be an additive left exact functor. Then*

1. $R^0F \cong F$ *(as functors).*
2. *For any exact sequence in $\mathscr{A}$;*

$$0 \longrightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \longrightarrow 0$$

*and for each $i \geq 0$, there are **connecting morphisms** $\delta^i\colon R^iF(A_3) \to R^{i+1}F(A_1)$ such that;*

$$\begin{aligned} 0 \longrightarrow F(A_1) \xrightarrow{F(f)} F(A_2) \xrightarrow{F(g)} F(A_3) \xrightarrow{\delta^0} \cdots \\ \xrightarrow{\delta^{i-1}} R^iF(A_1) \xrightarrow{R^iF(f)} R^iF(A_2) \xrightarrow{R^iF(g)} R^iF(A_3) \xrightarrow{\delta^i} \cdots \end{aligned}$$

*is an exact sequence in $\mathscr{B}$.*

3. *For given a commutative diagram of the form (where the rows are exact) in $\mathscr{A}$;*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \xrightarrow{g} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{f'} & B_2 & \xrightarrow{g'} & B_3 & \longrightarrow & 0 \end{array}$$

*and for any $i \geq 0$ the following diagram;*

$$\begin{array}{ccc} R^iF(A_3) & \longrightarrow & R^{i+1}F(A_1) \\ \downarrow{\scriptstyle R^iF(\gamma)} & & \downarrow{\scriptstyle R^{i+1}F(\alpha)} \\ R^iF(B_3) & \longrightarrow & R^{i+1}F(B_1) \end{array}$$

*is commutative in $\mathscr{B}$.*

4. *For each injective object $I \in \mathscr{A}$ and for any $i > 0$, $R^i F(I) = 0$.*

See a textbook on homological algebra for the proof of this proposition. The $\delta$-functor can be thought of as an extract of the above property.

**Definition 2.3.** Let $\mathscr{A}, \mathscr{B}$ be Abelian categories. A family of additive functors $T^\bullet := \{T^i\}$ is called a $\delta$-**functor** when the following conditions hold;

1. For any exact sequence in $\mathscr{A}$;

$$0 \longrightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \longrightarrow 0$$

and for each $i \geq 0$, there are **connecting morphisms** $\delta^i \colon T^i(A_3) \to T^{i+1}(A_1)$ such that;

$$0 \longrightarrow T^0(A_1) \xrightarrow{T^0(f)} T^0(A_2) \xrightarrow{T^0(g)} T^0(A_3) \xrightarrow{\delta^0} \cdots$$
$$\xrightarrow{\delta^{i-1}} T^i(A_1) \xrightarrow{T^i(f)} T^i(A_2) \xrightarrow{T^i(g)} T^i(A_3) \xrightarrow{\delta^i} \cdots$$

is an exact sequence in $\mathscr{B}$.

2. For given a commutative diagram of the form (where the rows are exact) in $\mathscr{A}$;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \xrightarrow{g} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{f'} & B_2 & \xrightarrow{g'} & B_3 & \longrightarrow & 0 \end{array}$$

and for any $i \geq 0$ the following diagram;

$$\begin{array}{ccc} T^i(A_3) & \longrightarrow & T^{i+1}(A_1) \\ \downarrow{\scriptstyle T^i(\gamma)} & & \downarrow{\scriptstyle T^{i+1}(\alpha)} \\ T^i(B_3) & \longrightarrow & T^{i+1}(B_1) \end{array}$$

is commutative in $\mathscr{B}$.

Let us define that two $\delta$-functors are isomorphic in the following way. Let $T^\bullet, U^\bullet$ be $\delta$-functors. A family of natural transformations $\theta^\bullet = \{\theta^i \colon T^i \Rightarrow U^i\}$ is called a **morphism of $\delta$-functors** if for each exact sequence ;

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

in $\mathscr{A}$, the following diagram;

$$\begin{array}{ccc} T^i(A_3) & \xrightarrow{\delta^i_T} & T^{i+1}(A_1) \\ \downarrow{\scriptstyle \theta^i_{A_3}} & & \downarrow{\scriptstyle \theta^{i+1}_{A_1}} \\ U^i(A_3) & \xrightarrow{\delta^i_U} & U^{i+1}(A_1) \end{array}$$

is commutative. An isomorphism is a morphism which has a two-sided inverse.

**Definition 2.4.** Let $\mathscr{A}, \mathscr{B}$ be Abelian categories. The $\delta$-functor $T^\bullet$ is called **universal** if for each $\delta$-functor $U^\bullet$ and natural transformation $\theta \colon T^0 \Rightarrow U^0$, there is an unique morphism of $\delta$-functors $\theta^\bullet \colon T^\bullet \to U^\bullet$ such that $\theta^0 = \theta$.

by the definition, two universal $\delta$-functors such that $T^0 = U^0$ are isomorphic up to unique isomorphism. So for each additive functor $F\colon \mathscr{A} \to \mathscr{B}$, if an universal $\delta$-functor $T^\bullet$ with $T^0 = F$ exists, it is unique up to unique isomorphism. An universal $\delta$-functor $T^\bullet$ with such property is called a right satellite functor of $F$.

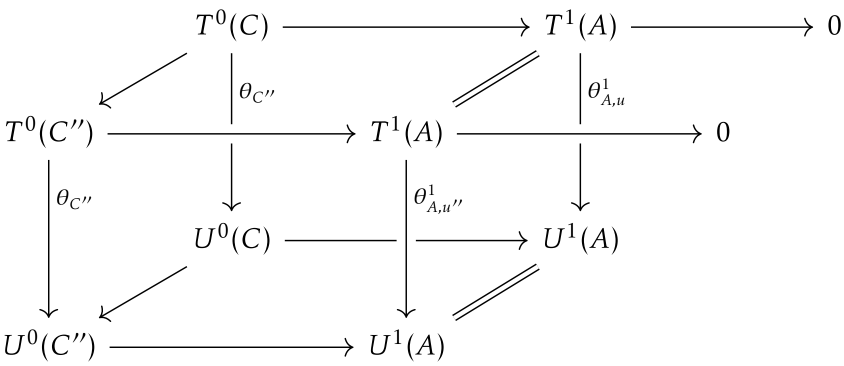
The following property gives a sufficient condition for the $\delta$-functor to be universal, which shows that the derived functor is also universal if $\mathscr{A}$ has enough injectives.

**Definition-Proposition 2.5.** Let $\mathscr{A}, \mathscr{B}$ be Abelian categories, $F$ an additive functor. $F$ is said to be **effaceable** if for each $A \in \mathscr{A}$, there are an $M \in \mathscr{A}$ and an injection (monomorphism) $u\colon A \to M$ such that $F(u) = 0$. A $\delta$-functor $T^\bullet$ is universal if for each $i > 0$, $T^i$ is effaceable.

*Proof.* Let $U^\bullet$ be a $\delta$-functor and $\theta\colon T^0 \Rightarrow U^0$ a natural transformation. We show that there exists uniquely a morphism of $\delta$-functors $\theta^\bullet\colon T^\bullet \to U^\bullet$ such that $\theta^0 = \theta$. We construct it inductively. For any $A \in \mathscr{A}$, there is an injection $u\colon A \to M$ such that $T^1(u) = 0$ since $T^1$ is effaceable. Let $C$ be the cokernel of $u$. We consider the long exact sequences induced by $0 \longrightarrow A \xrightarrow{u} M \xrightarrow{\pi} C \longrightarrow 0$ . So we get the following commutative diagram.

$$\begin{array}{ccccccc} T^0(M) & \xrightarrow{T^0(\pi)} & T^0(C) & \xrightarrow{\delta_T^0} & T^1(A) & \xrightarrow{T^1(u)=0} & 0 \\ \downarrow{\scriptstyle \theta_M} & & \downarrow{\scriptstyle \theta_C} & & \vdots & & \\ U^0(M) & \xrightarrow{U^0(\pi)} & U^0(C) & \xrightarrow{\delta_U^0} & U^1(A) & & \end{array}$$

Now $\theta^1_{A,u} := \delta^0_U \circ \theta_C \circ (\delta^0_T)^{-1}\colon T^1(A) \to U^1(A)$ is well-defined since the rows are exact. We show $\theta^1_{A,u}$ is independent of the choice of $u$. Let $u'\colon A \to M'$ be an injection such that $T^1(u') = 0$. $M \sqcup_A M'$ denote the cofibre product of $M$ and $M'$ on $A$. Then we get an injection $u''\colon A \to M \sqcup M'$ such that $T^1(u'') = 0$. Let $C''$ be the cokernel of $u''$. Then the following diagram;




is commutative. So we have $\theta^1_{A,u} = \theta^1_{A,u''}$, similarly we obtain $\theta^1_{A,u'} = \theta^1_{A,u''}$, then we get $\theta^1_{A,u} = \theta^1_{A,u'}$. So $\theta^1_A$ is independent of the choice of $u$.

Secondly, we show that for each $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ the following diagram;

$$\begin{array}{ccc} T^1(A) & \xrightarrow{T^1(f)} & T^1(B) \\ \downarrow{\scriptstyle \theta^1_A} & & \downarrow{\scriptstyle \theta^1_B} \\ U^1(A) & \xrightarrow{U^1(f)} & U^1(B) \end{array}$$

is commutative to prove $\theta^1$ is a natural transformation. For any injections $u\colon A \to M$ and $v\colon B \to N$

with $T^1(u) = T^1(v) = 0$, we take the cofibre product;

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "v\circ f"] & M \arrow[d] \\ N \arrow[r, "u'"] & M \sqcup_A N \end{tikzcd}$$

then $u'$ is injective. So we have an injection $u' \circ v\colon B \to M \sqcup N$ with $T^1(u' \circ v) = 0$. Then we replace $N$ by $M \sqcup N$ and get the following commutative diagram with exact rows;

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & B \arrow[r] & N \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

So $\theta^1$ is a natural transformation since the following diagram;

$$\begin{tikzcd} & T^0(C) \arrow[rr] \arrow[dl] \arrow[dd] & & T^1(A) \arrow[rr] \arrow[dl, "T^1(f)"'] \arrow[dd, "\theta^1_A"] & & 0 \\ T^0(C') \arrow[rr] \arrow[dd] & & T^1(B) \arrow[rr] \arrow[dd, "\theta^1_B"] & & 0 & \\ & U^0(C) \arrow[rr] \arrow[dl] & & U^1(A) \arrow[dl, "U^1(f)"] & & \\ U^0(C') \arrow[rr] & & U^1(B) & & & \end{tikzcd}$$

is commutative.

Finally, we show that $\theta^1_A$ is commutative with the connecting morphisms. Let

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

be a short exact sequence in $\mathscr{A}$, we use the same method as above for the injection $u\colon A_1 \to M$ with $T^1(u) = 0$ so that each row of the following commutative diagram is exact.

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d, equal] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_1 \arrow[r] & M \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

We consider the following diagram.

$$\begin{tikzcd} & T^0(A_3) \arrow[rr, "\delta"] \arrow[dl, "\theta_{A_3}"'] \arrow[dd] & & T^1(A_1) \arrow[dl, "\theta^1_{A_1}"] \arrow[dd, equal] \\ U^0(A_3) \arrow[rr, "\delta"] \arrow[dd] & & U^1(A_1) \arrow[dd, equal] & \\ & T^0(C) \arrow[rr, "\delta"] \arrow[dl, "\theta_C"'] & & T^1(A_1) \arrow[dl, "\theta^1_{A_1}"] \\ U^0(C) \arrow[rr, "\delta"] & & U^1(A_1) & \end{tikzcd}$$

The desired commutativity of $\theta^1_{A_1}, \theta_{A_3}$ and $\delta$ follows from commutativity of other squares, which follow from the construction of $\theta^1_{A_1}$ and facts that $T^\bullet$ and $U^\bullet$ are $\delta$-functors and that $\theta$ is a natural transformation.

This shows that $\theta^1$ is a natural transformation commutative with connecting morphisms, and its uniqueness can be seen from its construction (the universality of cokernel). In this way, $\theta^{i+1}$ can be constructed from $\theta^i$ inductively on $i$. □

**Corollary 2.6.** *Let $\mathscr{A}, \mathscr{B}$ be Abelian categories, and suppose $\mathscr{A}$ has enough injectives. Let $T^\bullet\colon \mathscr{A} \to \mathscr{B}$ be an universal $\delta$-functor, then $T^0$ is left-exact and for each $i \geq 0$ there is a natural isomorphism $T^i \cong R^i T^0$.*

*Proof.* $T^0$ is left exact by the definition of a $\delta$-functor, so there are right derived functors $R^i T^0$. For each $i > 0$, $R^i T^0$ is effaceable by 4 of Proposition 2.2, then $R^\bullet T^0$ is an universal $\delta$-functor. Now $R^0 T^0 = T^0$, so there is an unique isomorphism $R^\bullet T^0 \cong T^\bullet$ by universality. □

## 3 Čech cohomology and Koszul complex

In this section, we review the Čech cohomology of rings and modules. Let $A$ be a ring and fix a sequence $a_1, \dots, a_r$ of elements of $A$. For each $I = \{j_1, \dots, j_i\} \subset \{1, \dots, r\}$ $(j_1 < \cdots < j_i)$, let $a_I = a_{j_1} \dots a_{j_i}$. $e_1, \dots, e_r$ denotes the standard basis of $A^r$ and let $e_I = e_{j_1} \wedge \cdots \wedge e_{j_i}$.

**Definition 3.1.** Let $A$ be a ring, $\underline{a} = a_1, \dots, a_r \in A$. For each $1 \leq i \leq r$, $C^i(\underline{a})$ is the module defined by the following equation

$$C^i(\underline{a}) := \sum_{\#I = i} A_{a_I} e_I.$$

Then we define $C^\bullet(\underline{a})$ to be the complex defined by the following differentials

$$d^i\colon C^i(\underline{a}) \to C^{i+1}(\underline{a}); e_I \mapsto \sum_{j=1}^{n} e_I \wedge e_j.$$

It is called a **Čech complex**. $\check{H}^i(\underline{a})$ denotes the cohomology of this complex and it is called a **Čech cohomology**.

For an $A$-module $M$, we define $C^\bullet(\underline{a}, M) := C^\bullet(\underline{a}) \otimes M$. Here $\check{H}^i(\underline{a}, M)$ denotes the cohomology of $C^\bullet(\underline{a}, M)$.

**Proposition 3.2.** *Let $A$ be a ring. For each $\underline{a} = a_1, \dots, a_r \in A$, $\check{H}^\bullet(\underline{a}, -) = \{\check{H}^i(\underline{a}, -)\}_{i \geq 0}$ is a $\delta$-functor.*

*Proof.* Consider an exact sequence of $A$-modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0\ .$$

Since $C^\bullet(\underline{a}, M) = C^\bullet(\underline{a}) \otimes M$ and each component of the Čech complex is a flat $A$-module, the following sequence of complexes is exact;

$$0 \longrightarrow C^\bullet(\underline{a}, M_1) \longrightarrow C^\bullet(\underline{a}, M_2) \longrightarrow C^\bullet(\underline{a}, M_3) \longrightarrow 0$$

then there are connecting morphisms. So $\check{H}^\bullet(\underline{a}, -)$ is a $\delta$-functor. □

For the result we want, we need to look at the relationship between Čech complex and Koszul complex.

For $\underline{a} = a_1, \dots, a_r \in A$, let $\{e_i\}$ be the standard basis of a free $A$-module $A^r$. $f: A^r \to A; e_i \mapsto a_i$ induces a chain complex $K_\bullet(\underline{a})$. In other words, $K_\bullet(\underline{a})$ is the complex defined by following equations;

$$K_i(\underline{a}) := \bigwedge^i A^r,$$

$$d_i: K_i(\underline{a}) \to K_{i-1}(\underline{a}); x_1 \wedge \cdots \wedge x_i \mapsto \sum_{j=1}^{i} (-1)^{j+1} f(x_j) x_1 \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_i.$$

Note that $K_\bullet(\underline{a})$ does not depend on the order of $a_i$ .

We get a co-chain complex $K^\bullet(\underline{a})$ via the contravariant functor $\mathrm{Hom}(-, A)$;

$$K^\bullet(\underline{a}): \quad 0 \longrightarrow A \longrightarrow \mathrm{Hom}(K_1(\underline{a}), A) \longrightarrow \cdots .$$

$K^\bullet(\underline{a})$ is called a Koszul complex. For each $A$-module $M$, $K^\bullet(\underline{a}, M) = K^\bullet(\underline{a}) \otimes M$. Here $H^i(\underline{a}, M)$ denotes the cohomology of Koszul complex.

**Lemma 3.3.** *Let $A$ be a ring and $\underline{a} = a_1, \dots, a_r \in A$. For each $1 \le i \le r$;*

$$\varphi^i: K^i(\underline{a}) \to C^i(\underline{a}); (e_I)^* \mapsto (1/a_I) e_I$$

*is a morphism of complexes.*

*Proof.* Let $\delta^i$ be the differential of the Koszul complex. Then;

$$\delta^i(e_I^*)(e_J) = \begin{cases} a_j & (j \notin I, J = I \cup \{j\}) \\ 0 & (\text{otherwise}) \end{cases}$$

So;

$$\varphi^{i+1} \circ \delta^i(e_I^*) = \sum_{j \notin I} \frac{a_j}{a_I a_j} e_I \wedge e_j = \sum_{j \notin I} \frac{1}{a_I} e_I \wedge e_j$$

is equal to $d^i \circ \varphi^i(e_I^*)$. □

For any pair $n \le m$, we set;

$$\varphi^\bullet_{mn}: K^\bullet(\underline{a}^n) \to K^\bullet(\underline{a}^m); (e_I)^* \mapsto (a_I)^{m-n} (e_I)^*$$

then $\{K^\bullet(\underline{a}^n), \varphi^\bullet_{mn}\}_{n \in \mathbb{N}}$ is an inductive system.

**Proposition 3.4.** *Let $A$ be a ring, $a_1, \dots, a_r \in A$. Then;*

$$\varinjlim K^\bullet(\underline{a}^n) \cong C^\bullet(\underline{a}).$$

*Proof.* We define $\varphi^\bullet_n: K^\bullet(\underline{a}^n) \to C^\bullet(\underline{a}^n) = C^\bullet(\underline{a})$ in the same way as above lemma. Then $\varphi^\bullet_m \circ \varphi^\bullet_{nm} = \varphi^\bullet_n$ where $n \le m$. So we have $\varphi: \varinjlim K^\bullet(\underline{a}^n) \to C^\bullet(\underline{a})$. Each element of $C^i(\underline{a})$ is represented by a finite sum of $(b_I / a_I^{n_I}) e_I$, so it can be displayed as $\sum (1/a_I^n) b_I e_I$ by taking the maximum of $n$ and replacing $b_I$. Then it is an image of $\sum (b_I e_I) \in K^i(\underline{a}^n)$ , so $\varphi$ is surjective.

Secondly, we show $\varphi$ is injective. Assume $\varphi^i_n(x) = 0$ for $x \in K^i(\underline{a}^n)$. If $x = \sum b_I e_I^*$ then $\varphi^i_n(x) = \sum (b_I / a_I^n) e_I^* = 0$, so $b_I / a_I^n = 0$ in $A_{a_I^n}$. Therefore if we take a sufficiently large $l$, $a_I^l b_I = 0$. So $\varphi^l_{nm}(x) = 0$ by increasing $l$ if necessary, then $\varphi$ is injective. □

Since the functor of taking the inductive limit is exact, the following corollary follows.

**Corollary 3.5.** *Let $A$ be a ring, $a_1, \dots, a_r \in A$. For each $A$-module $M$;*

$$\check{H}^i(\underline{a}, M) \cong \varinjlim H^i(\underline{a}^n, M).$$

## 4 Weakly proregular sequences

In this section, we summarise the weakly proregular sequence following [6, Sect.2].

**Definition 4.1.** Let $\mathscr{A}$ be an Abelian category, $(X_n, \varphi_{mn})$ a projective system in $\mathscr{A}$. $(X_n)$ is said to be **essentially zero** or **pro-zero** if for each $n$, there is an $m \geq n$ such that $\varphi_{mn}\colon X_m \to X_n$ is the zero map.

Obviously, if $(X_n)$ is essentially zero then $\varprojlim X_n = 0$.

**Proposition 4.2.** *Let $\mathscr{A}$ be an Abelian category. We consider an exact sequence of projective systems in $\mathscr{A}$;*

$$0 \longrightarrow (X_n) \longrightarrow (Y_n) \longrightarrow (Z_n) \longrightarrow 0 \ .$$

*Then $(Y_n)$ is essentially zero if and only if the other two are essentially zero.*

*Proof.* If $(Y_n)$ is essentially zero, then it is clear that the other two are so. We show the opposite. For each $n$, there is an $m \geq n$ such that $X_m \to X_n$ is the zero map since $(X_n)$ is essentially zero. Similarly there is an $l \geq m$ such that $Z_l \to Z_m$ is the zero map, then we have the following commutative diagram with the exact rows;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_l & \longrightarrow & Y_l & \longrightarrow & Z_l & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \varphi_{lm}} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & X_m & \longrightarrow & Y_m & \longrightarrow & Z_m & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \varphi_{mn}} & & \downarrow & & \\
0 & \longrightarrow & X_n & \longrightarrow & Y_n & \longrightarrow & Z_n & \longrightarrow & 0
\end{array}$$

So we get $\varphi_{ln} = \varphi_{mn} \circ \varphi_{lm} = 0$ by an easy diagram chasing. □

We use same symbols as before for the Koszul and Čech complexes. Note that $(K_i(\underline{a}^n))_{n\in\mathbb{N}}$ is a projective system defined by $K_i(\underline{a}^m) \to K_i(\underline{a}^n); e_I \mapsto a_I^{m-n} e_I$ $(m \geq n)$.

**Definition 4.3.** Let $A$ be a ring. $\underline{a} = a_1, \ldots, a_r \in A$ is called a **weakly proregular sequence** if for each $1 \leq i \leq r$, the projective system $\{H_i(\underline{a}^n)\}$ is essentially zero.

The property of being weakly proregular does not depend on the order by the definition.

**Proposition 4.4.** *Let $A$ be a ring, $\underline{a} = a_1, \ldots, a_r \in A$. $\underline{a}$ is a weakly proregular sequence if and only if $\check{H}^\bullet(\underline{a}, -)$ is an effaceable functor for $i > 0$.*

*Proof.* Assume that $\underline{a}$ is a weakly proregular sequence. Let $I$ be an injective module. Now there is an isomorphism $H^i(\underline{a}^n, I) \cong \mathrm{Hom}(H_i(\underline{a}^n), I)$ since $K^\bullet(\underline{a}^n, I) = \mathrm{Hom}(K_\bullet(\underline{a}^n), I)$ and $\mathrm{Hom}(-, I)$ is an exact functor. For each $n \geq 0$, there is an $m \geq n$ such that $H_i(\underline{a}^m) \to H_i(\underline{a}^n)$ is the zero map since $H_i(\underline{a}^n)$ is essentially zero. So $\check{H}^i(\underline{a}, I) = \varinjlim H^i(\underline{a}^n, I) = 0$.

Secondly, assume that $\check{H}^\bullet(\underline{a}, -)$ is an effaceable functor for $i > 0$. For each $n \geq 0$, we have an injective module $I$ and an injection $\varepsilon\colon H_i(\underline{a}^n) \to I$. Then there is an $m \geq n$ such that;

$$H_i(\underline{a}^m) \longrightarrow H_i(\underline{a}^n) \xrightarrow{\ \varepsilon\ } I$$

is the zero map by $\varepsilon \in H^i(\underline{a}^n, I)$ and $\varinjlim H^i(\underline{a}^n, I) = 0$. □

Then Čech cohomology is the derived functor of $\check{H}^0(\underline{a}, -)$ if $\underline{a}$ is a weakly proregular sequence. So the next question of interest is when is a sequence weakly proregular. We introduce proregular sequences by [2], and review that each sequence $\underline{a}$ is weakly proregular in the Noetherian case.

**Definition 4.5.** Let $A$ be a ring, $\underline{a} = a_1,\ldots,a_r \in A$. $\underline{a}$ is called a **proregular sequence** if for each $1 \le i \le r$ and $n > 0$, there is an $m \ge n$ such that $((a_1^m,\ldots,a_{i-1}^m) : a_i^m A) \subset ((a_1^n,\ldots,a_{i-1}^n) : a_i^{m-n} A)$.

Note that a regular sequence is proregular.

**Proposition 4.6.** *Let $A$ be a Noetherian ring. For each $\underline{a} = a_1,\ldots,a_r \in A$, $\underline{a}$ is a proregular sequence.*

*Proof.* Let $J_m := ((a_1^m,\ldots,a_{i-1}^m) : a_i^m A), I_{n,m} := ((a_1^n,\ldots,a_{i-1}^n) : a_i^{m-n} A)$. For each $n$, $\{I_{n,m}\}_{m\ge n}$ is an ascending chain of ideals, then there is an $m_0 \ge n$ such that for each $m \ge m_0, I_{n,m_0} = I_{n,m}$. Let $m := m_0 + n$, then for each $a \in J_{m_0}$, $aa_i^{m-n} = aa_i^{m_0} \in (a_1^{m_0},\ldots,a_{i-1}^{m_0}) \subset (a_1^n,\ldots,a_{i-1}^n)$. So $a \in I_{n,m} = I_{n,m_0}$. □

**Proposition 4.7.** *Let $A$ be a ring. A proregular sequence is weakly proregular.*

*Proof.* We use an induction on $r$. When $r = 1$, let $a \in A$ be proregular. Then for each $n > 0$, there is an $m \ge n$ such that $\operatorname{Ann} a^m \subset \operatorname{Ann} a^{m-n}$. So $(H_1(a^n))$ is essentially zero since $H_1(a^n) = \operatorname{Ann} a^n$. Now we assume that claim up to $r-1$. Here $\underline{a} = a_1,\ldots,a_r$ and $\underline{a}' = a_1,\ldots,a_{r-1}$, the exact sequence of complexes;

$$0 \longrightarrow K_\bullet(\underline{a}'^n) \longrightarrow K_\bullet(\underline{a}^n) \longrightarrow K_\bullet(\underline{a}'^n)(-1) \longrightarrow 0$$

induces the exact sequence of homology;

$$\cdots \longrightarrow H_i(\underline{a}'^n) \xrightarrow{(-1)^i a_r^n}$$

$$H_i(\underline{a}'^n) \longrightarrow H_i(\underline{a}^n) \longrightarrow H_{i-1}(\underline{a}'^n) \xrightarrow{(-1)^{i-1} a_r^n}$$

$$H_{i-1}(\underline{a}'^n) \longrightarrow \cdots$$

Then we have the following exact sequence;

$$0 \longrightarrow H_0(a_r^n, H_i(\underline{a}'^n)) \longrightarrow H_i(\underline{a}^n) \longrightarrow H_1(a_r^n, H_{i-1}(\underline{a}'^n)) \longrightarrow 0$$

and this induces the exact sequence of projective systems. The first projective system is essentially zero by the assumption of induction. Also for each $i > 1$, the third system is essentially zero since $H_1(a_r^n, H_{i-1}(\underline{a}'^n)) = \{x \in H_{i-1}(\underline{a}'^n) \mid a_r^n x = 0\}$. If $i = 1$, the system with ;

$$H_1(a_r, H_0(\underline{a}'^n)) = \left\{x \in H_0(\underline{a}'^n) \,\middle|\, a_r^n x = 0\right\}$$

is essentially zero since $\underline{a}$ is proregular. So this completes the proof by the induction. □

**Corollary 4.8.** *Let $A$ be a Noetherian ring. For each $\underline{a} = a_1,\ldots,a_r \in A$, $\underline{a}$ is weakly proregular.*

## 5 Local cohomology

Let $A$ be a ring and $I$ an ideal of $A$. The functor $\Gamma_I$ is defined by;

$$\Gamma_I(M) := \{x \in M \mid I^n x = 0 \text{ for some } n \ge 0\}$$

for an $A$-module $M$. Note that $\Gamma_I(M) = \varinjlim \operatorname{Hom}_A(A/I^n, M)$ and this isomorphism is functorial in $M$. By the definition, $\Gamma_I$ is a left exact functor.

**Definition 5.1.** Let $A$ be a ring and $I$ an ideal of $A$. $H^i_I(-)$ denote the derived functor of $\Gamma_I(-)$ and it is called a **local cohomology**.

Note that $H^i(M) \cong \varinjlim \mathrm{Ext}^i(A/I^n, M)$. We summarise the relationship between local cohomology and Čech cohomology. First, we note that the 0-th part of each cohomologies are naturally isomorphic.

**Lemma 5.2.** *Let $A$ be a ring, $\underline{a} = a_1, \dots, a_r \in A$, and $I = (a_1, \dots, a_r)$. For each $A$-module $M$;*

$$\Gamma_I(M) \cong \check{H}^0(\underline{a}, M).$$

*Proof.* Here $\check{H}^0(\underline{a}, M)$ is the kernel of

$$M \to \bigoplus_{i=1}^{r} M_{a_i} e_i; x \mapsto (x/1)e_i.$$

Then for each $x \in \check{H}^0(\underline{a}, M)$ and $1 \le i \le r$, there is an $n_i \ge 0$ such that $a_i^{n_i} x = 0$. So we have $x \in \Gamma_I(M)$. Similarly the converse is true, so they are equal as submodules of $M$. □

With the preparations we have made above, we can prove the results we have been aiming for.

**Theorem 5.3** (An elementary proof of [6, Theorem 3.2])**.** Let $A$ be a ring, $\underline{a} = a_1, \dots, a_r \in A$ and $I = (a_1, \dots, a_r)$. $\underline{a}$ is a weakly proregular sequence if and only if for any $i$ and $A$-module $M$, $H^i_I(M) \cong \check{H}^i(\underline{a}, M)$ functorially on $M$.

*Proof.* Assume that $\underline{a}$ is weakly proregular. $\check{H}^\bullet(\underline{a}, -)$ is a $\delta$-functor by Proposition 3.2. Moreover $\check{H}^\bullet(\underline{a}, -)$ is universal by Proposition 4.4 and Definition-Proposition 2.5. So $H^i_I(M) \cong \check{H}^i(\underline{a}, M)$ by above lemma. The converse is true by Proposition 4.4. □

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