Moroccan Journal of Algebra and Geometry with Applications
Supported by Sidi Mohamed Ben Abdellah University, Fez,
Volume 1, Issue 1 (2022), pp 62-75
ISSN: 2820-7114

Title :

On the category of algebras over a finite direct product of commutative rings

Author(s):

David E. Dobbs

Moroccan Journal of Algebra and Geometry with Applications
Vol. 1(1) (2022), 62–75
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco **ISSN: 2820-7114**

On the category of algebras over a finite direct product of commutative rings

David E. Dobbs
Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1320, USA
e-mail: *ddobbs1@utk.edu*

Communicated by Najib Mahdou
(Received 20 April 2021, Revised 09 June 2021, Accepted 12 June 2021)

Abstract. A full statement and a rather detailed proof are given of a folklore result describing the category of (unital) algebras over a finite direct product of commutative rings. Following an extensive survey of some recent work on minimal ring extensions and chain conditions for (unital) ring extensions such as the FCP and FMC properties, including generalizations of these conditions and the FIP property to ring extensions involving noncommutative rings, a corollary of the folklore theorem for FIP, FCP and FMC is given for ring extensions $A \subseteq B$ where A is a finite direct product of commutative rings and B is a (not necessarily commutative) A-algebra.

Key Words: Commutative ring, unital algebra, finite direct product, categorical equivalence, minimal ring extension, FMC, FCP, FIP.
2010 MSC: Primary 16B50; Secondary 13B99, 18A05.

1 Introduction

All rings considered in this note are associative and unital; all modules, algebras, algebra homomorphisms and ring extensions are unital. As usual, $\mathbb{F}_q$ denotes a/the finite field of cardinality q (for any prime-power q); X and Y denote commuting indeterminates over the ambient ring(s); $\mathrm{U}(A)$ denotes the set of units of a ring A; $|\mathcal{S}|$ denotes the cardinal number of a set $\mathcal{S}$; and $\subset$ denotes proper inclusion.

For more than 50 years, it has been part of the folklore that if a ring R is a direct product of finitely many commutative rings R_i, then the category of R-modules (resp., of commutative R-algebras) is equivalent, as a category, to the product of the categories of the R_i-modules (resp., of the commutative R_i-algebras). A proof of this module-theoretic (resp., algebra-theoretic) fact was given (resp., sketched), using constructions involving tensor products, in [8, Propositions 1.2.1 and 1.2.2 and Remarks, page 8]. A different proof (not featuring tensor products explicitly) of the algebra-theoretic fact appeared later in [17, Proposition III.3]). Although the methods of the latter proof do not generalize at once to the context of arbitrary (not necessarily commutative) algebras, the methods from [8] that involve tensor products do generalize to the context of arbitrary algebras. This was stated by Kosters (and a proof was very lightly sketched) in [32, Theorem 2.1]. We revisit that result in Proposition 2.1 (a), with a statement and (sketch of a) proof which provide more details and which we hope will serve as a useful reference in the future.

The proof of Proposition 2.1 (a) will provide enough information to enable the reader to construct any diagram whose commutativity becomes relevant. However, we will typically leave to the reader the verification that each of those diagrams commute. In doing so, we are endorsing the view that in most cases, an author should not try to chase another person's diagrams for them. That view was more colorfully expressed (but in a gender-specific way that was, unfortunately, common in the 1970s) in Barry Mitchell's dictum that "one man can only confuse another by trying to do his diagram chasing for him" [34, page 600]. The background from category theory that is assumed in Section 2 is minimal (specifically, the definitions of a category, a functor, a natural equivalence of functors, and a categorical equivalence).

A concept that plays a natural role in both Section 2 and Section 3 is that of a minimal ring extension. Suppose that $A \subset B$ are (distinct) rings. (Such notation/usage will always mean that A is a (unital) subring of B; equivalently, that $A \subset B$ is a ring extension.) Then $A \subset B$ is said to be a *minimal ring extension* if there does not exist a ring C such that $A \subset C \subset B$. The notion of a minimal ring extension was introduced in 1970 in [23] by Ferrand and Olivier in case B is commutative. Ferrand and Olivier characterized, up to isomorphism, the commutative minimal ring extensions of any field [23, Lemme 1.2]. More generally, they obtained deep information about commutative minimal ring extensions, building on a fundamental result [23, Théorème 2.2] that established the existence and uniqueness of a maximal ideal of A that has come to be called the "crucial maximal ideal" of a given minimal ring extension $A \subset B$. To be complete, one should note that some of the results in [23] can also be proved by the methods in Section 2 of a 1968 paper of Gilmer and Heinzer [26], who studied what is now considered to be a special kind of (commutative) minimal ring extension.

Recently, there has been some noticeable work on minimal ring extensions $A \subset B$ where A (and, sometimes, B) is not commutative: cf. [22], [1]. In fact, every nonzero ring has a minimal ring extension. (This fact was proven by Dorsey and Mesyan [22, Lemma 2.4 and Remark 2.5], who deftly let the role of a suitable idealization in Dobbs' proof for the commutative case [9, Theorem 2.4 and Remark 2.9] be played instead by a suitable Dorroh extension.) In view of Proposition 2.1 (a), it is therefore natural to ask if one can reduce the study of the (not necessarily commutative) algebras that are minimal ring extensions of a finite direct product of commutative rings R_i to the study, for each i, of the (not necessarily commutative) algebras that are minimal ring extensions of R_i. The answer, which is affirmative, is established in Proposition 2.1 (c). That, in turn, follows from the more general consideration in Proposition 2.1 (b) concerning the structure of the (not necessarily commutative) faithful algebras over a finite direct product of commutative rings.

The details that are provided for the proof of Proposition 2.1 (a) do more than serve as a useful reference. They also serve to facilitate the statements and proofs of parts (b) and (c) of Proposition 2.1. An application of parts (b) and (c) of Proposition 2.1 is deferred until Section 3. The statement of that application (Corollary 3.6) has been designed to generalize, to a suitable noncommutative "algebraic" context, most of the conclusions of [12, Lemma 2.2]. The latter result has played key roles in some recent studies of minimal ring extensions $A \subset B$ where B is commutative. The purpose of Corollary 3.6 is to extend the potential of applying similar methods to ring extensions $A \subset B$ where A is commutative and B is an A-algebra. Specifically, Corollary 3.6 uses parts (b) and (c) of Proposition 2.1 to study how each of the properties AFMC, FMC, FCP and FIP for finitely many ring extensions $A_i \subseteq B_i$, where A_i is commutative and B_i is an A_i-algebra for each i, relates to the corresponding property for the ring extension $\prod_i A_i \subseteq \prod_i B_i$. Prior to the statement of Corollary 3.6, Section 3 gives a survey that summarizes enough about the definitions and known results on AFMC, FMC, FCP and FIP to make Corollary 3.6 (motivated and, modulo Proposition 2.1,) self contained. Suffice it to say at this point that the definitions of each of AFMC, FMC and FCP involve finite chains of minimal ring extensions; and studies of the properties of FCP and FIP have often been intertwined.

2 The categorical equivalence

In Proposition 2.1, it will be convenient, for any commutative ring A, to let $\mathbf{Alg}_A$ denote the category of (unital but not necessarily commutative) A-algebras and A-algebra homomorphisms; and to let id_A denote the identity map on A.

Proposition 2.1. (a) (cf. Kosters [32, Theorem 2.1]) Let $R_1, \dots, R_m$ be a finite list of nonzero commutative rings (where possibly $R_i \cong R_j$ for some $i \neq j$), for some integer $m \geq 1$. Consider the (commutative) ring $R := \prod_{j=1}^{m} R_j$. For each $i, 1 \leq i \leq m$, use the i^{th} projection map $p_i : R \to R_i$ to view R_i as an R-

module. Let $\mathbf{R}$ denote the category $\mathbf{Alg}_R$. Consider the product category

$$\mathbf{P} := \mathbf{Alg}_{R_1} \times \cdots \times \mathbf{Alg}_{R_m}.$$

Then $\mathbf{R}$ and $\mathbf{P}$ are equivalent categories. A categorical equivalence $F : \mathbf{R} \to \mathbf{P}$ can be defined on objects by

$$F(S) := (R_1 \otimes_R S, \ldots, R_m \otimes_R S)$$

(for any R-algebra S) and on morphisms by

$$F(h) := (\mathrm{id}_{R_1} \otimes h, \ldots, \mathrm{id}_{R_m} \otimes h)$$

(for any R-algebra homomorphism $h : T \to S$). A categorical inverse for F is the functor $G : \mathbf{P} \to \mathbf{R}$ that is defined on objects by

$$G(C_1, \ldots, C_m) := C_1 \times \cdots \times C_m$$

(whenever C_j is an R_j-algebra for $j = 1, \ldots, m$) and on morphisms by

$$G(f_1, \ldots, f_m) := (f_1, \ldots, f_m)$$

(whenever f_j is an R_j-algebra homomorphism for $j = 1, \ldots, m$).

(b) Let $A = A_1 \times \cdots \times A_m$ be a finite direct product of nonzero commutative rings (where possibly $A_i \cong A_j$ for some $i \neq j$), for some integer $m \geq 1$. Let B be an A-algebra. Then B is A-algebra isomorphic to $B_1 \times \cdots \times B_m$ where, for each j, B_j is an A_j-algebra that is uniquely determined up to A_j-algebra isomorphism. Moreover, if B is a faithful A-algebra (that is, if we can view $A \subseteq B$ as a ring extension such that A is contained in the center of B) then, for each j, the above-mentioned A_j-algebra B_j can be chosen so that $A_j \subseteq B_j$.

(c) Let $A = A_1 \times \cdots \times A_m$ as in (b) and let B be a faithful A-algebra. (Once again, view $A \subseteq B$ as a ring extension such that A is contained in the center of B.) Pick/fix $B_1, \ldots, B_m$ as in (b) such that $A_j \subseteq B_j$ for all j. Then $A \subset B$ is a minimal ring extension if and only if there exists a (necessarily unique) index i such that $A_i \subset B_i$ is a minimal ring extension and $A_j = B_j$ for all $j \neq i$. If these equivalent conditions hold, then B is commutative.

Proof. (a) Kosters' statement and proof of his version of (a) consisted of defining the functor G (on objects and on morphisms), defining the functor F on objects, and stating "The rest is easy." If only for reference purposes, we will say more about this for the benefit of readers interested in carrying out the pages of relevant (easy but necessary) calculations. Our remaining comments in this sketch of a proof of (a) are intended to indicate how the analysis of categorical equivalences that was given at the levels of R-modules and of commutative R-algebras in [8, pages 7-8] extends naturally to the setting of (not necessarily commutative) R-algebras. In doing so, we will point out a couple of the places where it is necessary to use hypothesis of "algebra" (and so that hypothesis cannot be weakened to "ring extension"). The importance and relevance of that will be examined further following Example 3.1.

To save space, we will leave most of the verifications involving morphisms to the reader. (This will include checking functoriality of F and/or G, the fully faithfulness of F, and the naturality of the categorical equivalences θ and φ that are described below at the level of object assignment.) To a very large extent, such details related to algebra homomorphisms can be handled by using the projection maps p_j as in the proof of the module-theoretic assertions in [8, Proposition 1.2.2].

We will prove the key fact that any R-algebra S is R-algebra isomorphic to $S_1 \times \cdots \times S_m$ where, for each j, S_j is an R_j-algebra that is uniquely determined up to R_j-algebra isomorphism. An easy calculation shows that $R_i \otimes_R R_j = 0$ if $i \neq j$. As in the proof of [8, Proposition 1.2.1], one has, for each j, that the canonical abelian group isomorphisms

$$S \cong R \otimes_R S = (\bigoplus_{i=1}^{m} R_i) \otimes_R S \cong \bigoplus_{i=1}^{m} (R_i \otimes_R S) = \prod_{i=1}^{m} (R_i \otimes_R S)$$

are R-module isomorphisms. One can check that the resulting R-module isomorphism $S \to \prod_{i=1}^{m} (R_i \otimes_R S)$ preserves products and 1, and so $S \cong \prod_{i=1}^{m} (R_i \otimes_R S)$ as R-algebras. Note that the preceding assertion would not even be meaningful without the assumption that S is an R-algebra, since that assumption is crucial in showing that each $R_i \otimes_R S$ is an R-algebra.

We will next show that the above is essentially the only way to express S as a direct product of R_i-algebras. More precisely put: we will show that if S is R-algebra isomorphic to $\prod_{i=1}^{m} C_i$ where C_i is an R_i-algebra for each i, then $C_j \cong R_j \otimes_R S$ as an R_j-algebra, for each j.

First, note that if $i \neq j$, then

$$R_j \otimes_R C_i \cong R_j \otimes_R (R_i \otimes_{R_i} C_i) \cong (R_j \otimes_R R_i) \otimes_{R_i} C_i = 0 \otimes_{R_i} C_i = 0.$$

Next, we claim that for each j, $R_j \otimes_R C_j \cong C_j$ as R_j-algebras. Indeed, we have R_j-module isomorphisms

$$C_j \cong R \otimes_R C_j = (\bigoplus_{i=1}^{m} R_i) \otimes_R C_j \cong \bigoplus_{i=1}^{m} (R_i \otimes_R C_j) \cong R_j \otimes_R C_j,$$

and it is clear that the resulting R_j-module isomorphism $C_j \to R_j \otimes_R C_j$ preserves products and 1. This proves the above claim. Next, note that we have R_j-module isomorphisms

$$R_j \otimes_R S = R_j \otimes_R (\bigoplus_{i=1}^{m} C_i) \cong \bigoplus_{i=1}^{m} R_j \otimes_R C_i \cong R_j \otimes_R C_j.$$

It is clear that this R_j-module isomorphism $R_j \otimes_R S \to R_j \otimes_R C_j$ preserves products and 1. Since S is an R-algebra and C_j is an R_j-algebra, it follows that $R_j \otimes_R S \cong C_j$ as R_j-algebras, thus completing the proof of the above uniqueness assertion.

Another proof of the uniqueness of the C_j (up to R_j-algebra isomorphism) is available by applying [11, Theorem 2.2]. As [11] was designed to apply to algebras over direct products whose index sets may be infinite, the notation in (and leading up to) the cited result is rather dense and may seem less accessible than the argument in the preceding paragraph. Nevertheless, that denseness provided for a precisely stated corollary that we will cite in the proof of (c) given below.

The upshot of the above isomorphisms is that, for each R-algebra S, we have found an R-algebra isomorphism $\theta_S : S \to (G \circ F)(S)$. With careful attention to the behavior of morphisms, one can verify that θ is a natural transformation and hence (because of the R-algebra isomorphisms θ_S), a natural equivalence from the identity functor on **R** to GF. Consequently, GF is naturally equivalent to the identity functor on **R**.

We will next sketch the construction of a natural equivalence φ from FG to the identity functor on **P**. Let C be any object of **P**; that is, $C = (C_1, \dots, C_m)$, where C_j is an R_j-algebra for $j = 1, \dots, m$. With $\mathcal{C} := G(C) = \prod_{i=1}^{m} C_i$, we have

$$(F \circ G)(C) = F(G(C)) = F(\mathcal{C}) = (R_1 \otimes_R \mathcal{C}, \dots, R_m \otimes_R \mathcal{C}) \cong C,$$

where the R-algebra isomorphism in the last step of the above display is given by (taking $S := \mathcal{C}$ in) the third paragraph of this proof. Let φ_C denote the resulting R-algebra isomorphism $(F \circ G)(C) \to C$. With careful attention to the behavior of morphisms, one can verify that φ is a natural transformation and hence (because of the R-algebra isomorphisms φ_C), a natural equivalence from FG to the identity functor on $\mathbf{P}$. Consequently, FG is naturally equivalent to the identity functor on $\mathbf{P}$. That would complete a proof of (a).

If one is interested only in showing that F is an equivalence of categories, one can proceed differently, without any mention of G, FG, GF, and the natural equivalence between GF (resp., FG) and an identity functor. Indeed, according to a well known characterization of categorical equivalences [5, Chapter II, 1.2] (whose proof uses a well-ordering of the universe), one need only prove that F is fully faithful and essentially surjective. The "essentially surjective" assertion means, by definition, that each object T of $\mathbf{P}$ is isomorphic in $\mathbf{P}$ to $F(W)$ for some object W of $\mathbf{R}$. By definition of $\mathbf{P}$, $T = (C_1, \dots, C_m)$, where C_j is an R_j algebra for each $j = 1, \dots, m$. We will show that $W := \prod_{j=1}^m C_j$ behaves as desired. Indeed, since we saw (in the above uniqueness proof) that $R_j \otimes_R C_k = 0$ whenever $j \neq k$ and that $R_j \otimes_R C_j \cong C_j$, we have the following isomorphisms in $\mathbf{P}$:

$$F(W) = (R_1 \otimes_R W, \dots, R_m \otimes_R W) \cong (\prod_{j=1}^m (R_1 \otimes_R C_j), \dots, \prod_{j=1}^m (R_m \otimes_R C_j))$$

$$\cong (R_1 \otimes_R C_1, \dots, R_m \otimes_R C_m) \cong (C_1, \dots, C_m) = T.$$

This completes the sketch of a second proof of (a).

(b) The first assertion follows from the existence and uniqueness result that was proved in the third, fourth and fifth paragraphs of the proof of (a). As for the "Moreover" assertion, also observe from the proof of the uniqueness result in (a) that for each j, we have, up to isomorphism, that the functor $A_j \otimes_A -$ carries the injection $A \hookrightarrow B$ to the morphism $A_j \to B_j$. Thus, we could arrange that $A_j \subseteq B_j$ by showing that this functor is exact, that is, that A_j is a flat A-module. This, in turn, follows since each A_j is a projective A-module (and hence is A-flat), a fact that follows from [8, Proposition 1.2.3] but also admits a direct proof since $\oplus_{i=1}^m A_i = A$.

(c) Pick/fix $B_1, \dots, B_m$ as in (b) such that $A_j \subseteq B_j$ for all j. To prove that $A \subset B$ is a minimal ring extension if and only if there exists an index i such that $A_i \subset B_i$ is a minimal ring extension and $A_j = B_j$ for all $j \neq i$, one can combine (b) and [11, Theorem 2.2 and Corollary 2.3] with the reasoning in the first 14 lines of the proof of [10, Theorem 15]. (Note that those 14 lines of that proof did not use the hypothesis in the cited result that the algebra is commutative.) Finally, the most expeditious way to establish the uniqueness of the index i is to apply [11, Corollary 2.3]. Finally, suppose that these equivalent conditions hold. To prove that B is commutative, it is enough to prove that B_i is commutative. This, in turn, holds because of the following three facts: the minimality of $A_i \subset B_i$ ensures that B_i is the ring generated by $A_i \cup \{\beta\}$ for any $\beta \in B_i \setminus A_i$; the algebraicity of $A_i \subset B_i$ ensures that $\alpha\beta = \beta\alpha$ for all $\alpha \in A_i$; and A_i is commutative. The proof is complete. □

3 Applications to ring extensions

The fifth paragraph of the Introduction to [15] "involve[d] extending the context for the FIP, FCP and FMC concepts to noncommutative ring extensions." It will be convenient to repeat, and to augment, some of that material in this paragraph and the next paragraph. Let $A \subseteq B$ be (possibly noncommutative) rings. Let $[A, B]$ denote the set of intermediate rings of the ring extension $A \subseteq B$; that is, $[A, B] := \{C \mid C \text{ is a ring and } A \subseteq C \subseteq B\}$. We say that $A \subseteq B$ *satisfies* FIP (the "finitely many intermediate rings" property) if $|[A, B]| < \infty$. According to one version of the classical Primitive Element Theorem of field theory, a field extension $K \subseteq L$ satisfies FIP if and only if $L = K(u)$ for some element $u \in L$ such that u is integral (i.e., algebraic) over K. For any field K, a characterization of the nonzero K-algebras T such that $K \subseteq T$ satisfies FIP was obtained for perfect K in [2, Theorem 3.8 (d)] and for arbitrary fields K in [17, Theorem III.2]. (In [32], an algebra B over a commutative ring A was called a *futile* A-algebra in case B has only finitely many A-subalgebras; so, if $A \subseteq B$ are nonzero commutative rings, then B is a futile A-algebra if and only if $A \subseteq B$ satisfies FIP. However, we will not mention the "futile" terminology again here, for two reasons: our interest is in rings and ring extensions (not only in algebras over commutative rings and algebra extensions); and we consider the terminology in question from [32] to be pejorative and hence inappropriate.)

Returning to the ambient rings $A \subseteq B$, note that $[A,B]$ is a poset under inclusion. We say that $A \subseteq B$ *satisfies* FCP (the "all chains are finite" property) if every chain in $[A,B]$ is finite; and we say that $A \subseteq B$ *satisfies* FMC (the "existence of a finite maximal chain" property) if there exists a finite maximal chain, $A = A_0 \subset \ldots \subset A_n = B$, in $[A,B]$ going from A to B. (Note that in any such chain, each step $A_{i-1} \subset A_i$ is a minimal ring extension.) It will be convenient to say that a ring extension *satisfies* n-FMC if the preceding condition holds. It is clear that a ring extension satisfies FMC if and only if it satisfies n-FMC for some integer $n \geq 0$. If K is a field and T is a nonzero K-algebra, then perhaps the most familiar example of such a T for which $K \subseteq T$ satisfies FCP and FMC arises when T is a (nonzero) finite-dimensional K-algebra (i.e., when T has a nonempty finite K-vector space basis). By an easy fact about chains in a poset, a ring extension $A \subseteq B$ satisfies FCP if and only if (the poset) $[A,B]$ satisfies both the ascending chain condition and the descending chain condition. For arbitrary ring extensions, it is clear that FIP $\Rightarrow$ FCP $\Rightarrow$ FMC, but neither of these implications has a valid converse. The classical example of a ring extension that satisfies FCP but not FIP is $\mathbb{F}_p[X^p, Y^p] \subset \mathbb{F}_p[X,Y]$. An example showing that FMC $\not\Rightarrow$ FCP was given in [7, Example 6.4]. (The latter example is striking. It features a quasi-local (commutative) integral domain R of Krull dimension 2 and an overring S of R (contained in the quotient field of R) such that there exists a maximal chain in $[R,S]$ that has length 2 and goes from R to S (so $R \subset S$ satisfies FMC) although $R \subset S$ does not satisfy FCP. In a sense, this example is best-possible, because the existence of a maximal chain of rings of length 1 going from A to B is equivalent to $A \subset B$ being a minimal ring extension, and of course, any minimal ring extension satisfies FCP.)

All rings in this paragraph and the next three paragraphs will, apart from one parenthetical passage, be commutative. Let $A \subseteq B$ be (commutative) rings, and let $\overline{A}$ denote the integral closure of A in B. It is clear that if $A \subset B$ is a minimal ring extension, then $A \subset B$ is either integrally closed (in the sense that $\overline{A} = A$) or integral (in the sense that $\overline{A} = B$). An integrally closed minimal ring extension $A \subset B$ is the same as a minimal ring extension $A \subset B$ such that (A,B) is a normal pair. Recall from [27] that if $A \subset B$ are (commutative) rings, then (A,B) is said to be a *normal pair* if each ring in $[A,B]$ is integrally closed in B. The most familiar example of a normal pair arises from any Prüfer domain A by taking B to be the quotient field of A. (In [13], Dobbs and Jarboui recently initiated a study of normal pairs of noncommutative rings that was based on a notion of integrality due to Atterton [3] but, as documented in [13, Remark 2.11], noncommutative ring theorists have, for approximately the last half-century, preferred a number of other notions of "integrality" whose studies are often restricted to special kinds of extensions of noncommutative rings and whose connections to the classical concept of integrality are less evident than was Atterton's.) Because the property of being a normal pair is a local property (in ways that are made explicit in [21, page 340]), a characterization of normal pairs (A,B) would be available if it were achieved in case A is quasi-local. This, in turn, was accomplished in [20, Theorem 3.1 and Corollary 3.2] by Dobbs and Shapiro with the aid of Kaplansky transforms. It is fair to say that much of the transform-related work on integrally closed minimal ring extensions of a commutative ring was motivated by Ayache's pioneering study on the possible existence and structure of the integrally closed minimal overrings (inside the quotient field) of an integrally closed commutative integral domain which is not a field [4, Theorems 2.4 and 3.1]. Many characterizations of integrally closed minimal ring extensions were accomplished (without assuming quasi-local base rings) by Cahen, Dobbs and Lucas in [6, Section 3] by using a generalized notion of the Kaplansky transform of a (commutative) ring extension relative to a non-nil ideal of the base ring. (For the definition of that generalized notion, see [6, page 1083].) For the construction of an earlier generalized Kaplansky transform (of an ideal of an arbitrary commutative ring) that was used in another approach by Dobbs and Shapiro to the case where the base ring is not quasi-local and has a total quotient ring that is von Neumann regular [20, Theorem 3.7] and that has an application to algebraic geometry (for a base ring whose total quotient ring is von Neumann regular), see [37]. As for integral minimal ring extensions, one obtains the following "ramified/decomposed/inert" trichotomy (cf. [17, Corollary II.2]), by combining [23, Lemme 1.2 and Théorème 2.2] with an easy pullback result [17, Lemma II.3]. An integral ring extension $A \subseteq B$ is a minimal ring extension if and only if there exists a maximal ideal M of A (which is necessarily both the crucial maximal ideal of the minimal ring extension $A \subset B$ and the conductor $(A : B) := \{b \in B \mid bB \subseteq A\}$) such that, if $K := A/M$, then $B/MB = B/M$ is K-algebra isomorphic to (exactly) one of the following: $K[X]/(X^2)$, $K \times K$, a minimal field extension of K. The (three) individual types of extensions in this trichotomy have each been the subject of study in at least one recent paper. Most of those studies are not especially relevant to the rest of this note, but there will be a role for the trichotomy in a couple of the results from [15] that will be stated below. Finally, we wish to point out that many characterizations of normal pairs (of commutative rings) can be found in [31, Theorem 5.2, page 47].

Any summary, no matter how brief it may be, of what is known about FCP and FIP for commutative rings $A \subset B$ would be incomplete without mentioning the pioneering work of Robert Gilmer. (Recall that [26] was mentioned above.) For a (commutative) integral domain R with quotient field K, Gilmer [25, Theorems 2.3 and 3.1] characterized (using other terminology) when $R \subseteq K$ satisfies FCP and when $R \subseteq K$ satisfies FIP. This work of Gilmer prompted several individuals to work on FCP and (without the terminology) FIP for various kinds of extensions of various kinds of commutative integral domains. (A representative sampling of that work was mentioned in [18, pages 392, 399, 400, 425, 427, 428]; for additional related background, the interested reader can look into sources cited in the relevant items in the bibliography of [18].) More generally, motivated by the behavior of minimal ring extensions, Dobbs, Picavet and Picavet-L'Hermitte proved in [18, Theorem 3.13] that if $A \subseteq B$ are (commutative) rings, then $A \subseteq B$ satisfies FCP (resp., FIP) if and only if both $A \subseteq \overline{A}$ and $\overline{A} \subseteq B$ satisfy FCP (resp., FIP). In this way, the study of the commutative ring extensions satisfying FCP (resp., FIP) was reduced to the study of the FCP (resp., FIP) property for (commutative) ring extensions that are either integral or integrally closed. Also, in retrospect, the "FCP" assertion in [18, Theorem 3.13] makes clear that any commutative ring extension $R \subset S$ satisfying FCP but not FMC would necessarily have, as was the case for the ring extension $R \subset S$ in an above-mentioned example [7, Example 6.4], the property that S is not integral over R.

In [18, Theorem 4.2 (a)], it was shown that if $A \subseteq B$ are commutative rings with B integral over A, then: $A \subseteq B$ satisfies FCP $\Leftrightarrow$ B is finitely generated as an A-module and $A/((A : B))$ is an Artinian ring $\Leftrightarrow$ $A \subseteq B$ satisfies FMC. With this result in hand, a characterization of the integral extensions that satisfy FIP was then reduced to the context of quasi-local base rings by the following result [18, Proposition 5.17]: if $A \subseteq B$ are commutative rings with B integral over A, then: $A \subseteq B$ satisfies FIP $\Leftrightarrow$ $A \subseteq B$ satisfies FCP and $A_M \subseteq B_{A\setminus M}$ satisfies FIP for all maximal ideals M of A $\Leftrightarrow$ $A_M \subseteq B_{A\setminus M}$ satisfies FIP for all maximal ideals M of A and there are only finitely many maximal ideals M of A such that the canonical map $A_M \to B_{A\setminus M}$ is not surjective. The final step in characterizing the integral extensions (involving commutative rings) $A \subseteq B$ that satisfy FIP was completed in [18, Theorem 5.18] which addressed the case of quasi-local A (and, coincidentally yielded a new proof of the Primitive Element Theorem). To complete our summary of results from [18], we next state [18, Theorem 6.3 and Proposition 6.9]: if $A \subseteq B$ is an integrally closed extension of commutative rings, then: $A \subseteq B$ satisfies FIP $\Leftrightarrow$ $A \subseteq B$ satisfies FCP $\Leftrightarrow$ $A \subseteq B$ satisfies FMC $\Leftrightarrow$ (A, B) is a normal pair and there exist only finitely many prime ideals P of A such that the canonical map $A_P \to B_{A\setminus P}$ is not surjective. The preceding result can be viewed as a far-reaching generalization of the following fact that has been proven often (cf. the corollary in the Corrigendum to [2]): if R is an integrally closed (commutative) integral domain with quotient field K, then R has only finitely many overrings (inside K) if and only if R is a semi-quasi-local Prüfer domain of finite Krull dimension.

In his doctoral dissertation [24], Gilbert introduced the following chain condition as a generalization of a minimal ring extension. An extension $A \subseteq B$ of commutative rings is called a *λ-extension* if the poset $[A,B]$ is linearly ordered (by inclusion). It is possible to restate [18, Corollary 6.4] as follows: if $A \subseteq B$ is an integrally closed extension (of commutative rings) such that A is quasi-local and $A \subseteq B$ satisfies FMC, then $A \subseteq B$ is a λ-extension (and $[A,B]$ is a finite chain). A number of recent papers have pursued the notion of a λ-extension, especially for (commutative) ring extensions satisfying FCP. Among the most recent of these works are [35] and [16]. Since the studies of λ-extensions do not seem to specifically relate to Corollary 3.6, we will, in the interest of brevity, not comment further on them here, other than to alert the reader that some authors have decided to refer to a λ-extension as a "chained extension."

Also in the interest of brevity and with the main goal of motivating Corollary 3.6, this survey is largely ignoring a considerable amount of research that has been done during the past decade on some special kinds of commutative ring extensions that satisfy FCP or FIP. Most of that research has been done jointly by Picavet and Picavet-L'Hermitte, and its has enriched the intersection of commutative ring theory and lattice theory. The interested reader can access some of that work by consulting [35] and the papers cited in the bibliography of that paper. We would also suggest that it may be fruitful to consider generalizations of some of that work to the context of noncommutative ring extensions (or at least to the context of ring extensions $A \subseteq B$ where A is commutative and B is an A-algebra).

Before discussing some work on studying FMC for extensions of noncommutative rings, we pause to restate (without pejorative terminology) five of Kosters' results concerning the ring extensions $A \subseteq B$ that satisfy FIP where A is commutative and B is a (possibly noncommutative) A-algebra. Before stating the first and second of those results, we recall that much of the early work on minimal ring extensions of commutative integral domains focused on minimal overrings (inside the quotient field of the given base ring). For instance, before the above-mentioned studies involving Kaplansky transforms, Dobbs and Shapiro [19, Theorem 2.7] showed that if $A \subseteq B$ is an integrally closed minimal ring extension, with A an integral domain and B commutative, then B is (A-algebra isomorphic to) an overring of A (inside the quotient field of A). More than a decade before that, Sato, Sugatani and Yoshida [36] showed that if $A \subset B$ is a minimal ring extension where B is a commutative integral domain and A is not a field, then B is an overring of A. In the same vein, in [32, Lemma 5.2], Kosters showed that if A is a commutative integral domain but not a field and $A \subset B$ satisfies FIP, where B is an A-algebra which is torsion-free as an A-module, then B is an overring of A. In [32, Corollary 5.3], Kosters noted the consequence that if $A \subseteq B$ satisfies FIP where A is a commutative integral domain which is distinct from its quotient field K and where B is an A-algebra, then $B \otimes_A K$ is (isomorphic to) either 0 or K. In [32, Theorem 1.3], Kosters examined another topic that had not been considered by the author or his collaborators, by showing that if A is a commutative ring, then $A \subseteq B$ satisfies FIP for all A-algebras B that contain A if and only if $|A/M| < \infty$ for each maximal ideal M of A. Next, we mention Kosters' reduction to the commutative case in [32, Theorem 1.2], where he showed that if $A \subseteq B$ are rings such that A is commutative and B is an A-algebra and if I denotes the commutator ideal of the (possibly noncommutative ring) B, then: $A \subseteq B$ satisfies FIP $\Leftrightarrow A/(I \cap A) \subseteq B/I$ satisfies FIP and $|I| < \infty$. (Note that B/I is commutative in general; and that $I = 0$ if and only if B is commutative.) Finally, given the role played by Artinian base rings in [18, Section 5], it seems interesting to mention the following fragment of Kosters' [32, Proposition 4.13]: if $A \subseteq B$ satisfies FIP where (A,M) is a commutative local Artinian ring such that A/M is infinite and where B is an A-algebra, then B is commutative. As noted in [32], some proofs in [32] use results and methods from [18]. While we found nothing really new in Kosters' treatment of the integrally closed commutative ring extensions that satisfy FIP, we are glad to also note that some of the assertions and methods of proof in [32] do contain genuine innovations. We invite the reader to compare [32] and [18] more fully.

The above attention given to defining FIP, FCP and FMC for ring extensions involving arbitrary (that is, possibly noncommutative) rings allowed Dobbs and Jarboui to state (and prove) the following result [15, Theorem 2.2]: if A is a nonzero ring, then: $\mathrm{U}(A) = \{1\}$ and $\mathbb{F}_2 \subseteq A$ satisfies FIP $\Leftrightarrow$ $\mathrm{U}(A) = \{1\}$ and $\mathbb{F}_2 \subseteq A$ satisfies FCP $\Leftrightarrow$ $\mathrm{U}(A) = \{1\}$ and $\mathbb{F}_2 \subseteq A$ satisfies FMC $\Leftrightarrow$ R is a finite Boolean ring. (Of course, one considered $\mathbb{F}_2 \subseteq A$ in the preceding result because any nonzero Boolean ring has characteristic 2.) It would seem natural to ask if there are any analogues of [15, Theorem 2.2] for rings of odd prime characteristic. For instance, since it is well known that the finite nonzero Boolean rings are the same as the (not necessarily commutative) rings which are isomorphic to finite nonempty direct products of copies of $\mathbb{F}_2$, one may ask the following question: in the universe of (not necessarily commutative) rings of arbitrary prime characteristic p, can one use FMC or FCP to characterize the rings that are isomorphic to finite nonempty direct products of copies of $\mathbb{F}_p$? In seeking such a result, it would be natural to argue inductively and to avoid chains involving rings having nonzero nilpotent elements. Indeed, if $\mathbb{F}_p = B_0 \subset B_1$ is a minimal ring extension, it is easy to see that B_1 is commutative (since B_0 is contained in the center of B_1, as all ring extensions are unital). Hence, by [23, Lemme 1.2] (and since we are avoiding nonzero nilpotent elements), we could take $B_1 \cong \mathbb{F}_p \times \mathbb{F}_p$. There would then be no harm in identifying B_1 with $\mathbb{F}_p \times \mathbb{F}_p$. If one could handle the induction step by showing that a nilpotent-free minimal ring extension of B_1 must be isomorphic to a direct product of three copies of $\mathbb{F}_p$, it would be reasonable to expect such an iterative argument to terminate successfully after invoking FCP. However, no such argument is possible. The fact is that the structure of an arbitrary minimal ring extension B_2 of B_1 is not known. After all, one cannot invoke [12, Lemma 2.2] in this regard, since we do not know, *a priori*, that B_2 is commutative. (Furthermore, we cannot invoke the above Proposition 2.1 (c) because we do not know that B_2 is an algebra over B_1; that is, we do not know if B_1 is contained in the center of B_2.) In fact, as the next example shows, B_2 need not be commutative!

Example 3.1. (Dobbs and Jarboui [15, Example 2.5]) Consider any integer $m \geq 2$. Then there exists a finite maximal chain of finite rings, $A = B_0 \subset \ldots \subset B_m = B$, such that B_0 and B_1 are commutative and B_m is noncommutative. For an example of such data in which $|B|$ is the minimum possible, take $m = 2$ and $A = \mathbb{F}_2$, with B the ring $\mathrm{U}_2(\mathbb{F}_2)$ of upper triangular 2×2 matrices over $\mathbb{F}_2$. A maximal chain of rings $\mathbb{F}_2 \subset B_1 \subset B_2 := \mathrm{U}_2(\mathbb{F}_2)$ going from $\mathbb{F}_2$ to $\mathrm{U}_2(\mathbb{F}_2)$ can be built using $B_1 := \{0, I, C, I + C\}$, where I is the 2×2 identity matrix and C is the 2×2 matrix whose only nonzero entry is $c_{11} = 1$. (In particular, in this example, $B_{m-1} \subset B$ is a minimal ring extension, B_{m-1} is commutative, B is noncommutative, and B_{m-1} is not contained in the center of B.) To construct an example for any $m > 2$, it suffices to prolong the above chain by inductively choosing B_{i+1} to be any minimal ring extension of B_i, for $i = 2, 3, \ldots, m - 1$.

Observe that in Example 3.1, the ring B_1 is isomorphic (as an $\mathbb{F}_2$-algebra, that is, as a ring) to $\mathbb{F}_2 \times \mathbb{F}_2$; and C is not in the center of B_2 (so, B_2 is not an algebra over B_1), even though $B_1 \subset B_2$ is a minimal ring extension. Thus, the “argument" preceding the statement of Example 3.1 does indeed fail.

Because of Example 3.1, it is natural to discuss minimal ring extensions such that A is commutative, B is noncommutative and B is not an A-algebra. Classical algebra knew of such examples early in the 20$^{\text{th}}$ century (if not earlier). For instance, let K be any field whose Brauer group, $\mathrm{Br}(K)$, is nonzero. (For instance, take K to be $\mathbb{R}$ or $\mathbb{Q}$, but not an algebraically closed field and not a finite field.) Then there exist $K \subset L \subset D$ such that D is a central simple division algebra over K (equivalently, a central separable K-division algebra, equivalently an Azumaya K-division algebra) and L is a maximal subfield of D. (Cf. [28, pages 89-94]. For instance, if $K = \mathbb{R}$, take D as the ring $\mathbb{H}$ of quaternions over K and take L as $\mathbb{C}$ (viewed as $\mathbb{R} + \mathbb{R}i \subset \mathbb{H} = D$). Then $L \subset D$ is a minimal ring extension, L is commutative, D is noncommutative, and D is not an L-algebra since i is not in the center of $\mathbb{H}$.) More recently, there has been renewed interest in the maximal commutative subrings of a given (noncommutative) ring. One such work was [29]. In fact, [29] had reason to consider (and to generalize) the ring extension that was denoted by $B_1 \subset B_2$ in Example 3.1, but the minimality assertion in Example 3.1 seems not to have been noticed prior to the appearance of [15].

Since the "argument" preceding the statement of Example 3.1 fails, we pause to explain a bit of how [15, Theorem 2.2] was proved. One question that arose in proving [15, Theorem 2.2] was to know that if A is a finite ring and $A \subset B$ is a minimal ring extension, then B is also finite. While this fact is not difficult to establish for commutative rings (as an application of [10, Proposition 7]), the proof of it in [15] used a deep noncommutative rng-theoretic result due independently to Klein [30] and Laffey[33]. The proof of [15, Theorem 2.2] also used the result [14, Corollary 2.5] that a (not necessarily commutative) ring A is a finite Boolean ring if and only if A is finite and $\mathrm{U}(A) = \{1\}$.

Any ring of odd prime characteristic has a unit that is distinct from 1 (since this is true of its prime subring). Therefore, in order to establish analogues of [15, Theorem 2.2], it seemed worthwhile to adjust the FMC property in order to facilitate induction arguments that could handle the difficulty that was noted above. To that end, Dobbs and Jarboui introduced the following definitions in [15]. (The rest of this paragraph, as well as all of the next paragraph, is taken nearly *verbatim* from [15].) Let $A \subseteq B$ be rings. We will say that $A \subseteq B$ satisfies AFMC (for "Adjusted FMC") if there exists (in $[A,B]$) a finite maximal increasing chain of rings going from A to B whose last step $B_{m-1} \subset B$ is such that B is a B_{m-1}-algebra. If the number of steps in such a chain is relevant, we will say, for an integer $m \geq 0$, that $A \subseteq B$ satisfies m-AFMC if there exists a finite maximal chain (in $[A,B]$), $A = B_0 \subset \ldots \subset B_m = B$, such that B is a B_{m-1}-algebra, and we will call any such chain an m-AFMC *chain* (*going from A to B*).

Some conclusions about the above new concepts are clear. For instance, $A \subseteq B$ satisfies AFMC if and only if $A \subseteq B$ satisfies m-AFMC for some $m \geq 0$. (Recall that $A \subseteq B$ satisfies FMC if and only if $A \subseteq B$ satisfies m-FMC for some $m \geq 0$.) Of course, m-AFMC $\Rightarrow$ m-FMC, and so AFMC $\Rightarrow$ FMC. However, as Example 3.1 shows, FMC $\not\Rightarrow$ AFMC; in fact, 2-FMC $\not\Rightarrow$ AFMC. A fact (which is central to the reason for introducing the AFMC property) is that if $A \subseteq B$ satisfies m-AFMC (for some m), then B is a commutative ring. (Here is a quick proof. By focusing on the last step in an m-AFMC chain going from A to B, it is enough to prove that if $A \subset B$ is a minimal ring extension and B is an A-algebra (and A is commutative), then B is commutative. This, in turn, can be easily proved by revisiting the paragraph that preceded the statement of Example 3.1 and adapting the correct part of the argument that was given in that paragraph to establish commutativity of what was denoted by B_1 in that paragraph.) Though easy, this fact is useful. For instance, it shows the AFMC property remedies a feature of the FMC property that was revealed by Example 3.1. In a sense, AFMC is the natural variant of FMC that should be studied by those who are (primarily but not exclusively) interested in commutative rings, as it is now clear that if $A \subseteq B$ are commutative rings, then $A \subseteq B$ satisfies m-AFMC (if and) only if $A \subseteq B$ satisfies m-FMC.

We will next state, as items 3.2-3.5, four results whose statements involve the above definitions. The statements of those results are clearly in the spirit of [15, Theorem 2.2]. Also, those statements (correctly) indicate that, as has often been the case for proofs involving the FCP and FMC properties, one can expect that proofs of assertions involving the AFMC property will often make serious use of minimal ring extensions, especially the "ramified/decomposed/inert" trichotomy.

As usual, we will let $\mathrm{Max}(R)$ denote the set of maximal ideals of a ring R.

Proposition 3.2. (Dobbs and Jarboui [15, Theorem 2.7]) Let R be a nonzero ring of characteristic $k > 0$ and view $\mathbb{Z}/k\mathbb{Z} \subseteq R$ as usual. Then:

(a) If A is a subring of R such that $A \subseteq R$ satisfies m-AFMC for some $m \geq 0$, then R is a commutative ring.

(b) If A is a finite subring of R such that $A \subseteq R$ satisfies m-AFMC for some $m \geq 0$, then R is a finite commutative ring.

(c) Let the prime-power decomposition of k be $k = \prod_{j=1}^{s} q_j^{\alpha_j}$, where $q_1, \dots, q_s$ are pairwise distinct prime numbers and each $\alpha_j \geq 1$. Using the Chinese Remainder Theorem, identify $A = \mathbb{Z}/k\mathbb{Z} = \prod_{j=1}^{s} \mathbb{Z}/q_j^{\alpha_j}\mathbb{Z} \subseteq R$. Suppose that $A \subseteq R$ satisfies m-AFMC, with $A = R_0 \subset \dots \subset R_m = R$ an m-AFMC chain. (So, by (b), R is a finite commutative ring.) Then $|\mathrm{Max}(R)| \leq s + m$. If $0 \leq \nu \leq m$, then $|\mathrm{Max}(R)| = s + \nu$ if and only if exactly ν of the steps of the form $R_{i-1} \subset R_i$ are decomposed (minimal ring) extensions (and the other $m - \nu$ steps of the form $R_{i-1} \subset R_i$ are ramified or inert (minimal ring) extensions). In particular, $|\mathrm{Max}(R)| = s + m$ if and only if $R_{i-1} \subset R_i$ is a decomposed extension for all $i = 1, \dots, m$. Also, R has no nonzero nilpotent elements if and only if $\alpha_1 = \cdots = \alpha_s = 1$ and each step ($R_{i-1} \subset R_i$ for $1 \leq i \leq m$) is either decomposed or inert.

Corollary 3.3. (Dobbs and Jarboui [15, Corollary 2.8]) Let p be a prime number and let R be a (necessarily nonzero) ring of characteristic p. View $\mathbb{F}_p \subseteq R$ as usual. Then:

(a) Let m be a non-negative integer. Then the following two conditions are equivalent:

(1) R is a (not necessarily commutative) integral domain and $\mathbb{F}_p \subseteq R$ satisfies m-AFMC;

(2) $R \cong \mathbb{F}_{p^{q_1 \cdots q_m}}$, for some finite list of prime numbers $q_1, \dots, q_m$ (possibly with $q_i = q_j$ for some $i \neq j$).

(b) The following three conditions are equivalent:

(i) R is a (not necessarily commutative) integral domain and $\mathbb{F}_p \subseteq R$ satisfies FCP;

(ii) R is a (not necessarily commutative) integral domain and $\mathbb{F}_p \subseteq R$ satisfies AFMC;

(iii) R is a finite field.

Corollary 3.4. (Dobbs and Jarboui [15, Corollary 2.9]) Let p be a prime number and let R be a (necessarily nonzero) ring of characteristic p. View $\mathbb{F}_p \subseteq R$ as usual. Then the following conditions are equivalent:

(1) There exist a non-negative integer m_1 and an m_1-AFMC chain, $\mathbb{F}_p = R_0 \subset \dots \subset R_m = R$, going from $\mathbb{F}_p$ to R, such that each step $R_{i-1} \subset R_i$ of that chain is either inert or decomposed;

(2) There exists a non-negative integer m_2 such that $\mathbb{F}_p \subseteq R$ satisfies m_2-AFMC and R has no nonzero nilpotent elements;

(3) R is isomorphic to a finite direct product of finite fields (of characteristic p);

(4) R is a finite commutative semisimple ring.

Corollary 3.5. (Dobbs and Jarboui [15, Corollary 2.10]) Let R be a nonzero ring of prime characteristic $p > 0$ and view $\mathbb{F}_p \subseteq R$ as usual. Then:

(a) The following three conditions are equivalent:

(1) $\mathbb{F}_p \subseteq R$ satisfies m-AFMC, with an m-AFMC chain $\mathbb{F}_p = R_0 \subset \dots \subset R_m = R$, each of whose steps $R_{i-1} \subset R_i$ is decomposed;

(2) $\mathbb{F}_p \subseteq R$ satisfies m-AFMC and $|\mathrm{Max}(R)| = m + 1$;

(3) R is isomorphic (as a ring, equivalently, as a vector space over $\mathbb{F}_p$) to a direct product of finitely many copies of $\mathbb{F}_p$.

(b) If the equivalent conditions in (a) hold and $n := |\mathrm{Max}(R)|$, then any finite maximal chain going from $\mathbb{F}_p$ to R (that is, any FMC chain going from $\mathbb{F}_p$ to R; equivalently, any AFMC chain going from $\mathbb{F}_p$ to R) has length $m = n - 1$ and the number of subrings of R is B_n, the n^{th} Bell number.

Although the FIP property and the FCP property were each characterized in [18] for ring extensions in which the top ring is (and hence both rings are) commutative, we are not aware of any characterizations of either of those properties for ring extensions involving arbitrary (that is, possibly noncommutative) rings. Also, apart from the work that was recalled above from [15], we are not aware of other studies of the FMC property for ring extensions involving arbitrary noncommutative rings.

With the above background in this section serving as motivation, we close this note with an application of Proposition 2.1 to the following four properties: AFMC, FMC, FCP and FIP.

Corollary 3.6. (a) Let $A = A_1 \times \cdots \times A_m$ as in Proposition 2.1 (b) and let B be a faithful A-algebra. (Once again, view $A \subseteq B$ as a ring extension such that A is contained in the center of B.) Pick/fix $B_1, \ldots, B_m$ as in Proposition 2.1 (b) such that $A_k \subseteq B_k$ for all k. Then $A \subseteq B$ satisfies AFMC if and only if $A_k \subseteq B_k$ satisfies AFMC for all $k = 1, \ldots, m$. More precisely put: if μ is a non-negative integer, then $A \subseteq B$ satisfies μ-AFMC if and only if there exist non-negative integers $\mu_1, \ldots, \mu_m$ such that $\sum_{k=1}^{m} \mu_k = \mu$ and $A_k \subseteq B_k$ satisfies μ_k-AFMC for all $k = 1, \ldots, m$. If these equivalent conditions hold, then B is commutative.

(b) Let $A = A_1 \times \cdots \times A_m$ as in Proposition 2.1 (b) and let B be a faithful A-algebra. (Once again, view $A \subseteq B$ as a ring extension such that A is contained in the center of B.) Pick/fix $B_1, \ldots, B_m$ as in Proposition 2.1 (b) such that $A_k \subseteq B_k$ for all k. Then $A \subseteq B$ satisfies FMC (resp., FCP; resp., FIP) if and only if $A_k \subseteq B_k$ satisfies FMC (resp., FCP; resp., FIP) for all $k = 1, \ldots, m$.

Proof. These assertions follow easily from Proposition 2.1 (c). □

References

[1] N. Al-Kuleab and N. Jarboui, A note on intermediate matrix rings, Far East Journal Math. Edu. **17** (4) (2018), 227–229.

[2] D. D. Anderson, D. E. Dobbs and B. Mullins, The primitive element theorem for commutative algebras, Houston J. Math. **25** (4) (1999), 603–623. Corrigendum, Houston J. Math. **28** (1) (2002), 217–219.

[3] T. W. Atterton, Definitions of integral elements and quotient rings over non-commutative rings with identity, J. Austral. Math. Soc. **13** (1972), 433–446.

[4] A. Ayache, Minimal overrings of an integrally closed domain, Comm. Algebra **31** (12) (2003), 5693–5714.

[5] H. Bass, Lectures on Topics in Algebraic K-Theory, Tata Institute of Fundamental Research, Bombay, 1967.

[6] P.-J. Cahen, D. E. Dobbs and T. G. Lucas, Characterizing minimal ring extensions, Rocky Mountain J. Math., 41 (4) (2011), 1081–1125.

[7] P.-J. Cahen, D. E. Dobbs and T. G. Lucas, Finitely valuative domains, J. Algebra Appl. **11** (6) (2012), 1250112, 39 pages; DOI: 10.1142/S0219498812501125.

[8] D. E. Dobbs, Cech cohomology and a dimension theory for commutative rings, Ph. D. thesis, Cornell University, Ithaca (NY), 1969.

[9] D. E. Dobbs, Every commutative ring has a minimal ring extension, Comm. Algebra **34** (10) (2006), 3875–3881.

[10] D. E. Dobbs, On the commutative rings with at most two proper subrings, Int. J. Math. Math. Sci., volume 2016, Article ID 6912360, 13 pages, 2016. doi:10.1155/2016/6912360.

[11] D. E. Dobbs, Is A x B isomorphic to B x A?, Far East J. Math. Sci. **108** (2) (2018), 217–228. dx.doi.org/10.17654/MS108020217.

[12] D. E. Dobbs, A minimal ring extension of a large finite local prime ring is probably ramified, J. Algebra Appl. **19** (1) (2020), 2050015 (27 pages); DOI: 10.1142/S0219498820500152.

[13] D. E. Dobbs and N. Jarboui, Normal pairs of noncommutative rings, Ric. Mat. **69** (1) (2020), 95–109. DOI: 10.1007/s11587-019-00450-2.

[14] D. E. Dobbs and N. Jarboui, Associative rings in which 1 is the only unit, Palest. J. Math. **9** (2) (2020), 604–619.

[15] D. E. Dobbs and N. Jarboui, Characterizing finite Boolean rings by using finite chains of subrings, Gulf J. Math. **10** (1) (2021), 69–94.

[16] D. E. Dobbs and N. Jarboui, Prüfer-closed extensions and FCP λ-extensions of commutative rings, Palest. J. Math., to appear.

[17] D. E. Dobbs, B. Mullins, G. Picavet and M. Picavet-L'Hermitte, On the FIP property for extensions of commutative rings, Comm. Algebra **33** (9) (2005), 3091–3119.

[18] D. E. Dobbs, G. Picavet and M. Picavet-L'Hermitte, Characterizing the ring extensions that satisfy FIP or FCP, J. Algebra **371** (2012), 391–429.

[19] D.E. Dobbs and J. Shapiro, A classification of the minimal ring extensions of an integral domain, J. Algebra **305** (1) (2006), 185–193.

[20] D. E. Dobbs and and J. Shapiro, A classification of the minimal ring extensions of certain commutative rings, J. Algebra **308** (2007), 800–821.

[21] D. E. Dobbs and and J. Shapiro, Normal pairs with zero-divisors, J. Algebra Appl. **10** (2011), 335-356.

[22] T. J. Dorsey and Z. Mesyan, On minimal extensions of rings, Comm. Algebra **37** (2009), 3463–3486.

[23] D. Ferrand and J.-P. Olivier, Homomorphismes minimaux d'anneaux, J. Algebra **16** (1970), 461–471.

[24] M. S. Gilbert, Extensions of commutative rings with linearly ordered intermediate rings, Ph.D. dissertation, University of Tennessee, Knoxville, TN, 1996.

[25] R. Gilmer, Some finiteness conditions on the set of overrings of an integral domain, Proc. Amer. Math. Soc. **131** (2003), 2337–2346.

[26] R. Gilmer and W. J. Heinzer, Intersections of quotient rings of an integral domain, J. Math. Kyoto Univ. **7** (1967), 133–150.

[27] M. Griffin, Prüfer rings with zero divisors, J. Reine Angew. Math. **239/240** (1969), 55–67.

[28] I. N. Herstein, Noncommutative Rings, Math. Assn. America, Carus Monographs **15**, distributed by John Wiley and Sons, Inc., New York, 1968.

[29] O. A. S. Karamzadeh and N. Nazari, On maximal commutative subrings of non-commutative rings, Comm. Algebra **46** (12) (2018), 5083–5115.

[30] A. A. Klein, The finiteness of a ring with a finite maximal subring, Comm. Algebra **21** (4) (1993), 1389–1392.

[31] M. Knebusch and D. Zhang, Manis valuations and Prüfer extensions I, Lecture Notes Math. **1791**, Springer, Berlin-Heidelberg, 2002.

[32] M. Kosters, Algebras with only finitely many subalgebras, J. Algebra Appl. **14** (6) (2015), 1550086, 19 pp.

[33] T. J. Laffey, A finiteness theorem for rings, Proc. Roy. Irish Acad. Sect. A **92** (2) (1992), 285–288.

[34] B. Mitchell, Spectral sequences for the layman, Amer. Math. Monthly **76** (1969), 599–605.

[35] G. Picavet and M. Picavet-L'Hermitte, FCP Δ-extensions of rings, Arab J. Math. **10** (2021), 211-238.

[36] J. Sato, T. Sugatani and K. Yoshida, On minimal overrings of a Noetherian domain Comm. Algebra **20** (6) (1992), 1735–1746.

[37] J. Shapiro, Flat epimorphisms and a generalized Kaplansky ideal transform, Rocky Mountain J. Math. **38** (1) (2008), 267–289.