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Generalized module groupoids

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Abstract. In this paper , we present the concept of module over a generalized ring which we call a "generalized module" and discuss some interesting properties of generalized modules together with examples. Further, we define generalized module groupoid and describe the categorical relations between the category gM of generalized modules and the category $gM\mathcal{G}$ of generalized module groupoids.

Key Words: Category, Functors, Groupoid, Generalized group, Generalized group groupoid, Generalized module, Generalized module groupoid.
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1 Introduction

The generalized groups introduced by Molaei is an interesting generalization of groups[4]. It is well known that identity element in a group is unique, but in a generalized group there exists an identity element for each element. Clearly every group is a generalized group. A groupoid is another generalization of a group is a small category in which every morphism is invertible and was first defined by Brandt in the year 1926. Groupoids are studied by many mathematicians with different objective, one of the different approach is the structured groupoid which is obtained by adding another structure in such a way that the added structure is compatible with the groupoid operation.

In this paper we introduce the module action on a generalized group and we call the resulting structure as a generalized module and then we discuss some interesting examples and properties of generalized modules. Further, in an analogous way to generalized group groupoid we describe the generalized module groupoid over a generalized ring and obtain a relation between the category of generalized module and the category of generalized module groupoids.

2 Preliminaries

In this section we briefly recall all basic definitions and the elementary concepts needed in the sequel. In particular we recall the definitions of categories, groupoids, generalized groups with examples and discuss some interesting properties of these structures. For a more detailed discussion, the reader is referred to the book of S.Maclane [3] and [4].

A category $\mathcal{C}$ is a class of objects denoted $v\mathcal{C}$ together with a collection of disjoint classes, denoted by $\mathcal{C}(a,b)$; one for each pair (a,b) of objects in $v\mathcal{C}$. An element f of $\mathcal{C}(a,b)$ is called a morphism from a to b. For each triple (a,b,c) of objects in $\mathcal{C}$, a composition function $\mathcal{C}(a,b)\times\mathcal{C}(b,c)\to\mathcal{C}(a,c)$ is defined. Given morphisms $f:a\to b$ and $g:b\to c$, their composition will be written fg. The associativity of composition and existence of identities are assumed in a category. Existence of identity means that for each object $a\in v\mathcal{C}$ there exists $1_a\in\mathcal{C}(a,a)$ such that $f1_a=f$ and $1_ag=g$ for all $f\in\mathcal{C}(b,a)$ and $g\in\mathcal{C}(a,c)$. We shall often identify the identity morphism 1_a at an object $a\in v\mathcal{C}$ with the object a.

With this convention, the morphisms of a category $\mathcal{C}$ completely determine $\mathcal{C}$, having this in mind, we shall denote the class of all morphisms of $\mathcal{C}$ by $\mathcal{C}$ itself.

A category $\mathcal{C}$ is called small if both $v\mathcal{C}$ and $\mathcal{C}$ are sets rather than proper classes. A morphism $f : a \to b$ is called an isomorphism if there exists a morphism $g : b \to a$ such that $fg = 1_a$ and $gf = 1_b$. A small category in which every morphisms are isomorphisms is called a groupoid. For a category $\mathcal{C}$, a subcategory of $\mathcal{C}$ is a category $\mathcal{D}$ whose objects are objects in $\mathcal{C}$ and morphisms are morphisms in $\mathcal{C}$, with the same identities and composition of morphisms.

Given two categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ consists of two functions: the object function denoted as vF which assigns to each object a of $\mathcal{C}$, an object $F(a)$ of the category $\mathcal{D}$ and the morphism function denoted by F itself which assigns to each morphism $f : a \to b$ of $\mathcal{C}$, a morphism $F(f) : F(a) \to F(b)$ in $\mathcal{D}$ which should preserves identities and composition, ie., $F(1_c) = 1_{F(c)}$ and $F(fg) = F(f)F(g)$.

Example 2.1. A group G can be regarded as a category $\mathcal{C}$ in the following way;
the object set of $\mathcal{C}$ say $v\mathcal{C} = G$. Define $\mathcal{C}(G, G) = G$ and composition in $\mathcal{C}$ is the binary operation in G. Identity element in the group will be the identity morphism on the vertex G.

Example 2.2. Every group can be regarded as a groupoid with only one object.

Example 2.3. For a set X the Cartesian product $X \times X$ is a groupoid over X with morphisms are the elements in $X \times X$ with the composition $(x, y) \cdot (u, v)$ exists only when $y = u$ and is given by $(x, y)(u, v) = (x, v)$. In particular (x, x) is the unique left identity and (y, y) is the unique right identity.

We denote by **Gpd** the category of groupoids in which objects are the groupoids and morphisms are the functors.

2.1 Generalized group and generalized group groupoids

In this section, we recall the elementary concepts of the generalized group theory. Moreover, we recall some basic properties of generalized group-groupoid which is a generalized group object in the category of groupoids.

Definition 2.4 ([4]). A generalized group G is a non-empty set together with a binary operation called multiplication subject to the set of rules given below:

1. $(ab)c = a(bc)$ for all $a, b, c \in G$.
2. For each $a \in G$ there exists unique $e(a) \in G$ with $ae(a) = e(a)a = a$.
3. For each $a \in G$ there exists $a^{-1} \in G$ with $aa^{-1} = a^{-1}a = e(a)$.

It is seen that for each element a in a generalized group the inverse is unique and both a and a^{-1} have the same identity. Every abelian generalized group is a group.

Definition 2.5 ([4]). A generalized group G is said to be normal generalized group if $e(ab) = e(a)e(b)$ for all elements $a, b \in G$.

Definition 2.6 ([4]). A non-empty subset H of a generalized group G is a generalized subgroup of G if and only if for all $a, b \in H$, $ab^{-1} \in H$.

Example 2.7. $G = \left\{ A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a \neq 0,\ a,\ b\ \in \mathbb{R} \right\}$
Then, G is a generalized group and for all $A \in G$,

$$e(A) = \begin{pmatrix} 1 & b/a \\ 0 & 0 \end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} 1/a & b/a^2 \\ 0 & 0 \end{pmatrix}$$

where $e(A)$ and A^{-1} are the identity and the inverse of a matrix A, respectively. Also $e(AB) = e(B)$ for all $A, B \in G$.

Example 2.8. Let G be a generalized group with the multiplication m. Then, $G \times G$ with the multiplication as follows

$$m_1((a,b),(c,d)) = (m(a,c), m(b,d))$$

is a generalized group. For any element $(a,b) \in G \times G$, the identity element is $e_1(a,b) = (e(a), e(b))$ and the inverse element is $(a,b)^{-1} = (a^{-1}, b^{-1})$.

Definition 2.9 ([4]). Let G and H be two generalized groups. A generalized group homomorphism from G to H is a map $f : G \to H$ such that

$$f(ab) = f(a)f(b).$$

Theorem 2.10 ([4]). Let $f : G \to H$ be a homomorphism of generalized groups G and H. Then

(a) $f(e(a)) = e(f(a))$ is an identity element in H for all $a \in G$.

(b) $f(a^{-1}) = f(a)^{-1}$.

(c) If K is a generalized subgroup of G then $f(K)$ is a generalized subgroup of H.

Now we recall the definition of generalized group groupoid which is a generalized group object in the category of groupoids.

Definition 2.11. [2]. A generalized group groupoid $\mathcal{G}$ is a groupoid endowed with a structure of generalized group such that the following maps:

1. $+ : G \times G \to G,\ (f,g) \to f + g,$
2. $u : G \to G,\ f \to -f,$
3. $e : G \to G,\ f \to e(f),$

are functorials.

Since + is a functorial we have,

$$\begin{aligned} (f \circ g) + (h \circ k) &= +(f \circ g, h \circ k) \\ &= +[(f,h) \circ (g,k)] \\ &= +(f,h) \circ +(g,k) \\ &= (f+h) \circ (g+k). \\ (f \circ g) + (h \circ k) &= (f+h) \circ (g+k). \end{aligned}$$

Thus the interchange law

$$(f \circ g) + (h \circ k) = (f+h) \circ (g+k)$$

exists between groupoid composition and generalized group operation.
In other words, a generalized group groupoid is a groupoid endowed with a structure of generalized group such that the structure maps of groupoid are generalized group homomorphisms.

Example 2.12 ([2]). Let G be a generalized group. Then $G \times G$ is a generalized group groupoid with object set G. It follows from the example 2.3 that $G \times G$ is a groupoid with object set G. For each morphism $(x,y) \in G \times G$ the identity arrow of (x,y) is $(e(x),e(y))$ and the inverse is $(-x,-y)$ and the interchange law also holds. For,

$$\begin{aligned}
[(f,g)\circ(g,h)]+[(f',g')\circ(g',k')] &= (f,h)+(f',h')\\
&= (f+f',h+h')\\
[(f,g)+(f',g')]\circ[(g,h)+(g',h')] &= (f+f',g+g')\circ(g+g',h+h')\\
&= (f+f',h+h')\\
[(f,g)\circ(g,h)]+[(f',g')\circ(g',k')] &= [(f,g)+(f',g')]\circ[(g,h)+(g',h')].
\end{aligned}$$

Now we give the definition of a generalized ring.

Definition 2.13 ([5]). A generalized ring R is a nonempty set R with two different operations addition and multiplication denoted by '$+$' *and* '$\times$'respectively in which $(R,+)$ is a generalized group and satisfies the following conditions.

1. multiplication is an associative binary operation.
2. For all $x,y,z \in R$, $x(y+z) = xy+xz$ and $(x+y)z = xz+yz$.

Example 2.14 ([6]). R^2 with the operations $(a,b)+(c,d) = (a,d)$ and $(a,b)(c,d) = (ac,bd)$ is a generalized ring.

A generalized ring with its operations is a ring if and only if e is a constant function. If there is $1 \in R$ such that $x \cdot 1 = 1 \cdot x = x$, for all $x \in R$, then R is called a generalized ring with identity. Also one can easily prove that identity of a generalized ring is unique.

3 Generalized module and generalized module groupoid

In this section we present the concept of generalized module and generalized module groupoid. We will consider these concepts under two headings: generalized module and generalized module groupoid.

3.1 Generalized module

Let M be a generalized group and R be a generalized ring, in the following we proceed to define a generalized module using the generalized group M.

Definition 3.1. Let R be a generalized ring with unity. A generalized group M is said to be *(left)* generalized R module if for each element r in R and each m in M we have a product rm in M such that for $r,s \in R$ and $m,n \in M$,

1. $(r+s)m = rm+sm$.
2. $r(m+n) = rm+rn$.
3. $r(sm) = (rs)m$.
4. $1.m = m$.

Lemma 3.2. *If M is a generalized R module then*

(1) $e(rm) = re(m)$ *for all* $r \in R, m \in M$.

(2) $(rm)^{-1} = r(m^{-1})$.

Proof. Let $r \in R$ and $m \in M$, then

$$rm + re(m) = r(m + e(m)) = rm.$$

Similarly consider,

$$re(m) + rm = r(e(m) + m) = rm.$$

Hence $e(rm) = re(m)$.
Also,

$$rm + rm^{-1} = r(m + m^{-1}) = re(m).$$

$$rm^{-1} + rm = r(m^{-1} + m) = re(m).$$

Thus $(rm)^{-1} = r(m^{-1})$. □

Example 3.3. Every generalized ring R can be considered as a generalized module over itself. The action of $r \in R$ on $x \in R$ is given by $r \cdot x = rx$ the product obtained by using the multiplication in R.

Example 3.4. Let $M = \mathbb{R} \times \mathbb{R}$ where $\mathbb{R}$ is the set of all real numbers. Then M with the operation $+$ defined by $(x, y) + (m, n) = (x, y + n)$ is a generalized group in which for all $(m, n) \in M$, $e(m, n) = (m, 0)$ and $(m, n)^{-1} = (m, -n)$.
Indeed,

$$(m, n) + (m, 0) = (m, n + 0) = (m, n).$$

$$(m, 0) + (m, n) = (m, 0 + n) = (m, n).$$

On the other hand,

$$(m, n) + (m, -n) = (m, n - n) = (m, 0).$$

And,

$$(m, -n) + (m, n) = (m, -n + n) = (m, 0).$$

Moreover M is a generalized $\mathbb{R}$-module with the scalar multiplication $r \cdot (m, n) = (m, rn)$. We have the map $\cdot : \mathbb{R} \times M \to M$, $r(m, n) \to (m, rn)$ satisfying the conditions in Definition 3.1. For if, let $r \in \mathbb{R}$ and $(m, n), (x, y) \in M$ we have

$$(r + s)(x, y) = (x, (r + s)y) = (x, ry + rs).$$

$$r(x, y) + s(x, y) = (x, ry) + (x, rs) = (x, ry + rs).$$

Thus $(r + s)(x, y) = r(x, y) + s(x, y)$.
On the other hand,

$$r[(x, y) + (m, n)] = r[(x, y + n)] = (x, r(y + n)) = (x, ry + rn).$$

And

$$r(x, y) + r(m, n) = (x, ry) + (m, rn) = (x, ry + rn).$$

Hence $r[(x, y) + (m, n)] = r(x, y) + r(m, n)$. Moreover we have,

$$rs(x, y) = (x, rsy) = (x, r(sy)) = r(x, sy) = r(s(x, y))$$

and $1 \cdot (x, y) = (x, 1.y) = (x, y)$. Thus we proved that $M = \mathbb{R} \times \mathbb{R}$ is a generalized $\mathbb{R}$- module.

Proposition 3.5. *Let M be an R- module then $M \times M$ is a generalized R module with the operations*

$$(x,y)+(m,n)=(x,y+n)$$

$$r(x,y)=(x,ry)$$

and for each $x \in M$ the subset $M_x = \{(x,y) : y \in M\}$ is an R module.

Proof. $(M,+)$ is a generalized group in which for all $m,n \in M$, $e(m,n)=(m,0)$ and $(m,n)^{-1}=(m,-n)$. Moreover M is a generalized R module since for $r,s \in R$ and $(x,y),(m,n) \in M \times M$,

$$(r+s)(x,y)=(x,(r+s)y)=(x,ry+rs).$$

$$r(x,y)+s(x,y)=(x,ry)+(x,rs)=(x,ry+rs).$$

Thus $(r+s)(x,y)=r(x,y)+s(x,y)$.

$$r[(x,y)+(m,n)]=r[(x,y+n)]=(x,r(y+n))=(x,ry+rn).$$

$$r(x,y)+r(m,n)=(x,ry)+(m,rn)=(x,ry+rn).$$

Hence $r[(x,y)+(m,n)]=r(x,y)+r(m,n)$.

$$rs(x,y)=(x,rsy)=(x,r(sy))=r(x,sy)=r(s(x,y))$$

and

$$re(x,y)=r(x,0)=(x,r0)=(x,0)=e(x,y).$$

Hence $re(x,y)=e(x,y)$.

$$1\cdot(x,y)=(x,y) \quad \forall (x,y) \in M.$$

To show that for each $x \in M$ the subset $M_x=\{(x,y) : y \in M\}$ is an R module it is enough to show that M_x is an abelian group and the scalar multiplication is closed. Let $m=(x,y)$ and $n=(x,z)$ be elements in M_x then

$$m+n=(x,y)+(x,z)=(x,y+z)$$

and

$$n+m=(x,z)+(x,y)=(x,z+y)=(x,y+z).$$

Therefore

$$m+n=n+m$$

and $+$ is a commutative binary operation on M_x. For every $m \in M_x$, $(x,y)+(x,0)=(x,0)+(x,y)=(x,y)$ and $(x,y)^{-1}=(x,-y)$. Thus M_x is an abelian subgroup of M and for any $r \in R$ $r(x,y)=(x,ry) \in M_x$. Hence M_x is a R- module. □

Definition 3.6. Let M and N be two generalized R-modules. A function $f : M \to N$ is called generalized module homomorphism if

$$f(m+n) = f(m)+f(n) \text{ for all } m,n \in M.$$

$$f(rm)=rf(m), \text{ for all } r \in R,\ m \in M.$$

Definition 3.7. Let M be a generalized R module and $N \subseteq M$ is said to be generalized submodule of M if N is a generalized subgroup of G and for each $r \in R$ *and* $m \in N$, $rm \in N$.

Proposition 3.8. *Let M be a generalized R module which is also a normal generalized group. Then the set of identity elements in M is a submodule of M and we call it as the zero submodule or trivial submodule.*

Proof. We denote the set of identity elements in M by

$$e(M) = \{e(x) \ : \ x \in M\}.$$

First we show that $e(M)$ is a generalized subgroup of M. For if, $m, n \in e(M)$ there exists $x, y \in M$ such that $m = e(x)$ and $n = e(y)$. Now consider,

$$mn^{-1} = e(x)e(y)^{-1} = e(x)e(y) = e(xy).$$

Hence $mn^{-1} \in e(M)$ and $e(M)$ is a generalized subgroup of M.
Let r be an element in the ring R and

$$rm = re(x) = e(rx).$$

Hence $rm \in e(M)$ for all $r \in R$ and $m \in M$. Thus $e(M)$, set of identity elements of M form a generalized submodule of M. □

Every nonzero generalized module M contains at least one submodule M itself.

Theorem 3.9. Let R is a generalized ring and M and N are normal generalized R- modules. If $f : M \to N$ is a generalized R- module homomorphism, then

$$kerf = \{x \in M : \ f(x) \in e(N)\}$$

where $e(N)$ denotes the set of identity elements of N, is a submodule of M and $Imf = \{f(x) : \ x \in M\}$ is a submodule of N.

Proof. Let M and N be two generalized modules over the generalized ring R and f be a homomorphism between M and N. To show that $kerf$ is a submodule of M it is suffices to prove that it is a generalized subgroup of M and is closed under scalar multiplication. Let $x, y \in kerf$ then $f(x) = e(n)$ and $f(y) = e(n')$ for some $n, \ n' \in N$. Now consider,

$$\begin{aligned} f(xy^{-1}) &= f(x)f(y^{-1}) \\ &= f(x)(f(y))^{-1} \\ &= e(n)e(n')^{-1} \\ &= e(n)e(n') \\ &= e(nn'). \end{aligned}$$

Hence $xy^{-1} \in kerf$ and $kerf$ is a generalized subgroup of M. For any $r \in R, m \in M$,

$$f(rx) = rf(x) = re(n) = e(rn).$$

Therefore $rx \in kerf$ and is a submodule of M for every $m \in M$. Similarly we can prove that Imf is a submodule of N. For; let $x, y \in Imf$ then there exists $m, n \in M$ such that $f(m) = x$ and $f(n) = y$. Now consider,

$$\begin{aligned} xy^{-1} &= f(m)f(n)^{-1} \\ &= f(m)f(n^{-1}) \\ &= f(mn^{-1}) \text{ and } mn^{-1} \in M. \end{aligned}$$

Hence $xy^{-1} \in Imf$ and Imf is a generalized subgroup of N and for any $r \in R$,

$$rx = rf(m) = f(rm).$$

Hence $rx \in Imf$, for any $r \in R$ and $x \in M$. Therefore Imf is a generalized submodule of N.

□

The generalized modules and their homomorphisms form a category in which objects are the generalized modules and morphisms are their homomorphisms denoted by $\mathcal{GM}$.

Product of generalized modules If M and N be two generalized modules over a ring R then their Cartesian product $M \times N$ defined by

$$M \times N = \{(m,n) : \ \ m \in M, \ \ n \in N\}$$

is a generalized module over R with respect to the component wise operations. *ie*; for any (m,n), $(x,y) \in M \times N$ and $r \in R$ the addition and scalar multiplication is given by

$$(m,n) + (x,y) = (m+x, n+y) \text{ and } r(m,n) = (rm, rn).$$

Definition 3.10. let $\mathcal{G}$ be a groupoid and R be a generalized ring. A groupoid $\mathcal{G}$ is said to be generalized module groupoid over R if it has a generalized module structure over R and it satisfies the following conditions.

(1) $\mathcal{G}$ is a generalized group groupoid.

(2) For each $r \in R$ the mapping

$$\eta_r : \mathcal{G} \to \mathcal{G}$$

defined by $\eta_r(g) = rg$ is a functor on $\mathcal{G}$.
ie; for any composable morphisms g, h in $\mathcal{G}$ and any $r \in R$ we should have

$$r(g \circ h) = rg \circ rh.$$

Example 3.11. Let M be a generalized module over a ring R. Then $M \times M$ is a generalized module groupoid over R with object set M. It follows from the example 2.12 that $M \times M$ with the operation $(x,y) + (m,n) = (x+m, y+n)$ is a generalized group groupoid. Also we have seen that the Cartesian product of two generalized modules are again a generalized module. Therefore $M \times M$ has a structure of both groupoid and generalized module. In order to show that $M \times M$ is a generalized module groupoid over R it is suffices to prove that for any $r \in R$, the map $\eta_r : M \times M \to M \times M$ defined by $\eta_r(x,y) = (rx, ry)$ is a functor. For, let $x \in M$ be any object in the category $M \times M$ then

$$\eta_r(1_x) = \eta_r(x,x) = (rx, rx) = 1_{\eta_r(x)}.$$

Consider two composable morphisms $(x,y), (y,z)$ in $M \times M$ then,

$$\eta_r[(x,y) \circ (y,z)] = \eta_r[(x,z)] = (rx, rz)$$

and

$$\eta_r(x,y) \circ \eta_r(y,z) = (rx, ry) \circ (ry, rz) = (rx, rz).$$

Thus,

$$\eta_r[(x,y) \circ (y,z)] = \eta_r(x,y) \circ \eta_r(y,z).$$

Hence $M \times M$ is a generalized module groupoid over R.

Definition 3.12. Let M and N be two generalized module groupoids over a ring R. A homomorphism $f : M \to N$ of generalized module groupoids is a functor of underlying groupoids preserving generalized module structure.

Thus the generalized module groupoids and their homomorphisms form a category denoted by $\mathcal{GMG}$

Theorem 3.13. Let gM be the category of generalized modules and $gM\mathcal{G}$ be the category of generalized module groupoids. Then $F : gM \to gM\mathcal{G}$ defined by

$$F(M) = M \times M$$

and for any generalized module homomorphism $f : M_1 \to M_2$

$$F(f) : M_1 \times M_1 \to M_2 \times M_2$$

such that

$$F(f)(m, n) = (f(m), f(n))$$

is a functor.

Proof. Let M be a generalized module over a ring R. Then the cartesian product $M \times M$ is a generalized module over R. If $f : M_1 \to M_2$ is a homomorphism of generalized modules then define $F : gM \to gM\mathcal{G}$ by

$$F(M) = M \times M$$

and

$$F(f) : M_1 \times M_1 \to M_2 \times M_2,$$

$$F(f)(m, n) = (f(m), f(n)).$$

It can be seen that $F(f)$ between $M_1 \times M_1$ and $M_2 \times M_2$ is a morphism in the category of generalized module groupoids. It can be seen that F is a functor between the category of generalized modules and the category of generalized module groupoids. For, $M \in gM$ we have,

$$F(1_M)(m, n) = (m, n).$$

$$F(1_M) = 1_{F(M)}.$$

And for $f : M \to N, g : N \to P$ the two composable homomorphisms in gM, for all $m, n \in M$, we have

$$\begin{aligned} F(fg)(m, n) &= (fg(m), fg(n)) \\ &= ((g(f(m)), g(f(n))). \\ F(f)F(g)(m, n) &= F(g)(F(f)(m, n)) \\ &= F(g)(f(m), f(n)) \\ &= (g(f(m)), g(f(n))). \\ F(fg)(m, n) &= F(f)F(g)(m, n). \end{aligned}$$

Hence, $F(fg) = F(f)F(g)$. □

Proposition 3.14. *Let $\{M_i\}_{i \in I}$ be a family of generalized module groupoids over the generalized ring R. Then $M = \prod M_i$ is a generalized R-module groupoid.*

Proof. $\prod \nu M_i$ has elements $(m_i)_{i\in I}$ where $m_i \in \nu M_i$ morphisms of M are all tuples $(f_i)_{i\in I}$ from $(dom f_i)$ to $(cod f_i)$ in M_i. The compositon of morphisms is

$$(f_i)_{i\in I} \cdot (g_i)_{i\in I} = (f_i g_i)_{i\in I}$$

whenever f_i and g_i are composable morphisms in M_i. Since each M_i is a groupoid each morphism f_i admits an inverse f_i^{-1} in M_i, hence $(f_i^{-1})_{i\in I}$ is the inverse of the morphism $(f_i)_{i\in I}$ in M. Thus the product $M = (\prod \nu M_i, \prod M_i)_{i\in I}$ is a groupoid. Moreover the product $M = \prod M_i$ has a structure of generalized module with respect to component wise operations,

$$(f_i)_{i\in I} + (g_i)_{i\in I} = (f_i + g_i)_{i\in I} \ \ and \ \ r(f_i)_{i\in I} = (r f_i)_{i\in I}.$$

It is easily proved that M is a generalized group groupoid. It remains to show that the map $\eta_r : M \to M$ by

$$\eta_r(f_i)_{i\in I} = (r f_i)_{i\in I}$$

is a functor on M. For; for each $(m_i)_{i\in I} \in \nu M$ consider,

$$\begin{aligned} \eta_r(1_{(m_i)})_{i\in I} &= (r 1_{(m_i)})_{i\in I} \\ &= (1_{(r m_i)})_{i\in I} \quad \text{(each } M_i \text{ is a generalized module groupid)} \\ &= 1_{\eta_r(m_i)_{i\in I}}. \end{aligned}$$

Let $(f_i)_{i\in I}, (g_i)_{i\in I}$ are two composable morphisms in M, then

$$\begin{aligned} \eta_r((f_i)_{i\in I} \circ (g_i)_{i\in I}) &= r((f_i)_{i\in I} \circ (g_i)_{i\in I}) \\ &= r(f_i \circ g_i)_{i\in I} \\ &= (r(f_i \circ g_i))_{i\in I} \\ &= (r f_i \circ r g_i)_{i\in I} \\ &= (r f_i)_{i\in I} \circ (r g_i)_{i\in I}. \\ \eta_r(f_i)_{i\in I} \circ \eta_r(g_i)_{i\in I} &= (r f_i)_{i\in I} \circ (r g_i)_{i\in I}. \\ \eta_r((f_i)_{i\in I} \circ (g_i)_{i\in I}) &= \eta_r(f_i)_{i\in I} \circ \eta_r(g_i)_{i\in I}. \end{aligned}$$

□

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