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Rings whose cyclics are DU-modules

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Abstract. A right R-module M is called a $D4$-Module if for any two direct summands A and B of M with $M = A + B$ and $M/A \cong M/B$, we have $A \cap B \subseteq^{\oplus} M$. The module M is called a Dual-Utumi-Module (DU-module) if M is a $D4$-module and for any two proper submodules A and B of M with $M/A \cong M/B$ and $A + B = M$, both A and B lie over direct summands of M. The notion of DU-modules is a simultaneous and strict generalization of both the quasi-discrete as well as the dual-square-free modules. In this paper, we study the modules whose factors are DU-modules ($D4$-modules), extending some of the known results on quasi-discrete modules and obtaining new ones.

Key Words: Projective, quasi-projective, quasi-discrete, Utumi, Dual-Utumi and dual-square-free modules.
2010 MSC: Primary 16D40, 16D50, 16D60; Secondary 16L30, 16L60, 16P20, 16P40, 16P60.

1 Introduction

In [23], Osofsky showed that rings whose cyclic modules are injective are precisely the semisimple artinian rings. Subsequently, and motivated by Osofsky's result, several authors studied the rings whose cyclic modules are quasi-injective, quasi-continuous, auto-invariant, or extending. In [11], the notion of Utumi modules (U-modules for short) was introduced as a strict and simultaneous generalization of the quasi-continuous, auto-invariant and square-free modules. In [16], rings whose cyclics are U-modules were introduced and thoroughly investigated, extending many of the known results on the subject and providing new ones. Dually, in [13], the notion of Dual-Utumi modules (DU-modules for short) was introduced as a strict and simultaneous generalization of the quasi-discrete, pseudo-discrete and dual-square-free modules. In [5], an attempt was made to study the rings whose cyclic modules are DU-modules. In this paper, we investigate thoroughly the rings R whose cyclic right R-modules are DU-modules (CDU-rings for short).

We start our work in section 2, by investigating the more general notions of $D4$- and $FD4$-modules, where a right R-module M is called $D4$ if, whenever A_1 and A_2 are submodules of M with $M = A_1 \oplus A_2$ and $f : A_1 \to A_2$ is an R-homomorphism with $\operatorname{Im} f \subseteq^{\oplus} A_1$, we have $\ker f \subseteq^{\oplus} A_2$. The module M is called $FD4$-module if every factor of M is a $D4$-module. In section 3, we provide an interesting decomposition for $FD4$-modules. More precisely, we prove in Theorem 3.1, that if M is an $FD4$-module whose local summands are summands, then $M = Q \oplus P$ where Q is a summand-dual-square-free, $P = D \oplus A \oplus B$ is semisimple with $A \cong B$, and D embeds in $A \oplus B$, P and Q are factor-orhogonal, $Hom(Q,P) = 0$, and if $g \in Hom(P,Q)$, then $\operatorname{Im} g \ll Q$.

In section 4, we turn our attention to the rings whose cyclics are DU-modules, which is the main topic of this paper. We start by considering the more general notion of FDU-modules, where M is called FDU-module if every factor of M is a DU-module. We prove in Theorem 4.5, that M is an FDU-module iff whenever $M = A + B$ with $M/A \cong M/B$, then $A \cap B \subseteq^{\oplus} M$, iff every epimorphism $f : M \longrightarrow N^2$ splits, where $N^2 := N \oplus N$. In Theorem 4.12, we show that if M is an FDU-module whose local summands are summands, then $M = Q \oplus P$, where Q and P retain the same properties as those in the above mentioned decomposition of $FD4$-modules with the additional attribute that Q is indeed a dual-square-free module.

Throughout, all rings R are associative with unity and all modules are unitary R-modules. For a module M, we use $rad(M)$, $E(M)$ and $End(M_R)$ to denote the Jacobson radical, the injective hull and the endomorphism ring of M, respectively. If $M = R$, we write $J(R) = rad(R)$. We write $N \subseteq M$ if N is a submodule of M, $N \subseteq^{ess} M$ if N is an essential submodule of M, $N \subseteq^{\oplus} M$ if N is a direct summand of M, and $N \ll M$ if N is a small submodule of M. A submodule N of M is called proper if $N \subsetneq M$. An element a of R is called proper if aR is a proper submodule of R_R. A submodule N of a right R-module M is said to lie over a direct summand of M if, there is a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq N$ and $N \cap M_2 \ll M$.

2 Rings whose cyclics are $D4$-modules

In [6], the notion of $C4$-modules was introduced as a strict and simultaneous generalization of the well-known $C3$- and square-free modules, where a right R-module M is called a $C4$-module if, whenever A_1 and A_2 are submodules of M with $M = A_1 \oplus A_2$ and $f : A_1 \to A_2$ is an R-homomorphism with $\ker f \subseteq^{\oplus} A_1$, we have $\operatorname{Im} f \subseteq^{\oplus} A_2$. Dually, in [7], a right R-module M is called a $D4$-Module if, whenever A_1 and A_2 are submodules of M with $M = A_1 \oplus A_2$ and $f : A_1 \to A_2$ is an R-homomorphism with $\operatorname{Im} f \subseteq^{\oplus} A_1$, we have $\ker f \subseteq^{\oplus} A_2$. The notion of $D4$-modules is a strict and simultaneous generalization of both the $D3$- and the dual-square-free modules. The next lemma was established in [7, Theorem 2.2] and will be used frequently throughout the paper.

Lemma 2.1. *The following statements are equivalent for a module M:*

1. *M is a $D4$-module.*
2. *If A and B are submodules of M with $A \subseteq B$ and $M/B \cong A \subseteq^{\oplus} M$, then $B \subseteq^{\oplus} M$.*
3. *If A and B are submodules of M with $M = A + B$, $A \subseteq^{\oplus} M$ and $M/A \cong M/B$, then $A \cap B \subseteq^{\oplus} M$.*
4. *If A and B are direct summands of M with $M = A + B$ and $M/A \cong M/B$, then $A \cap B \subseteq^{\oplus} M$.*
5. *If A and B are submodules of M with $M = A + B$, $A \subseteq^{\oplus} M$ and $M/A \cong M/B$, then $B \subseteq^{\oplus} M$.*
6. *If $M = A \oplus A' = B \oplus B' = A + B = A + B'$, where A, A', B and B' are submodules of M, then $A \cap B \subseteq^{\oplus} M$.*
7. *If A and B are direct summands of M with $M = A + B$ and $A \cong B$, then $A \cap B$ is a direct summand of M.*

Definition 2.2. A right R-module M is called an $FD4$-module, if every factor of M is a $D4$-module. A ring R is called a right $CD4$-ring if R, as a right R-module, is an $FD4$-module.

Remark 2.3. Observe that since any factor module of a summand of M is again a factor of M, it is not difficult to prove that every direct summand of an $FD4$-module is again an $FD4$-module.

Proposition 2.4. *Let M be a right $FD4$-module. If $N = A \oplus B$ is a factor module of M with $A \cong B$, then N is semisimple. In particular, if $M = A \oplus B$ is a right $FD4$-module with $A \cong B$, then M is semisimple.*

Proof. Let K be a submodule of A. By the hypothesis, since the factor module $\frac{A}{K} \oplus A \cong \frac{A}{K} \oplus B \cong \frac{A \oplus B}{K}$ is a $D4$-module, the natural epimorphism $A \longrightarrow A/K$ splits; that is $K \subseteq^{\oplus} A$. This shows that the module A is semisimple, and since $A \cong B$, it follows that $N = A \oplus B$ is semisimple. □

Recall that two right R-modules M and N are called (summand-)orthogonal, if they do not contain non-zero isomorphic (summands) submodules. The modules M and N are called factor-orthogonal if, no nonzero factor module of M is isomorphic to a factor module of N.

Lemma 2.5. *If $M = A \oplus B$ is a $D4$-module with A semisimple, then the following conditions are equivalent:*

1. *A and B are factor-orthogonal.*
2. *A and B are summand-orthogonal.*

Proof. $1 \Rightarrow 2$. Clear.

$2 \Rightarrow 1$. Let $\frac{A}{K} \overset{\sigma}{\cong} \frac{B}{L}$ be an isomprphsim, where $K \subseteq A$ and $L \subseteq B$. Since A is semisimple, $K \subseteq^{\oplus} A$, and so $K \oplus B \subseteq^{\oplus} M$. Clearly, $M = A \oplus B = (K \oplus B) + (L \oplus A)$ with

$$\frac{M}{K \oplus B} \cong \frac{A}{K} \cong \frac{B}{L} \cong \frac{M}{L \oplus A}$$

Since M is a $D4$-module, we infer from Lemma 2.1, that $L \oplus A \subseteq^{\oplus} M$. Consequently, $L \subseteq^{\oplus} B$. Since A and B are summand-orthogonal, $K = A$ and $L = B$, as required. □

The above lemma can be strengthened if we assume that the module M to be an $FD4$-module.

Lemma 2.6. *If $M = A \oplus B$ is an $FD4$-module, then the following conditions are equivalent:*

1. *A and B are factor-orthogonal.*
2. *A and B are summand-orthogonal.*

Proof. $1 \Rightarrow 2$. Clear.

$2 \Rightarrow 1$. Let $\frac{A}{K} \overset{\sigma}{\cong} \frac{B}{L}$ be an isomprphsim. Since $\frac{A}{K} \oplus A \cong \frac{B}{L} \oplus A \cong \frac{A \oplus B}{L}$ is a $D4$-module, the natural epimorphism $A \longrightarrow A/K$ splits; that is $K \subseteq^{\oplus} A$. Similarly, $L \subseteq^{\oplus} B$. Since A and B are summand-orthogonal, $K = A$ and $L = B$, as required. □

A module M is called square-free (SF-module) if it contains no non-zero isomorphic submodules A and B with $A \cap B = 0$. The module M is called summand-square-free (SSF for short) if the submodules A and B are summands of M. Dually, M is called dual-square-free (DSF) if M has no proper submodules A and B with $M = A + B$ and $M/A \cong M/B$. The module M is called summand-dual-square-free ($SDSF$ for short) if the submodules A and B are summands of M. A ring R is called right DSF-ring ($SDSF$-ring), if it is a DSF-module ($SDSF$-module) as a right R-module. Clearly, every DSF-module is a $D4$-module.

Corollary 2.7. *If $M = Q \oplus P$ is both an $SDSF$- and an $FD4$-module, then P and Q are factor-orthogonal.*

Proposition 2.8. *Let $M = A \oplus B$ is an $FD4$-module and $h : A \longrightarrow B$ be a non-zero homomorphism. Then the following hold:*

1. *Either $\operatorname{Im} h \ll B$ or there exist two non-zero direct summands $K \subseteq^{\oplus} A$ and $L \subseteq^{\oplus} B$ with $K \cong L$.*

2. *If M is an $SDSF$-module, then* $\operatorname{Im} h \ll B$.

3. *If A and B are indecomposable, then either* $\operatorname{Im} h \ll B$ *or* $A \cong B$.

4. *If A and B are indecomposable and $rad(M) = 0$, then $A \cong B$ and M is semisimple.*

Proof. (1) If we assume to the contrary that $\operatorname{Im} h$ is not small in B, then there exists a proper submodule $L \subseteq B$ with $\operatorname{Im} h + L = B$. Now, the following epimorphism

$$A \xrightarrow{h} \operatorname{Im} h \xrightarrow{\eta} \frac{\operatorname{Im} h}{\operatorname{Im} h \cap L} \overset{\theta}{\cong} \frac{B}{L} \to 0$$

induces the isomorphism $\frac{A}{\ker \theta\eta h} \cong B/L$. Therefore, both $B \oplus B/L$ and $A \oplus \frac{A}{\ker \theta\eta h}$ are factor modules of $M = A \oplus B$. By the hypothesis, both $B \oplus B/L$ and $A \oplus \frac{A}{\ker \theta\eta h}$ are $D4$-modules. This implies that both of the natural epimorphisms $B \longrightarrow B/L$ and $A \longrightarrow \frac{A}{\ker \theta\eta h}$ split. That is $L \subseteq^{\oplus} B$ and $\ker \theta\eta h \subseteq^{\oplus} A$. Write $B = L \oplus E$ and $A = \ker \theta\eta h \oplus C$, for submodules $E \subseteq B$ and $C \subseteq A$. Clearly, $E \cong C$ and both are nonzero, as required.

(2) & (3). Clear from (1).

(4). Follows from (3) and Proposition 2.4. □

Corollary 2.9. *Let e and f be indecomposable orthogonal idempotents of a right $CD4$-ring R. If $h : eR \longrightarrow fR$ is an R-homomorphism, then either h is an isomorphism or* $\operatorname{Im} h \ll fR$. *In particular, if $rad(R) = 0$ and h is non-zero, then $eR \cong fR$.*

Proof. Since eR and fR are orthogonal direct summands, $eR \oplus fR = (e+f)R$ is a direct summand of R. Therefore, $N := eR \oplus fR$ is a right $FD4$-module with $rad(N) \ll N$. Now the result follows directly from Proposition 2.8. The last statement is clear. □

Since the class of DSF-modules is closed under factors, and every DSF-module is a $D4$-module, the next corollary is an immediate consequence of Proposition 2.8 above and extends [12, Corollary 2.19].

Corollary 2.10. *If $M = A \oplus B$ is a right $FD4$- and an $SDSF$-module with $rad(M) = 0$, then $Hom(A,B) = Hom(B,A) = 0$. In particular, if $M = A \oplus B$ is a DSF-module then $Hom(A,B) = Hom(B,A) = 0$.*

A ring R is called abelian if all of its idempotents are central.

Lemma 2.11. *[11, Lemma 3.20] If e is a non-central idempotent of a ring R, then either eR or $(1-e)R$ is not a two-sided ideal of R.*

Proposition 2.12. *If R is both a right $CD4$- and a right $SDSF$-ring with $J(R) = 0$, then R is abelian.*

Proof. Assume to the contrary that R is not abelian, and let $e \in R$ be a non-central idempotent of R. By Lemma 2.11, either eR or $(1-e)R$ is not a two-sided ideal of R. Without loss of generality, assume that eR is not a two sided ideal of R. Therefore, $Re \not\subseteq eR$ and $(1-e)Re \neq 0$. Let $0 \neq (1-e)ae \in (1-e)Re$, $a \in R$, and define $h : eR \longrightarrow (1-e)R$ by $h(er) = (1-e)aer$, for all $r \in R$. Clearly h is a non-zero homomorphism, a clear contradiction to Corollary 2.10 above. □

Corollary 2.13. *If R is a right $CD4$-ring such that R/J is a right $SDSF$-ring, then R/J is abelian. In particular if R is a right DSF-ring, then R/J is abelian.*

A ring R is called I-finite, if it contains no infinite sets of orthogonal idempotents. A non-zero idempotent e of a ring R is called primitive if eRe has no non-trivial idempotents, equivalently, $(eR)_R$ is indecomposable. A non-zero idempotent e of a ring R is called local if eRe is a local ring. Clearly, every local idempotent is primitive. A ring R is called semiperfect if 1 is the sum of local orthogonal idempotents; equivalently if $R/J(R)$ is semisimple artinian and idempotents lift modulo $J(R)$. Every primitive idempotent of a semiperfect ring is local.

Lemma 2.14. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a finite direct product of rings. Then R is a right $CD4$-ring if and only if every R_i is a right $CD4$-ring.*

Proof. Routine. □

Proposition 2.15. *Let R be an I-finite ring with $J(R) = 0$. Then R is a right $CD4$-ring if and only if $R = R_1 \times R_2$, where R_1 is a semisimple artinian ring and R_2 is a finite direct product of right $CD4$-rings with no nontrivial idempotents*

Proof. The sufficiency follows from Lemma 2.14. For the necessity, let R be a right $CD4$-ring. Since R is I-finite, R_R has an indecomposable decomposition $R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$, where the set $\{e_i\}_{i=1}^n$ consists of pairwise orthogonal primitive idempotents. Let $[e_tR] = \oplus_{i\in I_t}\{e_iR : e_iR \cong e_tR\}$. Renumbering if necessary, we may write $R = [e_1R] \oplus [e_2R] \oplus \cdots \oplus [e_kR]$. Clearly each homogeneous component $[e_tR] = eR$ for some idempotent e of R, and $R = eR \oplus (1-e)R$ where $(1-e)R = \oplus_{k\neq t}[e_kR]$. We claim that each $[e_tR]$ is a two-sided ideal of R. Assume to the contrary that the following homogeneous component $[e_iR] = eR$ is not a two-sided ideal of R. In this case, $Re \not\subseteq eR$ and $(1-e)Re \neq 0$. Let $0 \neq (1-e)ae \in (1-e)Re$ and define $h : eR \longrightarrow (1-e)R$ by $h(er) = (1-e)aer$. Clearly h is a non-zero R-homomorphism, and so $h(e_jR) \neq 0$ for some $j \in I$. Let $g := h|_{e_jR} : e_jR \longrightarrow (1-e)R$. Now, since $(1-e)R = \oplus_{k\neq t}[e_kR]$, there is an $s \neq t$ such that if $\pi_s : (1-e)R \longrightarrow [e_sR]$ is the natural projection map, then $\alpha := \pi_s g : e_jR \to [e_sR]$ is a non-zero R-homomorphism. Again, if $[e_sR] = \oplus_{i\in I_s}\{e_iR : e_iR \cong e_sR\}$, there is an $r \in I_s$ such that $\beta := \eta_r h : e_jR \xrightarrow{\alpha} [e_sR] \xrightarrow{\eta_r} e_rR$ is a non-zero R-homomorphism, where $\eta_r : [e_sR] \to e_rR$ is the natural projection. Now, since $J(R) = 0$, we infer from Corollary 2.9 that the map $\beta : e_jR \to e_rR$ is an isomorphism, a clear contradiction. This shows that $eR = [e_iR]$ is a two-sided ideal of R, proving the claim. Now, if $[e_iR]$ contains more than one direct summand, then by Proposition 2.4, $[e_iR]$ is a semisimple artinian ring. If $[e_iR]$ consists of exactly one direct summand, then $[e_iR] = e_iR = e_iRe_i$ is a right $CD4$-ring with no nontrivial idempotents, completing the proof. □

Corollary 2.16. *Let R be an I-finite ring such that idempotents lift modulo $J(R)$. Then $\bar{R} := R/J$ is a right $CD4$-ring if and only if $\bar{R} = R_1 \times R_2$, where R_1 is a semisimple artinian ring and R_2 is a finite direct product of right $CD4$-rings with no nontrivial idempotents.*

Proof. Since R is an I-finite ring with idempotents lift modulo J, R/J is an I-finite ring. Hence the result follows from Corollary 2.16. □

3 A Decomposition Theorem For $FD4$-Modules

Recall that a local summand of a module M is a direct sum $L := \oplus_{i\in I} X_i$ of submodules of M such that $\oplus_{i\in F} X_i$ is a summand of M for any finite subset F of I. A module M is said to satisfy the (full) internal exchange property if for every internal direct sum decomposition $M = \oplus_{i\in I} M_i$ and every summand $N \subseteq^{\oplus} M$, there exist submodules $M_i' \subseteq M_i$, $i \in I$, such that $M = \oplus_{i\in I} M_i' \oplus N$.

Theorem 3.1. Let M be a right $FD4$-module whose local summands are summands. Then $M = Q \oplus P$ where

1. Q is an $SDSF$-module.
2. $P = D \oplus A \oplus B$ is semisimple with $A \cong B$, and D embeds in $A \oplus B$.
3. P and Q are factor-orthogonal.
4. $Hom(Q,P) = 0$.
5. If $f \in Hom(P,Q)$, then $\operatorname{Im} f \ll Q$.

Proof. Consider the set

$$F = \{(A,B,f) : A \oplus B \subseteq^{\oplus} M \text{and} A \overset{f}{\cong} B\}.$$

Order F as follows: $(A,B,f) \leqslant (C,D,g)$ if $A \subseteq C$, $B \subseteq D$ and g extends f. Clearly, F is a non-empty inductive set. Let $(A_1,B_1,f_1) \leqslant (A_2,B_2,f_2) \leqslant \cdots$ be a chain of elements in F. Since local summands of M are summands, we infer, from [20, Lemma 2.16], that $\cup_{i\in I} A_i$, $\cup_{i\in I} B_i$ and $\cup_{i\in I} A_i \oplus \cup_{i\in I} B_i = \cup_{i\in I}(A_i \oplus B_i)$ are direct summands of M. Now, if we define $f : \cup_{i\in I} A_i \longrightarrow \cup_{i\in I} B_i$ by $f(a) = f_i(a)$ if $a \in A_i$, then f is an isomorphism that extends f_i, $i \in I$. Therefore, $< \cup_{i\in I} A_i, \cup_{i\in I} B_i, f >$ is an upper bound of the given chain, and by Zorn's Lemma, F has a maximal element $< A,B,f >$, say. Write $M = K \oplus A \oplus B = K \oplus L$, where $L := A \oplus B$ and $K \subseteq M$. By Proposition 2.4, clearly L is semisimple. We claim that K is an $SDSF$-module. Otherwise, there exist two proper direct summands X and Y of K with $K = X + Y$ and $K/X \cong K/Y$. Since direct summands of $D4$-modules are again $D4$, it follows from Lemma 2.1 that $X \cap Y \subseteq^{\oplus} K$. Write, $K = (X \cap Y) \oplus T$, for a submodule $T \subseteq K$. Clearly, $X = (X \cap Y) \oplus (X \cap T)$ and $Y = (X \cap Y) \oplus (Y \cap T)$. Also, $K = X + Y = [(X\cap Y)\oplus(X\cap T)]+[(X\cap Y)\oplus(Y\cap T)] = (X\cap Y)\oplus(X\cap T)\oplus(Y\cap T)$. Now, $(X \cap T) \cong K/[(X \cap Y) \oplus (Y \cap T)] = K/Y \cong K/X = K/[(X \cap Y) \oplus (X \cap T)] \cong (Y \cap T)$. Let $(X \cap T) \overset{g}{\cong} (Y \cap T)$ be such an isomorphism. If we define:

$$\alpha : [(X \cap T) \oplus A] \longrightarrow [(Y \cap T) \oplus B]$$

by $\alpha(x + a) = g(x) + f(a)$ for all $x \in (X \cap T)$ and $a \in A$, then α is a well-defined isomorphism. Now, if we set $V = (X \cap T) \oplus A$ and $W = (Y \cap T) \oplus B$, then:

$$M = K \oplus A \oplus B = (X \cap Y) \oplus (X \cap T) \oplus (Y \cap T) \oplus A \oplus B = (X \cap Y) \oplus V \oplus W.$$

This shows that $V \oplus W \subseteq^{\oplus} M$. Inasmuch as $V \cong W$, we infer that $(V,W,\alpha) \in F$. But since $(A,B,f) \leqslant (V,W,\alpha)$, we infer from the maximality of (A,B,f), that $(X \cap T) \oplus A = A$ and $(Y \cap T) \oplus B = B$. But this implies that $(X \cap T) = (Y \cap T) = 0$, which in turn gives $X = X \cap Y = Y = K$, a contradiction. So far, we have proved that $M = K \oplus L$, where K is an $SDSF$-module and $L := A \oplus B$ is semisimple.

Now, consider the set:

$$\Omega = \{(C,D,h) : C \subseteq^{\oplus} K, D \subseteq^{\oplus} L \text{ and} C \overset{h}{\cong} D\}.$$

Order Ω as follows: $(C,D,h) \leqslant (C_1,D_1,h_1)$ if $C \subseteq C_1$, $D \subseteq D_1$ and h_1 extends h. Clearly, Ω is a non-empty inductive set. Again with an argument similar to the one provided in the first part of the proof we can show that every chain of elements in Ω has an upper bound, and so by Zorn's lemma, Ω has a maximal element (C,D,h), say. Write $M = Q \oplus C \oplus A \oplus B$ where $K = Q \oplus C$ and $Q \subseteq M$. Set $P = C \oplus A \oplus B$. Clearly, Q is an $SDSF$-module; and since $C \cong D \subseteq^{\oplus} A \oplus B$, we infer that $P = C \oplus A \oplus B$ is semisimple, completing the proof of both (1) & (2).

Next, we will show that Q and P are factor-orthogonal. By the Lemma 2.6, we only need to show that Q and P are summand-orthogonal. To see this, let $X \cong Y$ with $X \subseteq^{\oplus} Q$ and $Y \subseteq^{\oplus} P$. Write, $L = A \oplus B = D \oplus E$ for a submodule E of L. Thus $P = C \oplus D \oplus E$ with $C \cong D$. Since P is an exchange module, it satisfies the internal exchange property, and so we can write $P = Y \oplus C_1 \oplus D_1 \oplus E_1$ where

$C = C_1 \oplus C_2$, $D = D_1 \oplus D_2$ and $E = E_1 \oplus E_2$. Clearly, $Y \cong C_2 \oplus D_2 \oplus E_2$, and so we can write $Y = Y_C \oplus Y_D \oplus Y_E$ with $Y_C \cong C_2$ $Y_D \cong D_2$ and $Y_E \cong E_2$. Since $C \cong D$, we can also write $C = C_3 \oplus C_4$ with $Y_D \cong D_2 \cong C_3$. Now, as $X \cong Y$, write $X = X_C \oplus X_D \oplus X_E$ with $X_C \cong Y_C \cong C_2$, $X_D \cong Y_D \cong D_2 \cong C_3$ and $X_E \cong Y_E \cong E_2$. Since X_C and C_2 are isomorphic direct summands of the $SDSF$-module $K = Q \oplus C$, it follows that $X_C = C_2 = 0$. Similarly, $X_D = C_3 = 0$. Therefore, $X_C = Y_C = C_2 = X_D = Y_D = D_2 = C_3 = 0$, from which it follows that $X = X_E$, $Y = Y_E$, $C = C_1$ and $D = D_1$. Now, since $X \cong E_2$ and $C \overset{h}{\cong} D$, there is an obvious isomorphism $g : X \oplus C \to E_2 \oplus D$ extending h. Inasmuch as $X \oplus C \subseteq^{\oplus} K$ and $E_2 \oplus D \subseteq^{\oplus} L$, it follows that $(X \oplus C, E_2 \oplus D, g) \in \Omega$. Now, since (C, D, h) is a maximal element of Ω, we must have $X \oplus C = C$, and so $X = 0$, completing the proof of (3).

Now, to establish (4), let $g : Q \longrightarrow P$ be a homomorphism. Therefore, $Q/\ker g \cong \operatorname{Im} g \subseteq^{\oplus} P$ since P is semisimple. So $Q/\ker g \cong P/T$ for a submodule T of P. Since P and Q are factor orthogonal, $\ker g = Q$ and hence $g = 0$, as required.

Finally, to prove (5), let $f \in Hom(P, Q)$ and assume to the contrary that $\operatorname{Im} f$ is not small in Q. Let L be a proper submodule of Q with $Q = \operatorname{Im} f + L$. Clearly, $\frac{Q}{L} \cong \frac{\operatorname{Im} f}{\operatorname{Im} f \cap L}$ and so the obvious epimorphism $P \longrightarrow \operatorname{Im} f \longrightarrow \frac{\operatorname{Im} f}{\operatorname{Im} f \cap L} \cong \frac{Q}{L} \to 0$ induces an isomorphism $\frac{P}{K} \cong \frac{Q}{L}$ for some submodule K of P, a clear contradiction to (3). This shows that $\operatorname{Im} f \ll Q$, as required. □

Corollary 3.2. *Let M be a right $FD4$-module whose local summands are summands. If $rad(M) = 0$, then in addition to the aforementioned decomposition of M in Theorem 3.1 above, we also have $Hom(P, Q) = 0$.*

Corollary 3.3. *Let M be a right $FD4$-module whose local summands are summands. If M is a $C4$-module, then in addition to the aforementioned decomposition of M in Theorem 3.1 above, we also have Q is an SSF-module and $Hom(P, Q) = 0$.*

Proof. Since M is a $C4$-module, Q is a $C4$-module. Inasmuch as Q is both a $C4$- and an $SDSF$-module, we infer form [2, Proposition 5.13], that Q is an SSF-module. Now, let $f \in Hom(P, Q)$. Since P is semisimple, $\ker f \subseteq^{\oplus} P$ and so $\operatorname{Im} f \cong T \subseteq^{\oplus} P$. Since M is a $C4$-module, $\operatorname{Im} f \subseteq^{\oplus} Q$. By (3) of Theorem 3.1, $\operatorname{Im} f = T = 0$, and so $Hom(P, Q) = 0$, as required. □

Corollary 3.4. *Let R be a ring. If R/J is a right $CD4$-ring whose local summands are summands, then R/J is a direct product if a semisimple artinian ring and a right $SDSF$-ring.*

4 Rings whose cyclics are DU-modules

In [11], a right R-module M is called U-module (after the well-known mathematician Utumi) if, whenever A and B are submodules of M with $A \cong B$ and $A \cap B = 0$, there exist two summands K and T of M such that $A \subseteq^{ess} K$, $B \subseteq^{ess} T$ and $K \oplus T \subseteq^{\oplus} M$. A ring R is called a right U-ring if R_R is a U-module. In [16], a ring R is called right CU-ring if every cyclic right R-module is a U-module.

In [13], a right R-module M is called Dual-Utumi module (DU-module for short) if, for any two proper submodules A and B of M with $M/A \cong M/B$ and $A + B = M$, there exist two summands K and L of M such that A lies over K, B lies over L and $K \cap L \subseteq^{\oplus} M$. A ring R is called a right DU-ring if R_R is a DU-module.

Definition 4.1. A right R-module M is called an FDU-module, if every factor of M is a DU-module. A ring R is called a right CDU-ring if R, as a right R-module, is an FDU-module.

Example 4.2. Since the class of DSF-modules is closed under factors, and every DSF-module is a DU-module, then clearly every DSF-module is an FDU-module.

Lemma 4.3. *If $M = X + Y$ with $X \cap Y \subseteq^{\oplus} M$, then $X \subseteq^{\oplus} M$ and $Y \subseteq^{\oplus} M$.*

Proof. Write, $K = (X \cap Y) \oplus T$. Therefore, clearly, $X = (X \cap Y) \oplus (X \cap T)$ & $Y = (X \cap Y) \oplus (Y \cap T)$ and $M = X + Y = [(X \cap Y) \oplus (X \cap T)] + [(X \cap Y) \oplus (Y \cap T)] = (X \cap Y) \oplus (X \cap T) \oplus (Y \cap T)$. Hence $X \subseteq^{\oplus} M$ and $Y \subseteq^{\oplus} M$, as required. □

Lemma 4.4. *If $A \subseteq^{\oplus} M$ and $K \subseteq A$, then $A/K \subseteq^{\oplus} M/K$.*

Proof. Write, $M = A \oplus T$ for a submodule T of M. Consequently, $M/K = A/K + (T+K)/K$, and we are done since $A/K \cap (T+K)/K = 0$. □

A module M is called lifting if every submodule N of M lies over a direct summand of M, and in [13], M is called a Generalized Lifting Module (GL-module) if for any two proper submodules A and B of M with $M/A \cong M/B$ and $A + B = M$, both A and B lie over direct summands of M.

Theorem 4.5. The following are equivalent for a right R-module M:

1. M is an FDU-module.
2. M is both a GL-module and an $FD4$-module.
3. If $M = A + B$ with $M/A \cong M/B$, then $A \cap B \subseteq^{\oplus} M$.
4. If $M = A + B$ with $M/A \cong M/B$, then $A \subseteq^{\oplus} M$ and $B \subseteq^{\oplus} M$.
5. Every epimorphism $f : M \longrightarrow N^2$ splits, where $N^2 := N \oplus N$.

Proof. (1) ⇒ (2). Clear.

(2) ⇒ (3). Let $M = A+B$ with $M/A \cong M/B$, where A and B are submodules of M. By the hypothesis, since M is a DU-module, there exist two direct summands K and L of M such that A lies over K, B lies over L; that is, $A/K \ll M/K$ and $B/L \ll M/L$ with $M = L+K$, $M/K \cong M/L$ and $K \cap L \subseteq^{\oplus} M$. Now, since $M/K \cap L \cong (K/K \cap L) \oplus (L/K \cap L)$ is an $FD4$-module with $M/L \cong (K/K \cap L) \cong (L/K \cap L) \cong M/K$, it follows from Proposition 2.4, that $M/K \cap L$ is semisimple. Therefore, both M/K and M/L are semisimple. This means that, $A/K \subseteq^{\oplus} M/K$ and $B/L \subseteq^{\oplus} M/L$. Inasmuch as $A/K \ll M/K$ and $B/L \ll M/L$, we infer that $A/K = B/L = 0$. Thus $A = K$ and $B = L$, and consequently $A \cap B = K \cap L \subseteq^{\oplus} M$, as required.

(3) ⇒ (4). This follows from Lemma 4.3

(4) ⇒ (1). Let $M/K = A/K + B/K$ with $\frac{M/K}{A/K} \cong \frac{M/K}{B/K}$. Clearly, $M = A+B$ with $\frac{M}{A} \cong \frac{M}{B}$. By the hypothesis, $A \subseteq^{\oplus} M$ and $B \subseteq^{\oplus} M$. Now, by Lemma 4.4 $A/K \subseteq^{\oplus} M/K$, $B/K \subseteq^{\oplus} M/K$ and $\frac{A}{K} \cap \frac{B}{K} = \frac{A \cap B}{K} \subseteq^{\oplus} \frac{M}{K}$. This shows that $\frac{M}{K}$ is a DU-module, and so M is an FDU-module.

(3) ⇒ (5). Let $f : M \to K_1 \oplus K_2$ be an epimorphism with $K_1 \cong K_2$. Then $M = f^{-1}(K_1) + f^{-1}(K_2) = A_1 + A_2$ where $A_i := f^{-1}(K_i)$, $1 \leq i \leq 2$. Clearly, $A_1 \cap A_2 = \ker f$. Thus:

$$M/A_1 = (A_1 + A_2)/A_1 \cong A_2/kerf \cong f(A_2) = K_2 \cong K_1 \cong A_1/kerf \cong M/A_2.$$

Now, by (3), $\ker f = A_1 \cap A_2 \subseteq^{\oplus} M$, as required.

(5) ⇒ (3). Let A and B are submodules of M with $M = A + B$ and $M/A \simeq M/B$. Clearly,

$$M/(A \cap B) = A/(A \cap B) \oplus B/(A \cap B) \cong M/B \oplus M/A \cong (M/A)^2.$$

Therefore, there exists an obvious epimorphism $f : M \longrightarrow (M/A)^2$ with $\ker f = A \cap B$. Now, by (4), $\ker f = A \cap B \subseteq^{\oplus} M$, as required. □

Example 4.6. [12, Example 2.9] If $\{p_i\}_{i \in I}$ is a finite set of distinct prime numbers, then the following are examples of FDU-modules:

1. $M_{\mathbb{Z}} := \mathbb{Z}_{p^\infty} \oplus (\oplus_{i \in I} \mathbb{Z}_{p_i})$;

2. $M_{\mathbb{Z}} := \mathbb{Q} \oplus (\oplus_{i\in I}\mathbb{Z}_{p_i})$;

3. $M_{\mathbb{Z}} := \mathbb{Q}/\mathbb{Z} \oplus (\oplus_{i\in I}\mathbb{Z}_{p_i})$.

Example 4.7. For any prime p, the $\mathbb{Z}$-module $M := \mathbb{Q} \oplus \mathbb{Z}_p$ is an FDU-module that is not quasi-discrete, where a lifting module is called quasi-discrete if it is also a $D3$-module.

Proposition 4.8. *Being FDU is a Morita invariant property of modules.*

Proof. Let R and S be two Morita equivalent rings. Assume that $\mathcal{F} : Mod\text{-}R \to Mod\text{-}S$ and $\mathcal{G} : Mod\text{-}S \to Mod\text{-}R$ are two category equivalences. Let M_R be an FDU-module and $f : \mathcal{F}(M) \to T \oplus T$ be an epimorphism with $T \in Mod\text{-}S$. Now we have the exact sequence

$$0 \to \ker f \to \mathcal{F}(M) \to T \oplus T \to 0.$$

By [1, Proposition 21.4],

$$0 \to \mathcal{G}(\ker f) \to M \to \mathcal{G}(T \oplus T) \cong \mathcal{G}(T) \oplus \mathcal{G}(T) \to 0$$

is exact in $Mod\text{-}R$. Since M_R is a FDU-module, $\mathcal{G}(\ker f) \subseteq^{\oplus} M$, and so $\ker f \subseteq^{\oplus} \mathcal{F}(M)$. By Theorem 4.5, $\mathcal{F}(M)$ is an FDU-module, as reqiured. □

Proposition 4.9. *For $n > 1$, the following conditions on a ring R are equivalent:*

1. *R is a semisimple artinian ring.*
2. *$M_n(R)$ is a right CDU-ring.*
3. *$M_n(R)$ is a right $CD4$-ring.*
4. *$M_2(R)$ is a right $CD4$-ring.*

Proof. $1 \Rightarrow 2$. Clear, since $M_n(R)$ is a semisimple artinian ring.

$2 \Rightarrow 3 \Rightarrow 4$. Clear.

$4 \Rightarrow 1$.Since $M_n(R) = \begin{bmatrix} R & R \\ R & R \end{bmatrix} = \begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ R & R \end{bmatrix}$ is a direct sum of two isomorphic right ideals, we infer from the hypothesis and Proposition 2.4, that $M_n(R)$ is semisimple artinian. Thus R is semisimple artinian. □

Remark 4.10. Being a CDU-ring or a $CD4$-ring is not a Morita Invariant property for rings. For, the ring of integers $\mathbb{Z}$ is a CDU-ring (in particular a $CD4$-ring) that is not semisimple artinian; and so by Proposition 4.9, $M_n(\mathbb{Z})$ is neither a CDU-ring nor a $CD4$-ring.

Proposition 4.11. *If M is a right R-module with $Soc(M) = 0$, then the following statements are equivalent:*

1. *M is a DSF-module;*
2. *M is an FDU-module;*
3. *M is both a DU-mdule and an $FD4$-module.*

Proof. $1 \Rightarrow 2 \Rightarrow 3$. Clear.

$3 \Rightarrow 1$. Let A and B be submodules of M with $M = A + B$ and $M/A \cong M/B$. By [13, Theorem 3.6], since M is a DU-module, there exist two summands T and L of M such that A lies over T, B lies over L, $M/T \cong M/L$, $M = T + L$ and $T \cap L \subseteq^{\oplus} M$. Now, by proposition 2.4, both M/T and M/L are semisimple. If we write $M = T \oplus T'$, for a submodule $T' \subseteq M$, then T' is semisimple. But, since $soc(M) = 0$, $T' = 0$. This shows that $M = T = A$, and M is a DSF-module, as required. □

Theorem 4.12. If M is an FDU-module whose local summands are summands, then $M = Q \oplus P$ where

1. Q is a DSF-module.
2. $P = C \oplus A \oplus B$ is semisimple with $A \cong B$ and C isomorphic to a direct summand of $A \oplus B$.
3. P and Q are factor-orthogonal.
4. $\mathrm{Hom}(Q, P) = 0$.
5. If $f \in Hom(P, Q)$, then $\mathrm{Im}\, f \ll Q$.

   Moreover, if in addition M is a $C4$-module, then $\mathrm{Hom}(P, Q) = 0$.

Proof. From Theorem 3.1 and Corollary 3.3, we only need to prove (1). But this follows from [13] and the fact that Q is both an $SDSF$-module and a DU-module. □

Since in a quasi-discrete module, local summands are summands, see [20, Corollary 4.13], the next corollary is an immediate consequence of Theorem 4.12.

Corollary 4.13. *Let M be a right R-module whose factors are quasi-discrete modules, then every factor module N of M can be written as $N = Q \oplus P$ where Q and P satisfy the conditions (1) through (5) of Theorem 4.12.*

Lemma 4.14. *If $R = R_1 \times R_2$, then R is a right CDU-ring if and only if both R_1 and R_2 are right CDU-rings.*

Proof. ($\Longrightarrow$). This is clear.

($\Longleftarrow$). We use a similar argument to the one provided in [21, Lemma 3.6.] with some modifications. Let I be a right ideal of R. Then $M := R_1/IR_1 \times R_2/IR_2$ is a right R-module, where $(r_1 + IR_1, r_2 + IR_2)r = (r_1 r + IR_1, r_2 r + IR_2)$ for $r_1 \in R_1, r_2 \in R_2$ and $r \in R$. Define the R-isomorphism $\theta : R/I \to M$ by letting $\theta(r+I) = (e_1 r + IR_1, e_2 r + IR_2)$, where $e_1 = 1_{R_1}$ and $e_2 = 1_{R_2}$. Suppose that $K/I, L/I$ are two submodules of $(R/I)_R$ with $R/I = K/I + L/I$ and $R/K \cong \frac{R/I}{K/I} \cong \frac{R/I}{L/I} \cong R/L$. Then $\theta(K/I) = (e_1 K/IR_1, e_2 K/IR)$, $\theta(L/I) = (e_1 L/IR_1, e_2 L/IR)$ and thier sum is M. It follows that, as R_1-modules, $e_1 K/IR_1, e_1 L/IR_1$ are submodules of R_1/IR_1 with $R_1/IR_1 = e_1 K/IR_1 + e_1 L/IR_1$ and that, as R_2-modules, $e_2 K/IR_2, e_2 L/IR_2$ are submodules of R_2/IR_2 with $R_2/IR_2 = e_2 K/IR_2 + e_2 L/IR_2$. Since $\theta : R/I \to M$ is an R-isomorphism, we have an ismorphisms $f : \frac{R/I}{K/I} \to \frac{M}{\theta(K/I)}$ and $g : \frac{R/I}{L/I} \to \frac{M}{\theta(L/I)}$ given by $f[(r+I) + K/I] = \theta(r+I) + \theta(K/I)$ and $g[(r+I)+L/I] = \theta(r+I)+\theta(L/I)$, respictively. Now, since $R/K \cong \frac{R/I}{K/I} \cong \frac{R/I}{L/I} \cong R/L$, we have $\frac{M}{\theta(K/I)} \cong \frac{M}{\theta(L/I)}$. But $\frac{M}{\theta(K/I)} = \frac{R_1/IR_1 \times R_2/IR_2}{(e_1 K/IR_1, e_2 K/IR)} = \frac{R_1/IR_1 \times R_2/IR_2}{e_1 K/IR_1 \times e_2 K/IR} \cong \frac{R_1/IR_1}{e_1 K/IR_1} \times \frac{R_2/IR_2}{e_2 K/IR} \cong \frac{R_1}{e_1 K} \times \frac{R_2}{e_2 K}$, and similarly, $\frac{M}{\theta(L/I)} \cong \frac{R_1}{e_1 L} \times \frac{R_2}{e_2 L}$. Thus $\frac{R_1}{e_1 K} \times \frac{R_2}{e_2 K} \cong \frac{R_1}{e_1 L} \times \frac{R_2}{e_2 L}$. Let $\varphi : \frac{R_1}{e_1 K} \times \frac{R_2}{e_2 K} \longrightarrow \frac{R_1}{e_1 L} \times \frac{R_2}{e_2 L}$ be the obtained isomorphsim and define $\lambda : \frac{R_1}{e_1 K} \longrightarrow \frac{R_1}{e_1 L}$ by $\lambda(r_1 + e_1 K) = r_1' + e_1 K$ where $\varphi(r_1 + e_1 K,\, 0 + e_1 L) = (r_1' + e_1 K,\, 0 + e_1 L)$ (Note that if $\varphi(r_1 + e_1 K,\, 0 + e_1 L) = (r_1' + e_1 K,\, r_2 + e_1 L)$, then $\varphi(r_1 + e_1 K,\, 0 + e_1 L) = \varphi(r_1 e_1 + e_1 K,\, 0.e_1 + e_1 L) = \varphi(r_1 + e_1 K,\, 0 + e_1 L)e_1 = (r_1' + e_1 K, r_2 + e_1 L)e_1 = (r_1' e_1 + e_1 K, r_2 e_1 + e_1 L) = (r_1' + e_1 K, 0 + e_1 L))$. Now, clearly, λ is an R_1-isomorphism. Inasmuch as $R_1/IR_1 = e_1 K/IR_1 + e_1 L/IR_1$, $\frac{R_1/IR_1}{e_1 K/IR_1} \cong \frac{R_1}{e_1 K} \cong \frac{R_1}{e_1 L} \cong \frac{R_1/IR_1}{e_1 L/IR_1}$ and R_1 is a CDU-ring, we infer that $e_1 K/IR_1 \cap e_1 L/IR_1$ is a direct summand of R_1/IR_1 by Theorem 4.5. Similarly, $e_2 K/IR_2 \cap e_2 L/IR_2$ is a summand of R_2/IR_2.Therefore, $\theta(K/I \cap L/I) = \theta(K/I) \cap \theta(L/I)$ is a direct summand of M. Thus $K/I \cap L/I$ is a direct summand of $(R/I)_R$. Consequently, by Theorem 4.5, $(R/I)_R$ is a DU-module and Hence R is a CDU-ring. □

Theorem 4.15. If R is a right $C4$-ring whose local summands are summands, then the following are equivalent:

1. R is a right CDU-ring.

2. R is a direct product of a semisimple artinian ring and a right DSF-ring.

Proof. $(2) \Rightarrow (1)$. Follows from Lemma 4.14.

$(1) \Rightarrow (2)$. By Theorem 4.12, $R = eR \oplus (1-e)R$ with eR semisimple artinian and $(1-e)R$ is a right DSF-ring. Moreover, $Hom(eR,(1-e)R) = Hom((1-e)R, eR) = 0$. The later statement says that both eR and $(1-e)R$ are ideals, completing the proof. □

Recall that a ring R is called right quasi-duo if maximal right ideals of R are two-sided. It was shown in [12, Corollary 2.17] that, R is right quasi-duo iff R is right DSF-ring.

Corollary 4.16. *If R is a semiperfect right $C4$-ring, then the following are equivalent:*

1. *R is a right CDU-ring.*

2. *R is a right $CD4$-ring.*

3. *R is a direct product of a semisimple artinian ring and a right quasi-duo ring.*

4. *R is a direct product of a semisimple artinian ring and a left quasi-duo ring.*

   Moreover, in this case R is also a left CDU-ring.

Proof. $(1) \Rightarrow (2)$. Clear.

$(2) \Rightarrow (3)$. By Corollary 3.3, $R = eR \oplus (1-e)R$, where eR is semisimple artinian, $(1-e)R$ is a right $SDSF$-ring and $Hom(eR,(1-e)R) = Hom((1-e)R, eR) = 0$. Since R is semiperfect, $(1-e)R$ is a lifting module. Now, by [7, Lemma 5.5], $(1-e)R$ is a right DSF-module. Inasmuch as eR & $(1-e)R$ are ideals of R, we infer that eR is a semisimple artinian ring and $(1-e)R$ is a right DSF-ring. Now, by [12, Corollary 2.17], $(1-e)R$ is right quasi-duo.

$(3) \Rightarrow (1)$. By Lemma 4.14.

$(3) \Leftrightarrow (4)$. By [12, Corollaries 2.17 and 3.7], a semiperfect ring is right quasi-duo if and only if it is left quasi-duo.

Now, by (4) and Lemma 4.14, R is also a left CDU-ring. □

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