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Trace properties in integral domains, a survey

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Abstract. An integral domain R is a TP domain (or satisfies the trace property) if the trace of each R -module is equal to either R or a prime ideal of R . Equivalently, every trace ideal is a prime ideal of R , that is, for every non-zero non-invertible (fractional) ideal I of R , $I(R : I)$ is a prime ideal of R . The notion of radical trace property relaxed the requirement that each trace ideal be a prime ideal to require only that each trace ideal is a radical ideal. Equivalently, a domain R is an RTP domain (or has the radical trace property) if $I(R : I)$ is a radical ideal for each nonzero non-invertible ideal I . Two other notions related to trace property are the notion of trace property for primary ideals and L-trace property. A domain is a TPP (resp. LTP) domain if $Q(R : Q) = R$ or $Q(R : Q)$ is a prime ideal of R for every primary ideal Q of R (resp. $I(R : I)R_P = PR_P$ for each minimal prime P of $I(R : I)$). Clearly each TP domain is an RTP domain, but not conversely. Also each RTP domain is a TPP domain and each TPP domain is an LTP domain, but whether the three notions RTP, TPP and LTP are equivalent is open except in certain special cases. This survey paper tracks some old/recent works investigating these notions in different contexts of integral domains such as integrally closed domains (namely valuation and Prüfer domains), Noetherian and Mori domains, pseudo-valuation domains and pullbacks, and Nagata and Serre's conjecture rings.

Key Words: Trace ideal, radical trace property, RTP domain, LTP domain.

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1 Introduction

Throughout this paper, R denotes an integral domain with quotient field K , integral closure R' and complete integral closure \bar{R} . The trace of an R -module M is the ideal of R generated by the set $\gamma(M) = \{\varphi(m) \mid m \in M, \varphi \in \text{Hom}_R(M, R)\}$ (see, [14]). For a nonzero fractional ideal I of R , $I^{-1} = (R : I) = \{x \in K \mid xI \subseteq R\}$ is the inverse (or the dual) of I and $I_v = (R : (R : I))$ is the v -closure of I . An ideal I of R is said to be a *trace ideal* if it is the trace of some R -module. In such a case, I will in fact be its own trace [7 Proposition 7.2], equivalently $(R : I) = (I : I)$ and $\gamma(I) = II^{-1} = I$.

In 1987, Anderson, Huckaba and Papick proved that the trace of each non-invertible ideal of a valuation domain is prime [1 Theorem 2.8]. Soon after, Fontana, Huckaba and Papick generalized this result to any module over a valuation domain [14 Proposition 2.1]. This led them to introduce the *trace property* and *TP domains* [14]. A domain R is said to satisfy the trace property (R satisfies *TP* or a *TP-domain* for short) provided that $\gamma(M)$ either equals R or a prime ideal of R for each R -module M . They proved that the definition of the trace property can be restricted to integral ideals, that is, a domain R has the trace property if the trace of each non-invertible ideal is prime. Two main classes of integral domains with trace property were characterized. Namely, Noetherian and Prüfer domains. A complete characterization of Noetherian domains with the trace property stated that a Noetherian domain R is a TP domain if and only if either (a) R is a Dedekind domain, or (b) $\dim(R) = 1$ and R has a unique non-invertible maximal ideal M with $M^{-1} = \bar{R}$ (i.e., all the other maximal ideals are invertible), [14 Theorem 3.5]. They also proved that a Prüfer domain R has the trace property if

and only if R satisfies (##) and the non-invertible prime ideals are linearly ordered [14, Theorem 4.6].

In 1988, Heinzer and Papick introduced the notion of radical trace property by relaxing the requirement that each trace ideal be a prime ideal to require only that each trace ideal is a radical ideal. That is, a domain R has the *radical trace property* (or R is an *RTP domain*) if the trace of each non-invertible ideal is a radical ideal [21]. For Noetherian domains, they proved that a Noetherian domain R is an RTP domain if and only if R_P is a TP domain for each prime ideal P of R , [21, Proposition 2.1]. They also proved that a Prüfer domain R with the *acc* on prime ideals has the radical trace property if and only if R has a Noetherian prime spectrum if and only if R satisfies (##), [21, Theorem 2.7]. Later, in 1992, Gabelli extended to Mori domains the characterization of RTP Noetherian domains given in [21]. She proved that a Mori domain R satisfies the radical trace property if and only if R_P is an RTP domain for each non-zero prime ideal P of R if and only if R_P is a TP domain for each non-zero prime ideal P of R if and only if R has dimension one and either PR_P is principal or $\overline{R_P} = (R_P : PR_P)$ for each non-zero prime ideal P of R , [15, Theorem 2.14].

Two types of domains that are related to RTP domains are TPP domains and LTP domains, introduced in [27] and [24], respectively. A *TPP domain* is one for which the trace of each non-invertible primary ideal is prime (in fact is its radical [27, Corollary 8]) and an *LTP domain* is a domain R for which $IR_P = PR_P$ for each minimal prime P of a trace ideal I . Evidence suggests that the three notions may be equivalent. It is known that RTP implies TPP [27, Theorem 4], and TPP implies LTP [24, Corollary 3]. Also the three are equivalent for Prüfer domains ([27, Theorem 23] and [24, Theorem 10]), one-dimensional domains ([27, Corollary 6] and [24, page 422]), and Mori domains ([27, Theorem 12] and [24, Theorem 18]); and domain with Prüfer integral closure ([31, Theorem 2.8]).

The purpose of this paper is to review the notions of TP, RTP and LTP properties in different contexts of integral domains such as integrally closed domains, Noetherian and Mori domains, pseudo-valuation domains, pullbacks, Nagata ring $R(x)$, the ring of Serre's conjecture $R\langle x \rangle$ and more. Section 2 deals with valuation and Prüfer domains; and section 3 deals with Noetherian and Mori domains. Section 4 investigates the above properties in different types of pullback constructions and Section 5 investigates their transfer to Nagata rings and Serre's conjecture rings. It is worth mentioning that these notions (TP, RTP, TPP and LTP) were extended to rings with zero-divisors and several characterizations in Noetherian, Mori and Prüfer rings with zero-divisors were given, see [29]. However, our review concentrates only on trace properties on integral domains.

2 Valuation and Prüfer domains

We start this section by the classical result on valuation domains.

Proposition 2.1. ([14, Proposition 2.1]) *Any valuation domain is a TP-domain.*

The next Theorem characterizes Prüfer domains with the trace property. A crucial component of this characterization is the notion of (##) property. Recall that a domain R is said to be a (#)-domain if whenever Δ_1 and Δ_2 are distinct subsets of $\text{Max}(R)$, then $\bigcap_{M \in \Delta_1} R_M \neq \bigcap_{M \in \Delta_2} R_M$, and R is called a (##)-domain (or has the (##) property) in case each overring of R is a (#)-domain, [17].

Theorem 2.2. ([14, Theorem 4.6]) *Let R be a finite dimensional Prüfer domain. Then, R satisfies TP if and only if R is a (##)-domain and the non-invertible prime ideals of R are linearly ordered.*

The next theorem characterizes the class of Prüfer domains with the “radical trace property”. Recall that a prime P is branched if proper P -primary ideals exist [11]. For a prime P of a Prüfer domain, P is branched if and only if it is minimal over a finitely generated ideal [11, Theorem 23.3].

Theorem 2.3. ([27, Theorem 23]) Let R be a Prüfer domain. Then the following are equivalent.

- (1) R is an RTP domain.
- (2) R is a TPP domain.
- (3) Each branched prime is the radical of a finitely generated ideal.

In [17], Gilmer and Heinzer proved two important theorems on Prüfer domains satisfying (#) and (##) properties respectively. The first one stated that a Prüfer domain R is a (#)-domain if and only if for each maximal ideal M there is a finitely generated ideal I such that M is the only maximal ideal containing I , [17, Theorem 1]. The second stated that a Prüfer domain R is a (##)-domain if and only if for each prime ideal P there exists a finitely generated ideal $I \subseteq P$ such that each maximal ideal containing I contains P , [17, Theorem 3]. Combining these two results with Theorem 2.3, we obtain the following corollary.

Corollary 2.4. Let R be a Prüfer domain. Then:

- (1) If R is an RTP domain, then every overring of R is an RTP domain ([27, Corollary 24]).
- (2) If R has (##), then it is an RTP domain ([27, Corollary 25]).
- (3) Assume that R is an RTP domain. If every maximal ideal of R is branched, then R has (#), and if every prime ideal of R is branched, then R has (##) ([27, Corollary 26]).

Combining Theorem 2.2 and Corollary 2.4, we obtain the following Theorem.

Theorem 2.5. ([27, Theorem 28]) Let R be a Prüfer domain. Then the following are equivalent.

- (1) R is a TP domain.
- (2) R is an RTP domain and the non-invertible prime ideals are linearly ordered.
- (3) R is a TPP domain and the non-invertible prime ideals are linearly ordered.
- (4) Each branched prime is the radical of a finitely generated ideal and the non-invertible prime ideals are linearly ordered.

In [30] it is shown that if I is a trace ideal of an RTP domain R , then $IJ(R : IJ) = I$ for each trace ideal J of R containing I and $IB(R : IB) = I$ for each trace ideal B of $(R : I)$ containing I . In some cases, the converse holds. In particular, if R is restricted to being a Prüfer domain, then each of these conditions is equivalent to R being an RTP domain.

Theorem 2.6. ([30, Theorem 2.10]) Let R be a Prüfer domain. Then the following are equivalent.

- (1) R is an RTP domain.
- (2) $IB(R : IB) = I$ for each trace ideal I of R and each ideal B of $(R : I)$ that contains I .
- (3) $IJ(R : IJ) = I$ for each trace ideal I of R and each ideal J of R that contains I .

The next theorem establishes a characterization of LTP domains in terms of primary ideals.

Theorem 2.7. ([24, Theorem 2]) Let R be an integral domain. The following are equivalent.

- (1) R is an LTP domain.
- (2) For each non-invertible primary ideal Q , $Q(R : Q)R_P = PR_P$ where $P = \sqrt{Q}$.
- (3) If a primary ideal is a trace ideal, then it is prime.

The next theorem collects information concerning prime ideals of an LTP domain. Recall that for a nonzero (fractional) ideal I of R , the t -closure of I is the ideal $I_t = \cup\{J_v | J \text{ is a finitely generated sub-ideal of } I\}$; and I is said to be a t -ideal if $I = I_t$.

Theorem 2.8. ([24, Theorem 5]) Let R be an LTP domain. Then:

- (1) Each maximal ideal is a t -ideal.
- (2) Each non-maximal prime ideal is a divisorial trace ideal.
- (3) Each maximal ideal is either idempotent or divisorial.

Recall that a domain is a Prüfer v -Multiplication domain ($PvMD$ for short) if R_P is a valuation domain for every t -prime ideal P of R . Thus an integral domain R is a Prüfer domain if and only if R is a $PvMD$ and each maximal ideal is a t -ideal.

Corollary 2.9. ([24, Corollary 12]) Let R be a $PvMD$. If R is an LTP domain, then it is a Prüfer domain.

The next theorem shows that the notions RTP, TPP and LTP are equivalent over a Prüfer domain.

Theorem 2.10. ([24, Theorem 10]) Let R be a Prüfer domain. Then the following are equivalent.

- (1) R is an RTP domain.
- (2) R is a TPP domain.
- (3) R is an LTP domain.

In [14, Proposition 2.9], Fontana, Huckaba and Papick proved that Krull domains with the TP property are Dedekind domains; and in [21], Heinzer and Papick extended this result to Krull domains with the radical trace property. Recall that a domain R is an almost Krull domain if R_M is a Krull domain for each maximal ideal M of R . The next corollary extends Heinzer-Papick's result to almost Krull domains.

Corollary 2.11. ([24, Corollary 13]) Let R be an almost Krull domain. The following are equivalent.

- (1) R is a TP domain.
- (2) R is an RTP domain.
- (3) R is a TPP domain.
- (4) R is an LTP domain.
- (5) R is a Dedekind domain.

It is well-known that an integral domain R has Prüfer integral closure if and only if for each overring S of R , the extension $R \subseteq S$ satisfies INC (that is, whenever $Q \subsetneq Q'$ are prime ideals in S , then $Q \cap R \subsetneq Q' \cap R$, [34, Proposition 2. 26]). Also if R is an LTP domain such that R and $(R : I)$ satisfy INC for each trace ideal I , then R is an RTP domain ([24, Corollary 9]). Thus LTP and RTP are equivalent for domains with Prüfer integral closure.

Theorem 2.12. ([31, Theorem 2.8]) If the integral closure of R is a Prüfer domain, then R is an LTP domain if and only if it is an RTP domain.

3 Noetherian-like setting

The first proposition in this section is an important component in the characterizations of Noetherian (and Prüfer) domains with the trace property.

Proposition 3.1. ([14, Proposition 2.9]) If R is a TP domain and M a non-invertible maximal ideal of R , then each non-invertible ideal of R is contained in M . In particular, if R is a domain satisfying TP and R has a non-invertible maximal ideal, then all other maximal ideals of R are invertible.

Next, we give a complete characterization for Noetherian domains satisfying TP.

Theorem 3.2. ([14, Theorem 3.5]) Let R be a Noetherian domain. Then R satisfies TP if and only if either (a) R is a Dedekind domain, or (b) $\dim(R) = 1$ and R has a unique non-invertible maximal ideal M with $M^{-1} = \bar{R}$ (i.e., all other maximal ideals of R are invertible.)

Example 3.3. Let k be a field, X an indeterminate over k and $R = k[[X^2, X^3]]$. Clearly R is a one-dimensional Noetherian local domain with maximal ideal $M = (X^2, X^3)$ and $M^{-1} = k[[X]] = \bar{R}$. By Theorem 3.2, R is a TP domain which is not integrally closed.

The next theorem shows how arbitrary Noetherian TP domains are constructed.

Theorem 3.4. ([14, Theorem 3.6]) A domain R is a Noetherian TP domain if and only if there exists a Dedekind domain T containing R and an ideal I of T such that:

- (a) T/I is a finitely generated k -module, where k is a subfield of the ring T/I ;
 (b)

$$\begin{array}{ccc} R := & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \xrightarrow{v} & T/I. \end{array}$$

is a pullback diagram, where the down arrow map u is the inclusion map and v is the canonical surjection.

The first example shows how to construct Noetherian TP domains that are not Gorenstein domains and the second example shows how to construct Noetherian TP domains with infinitely many maximal ideals.

Example 3.5. ([14, Example 3.7]) (1) Let $k \subseteq K$ be an algebraic extension of fields such that $[K : k] = n \geq 3$, X an indeterminate over k and $R = k + XK[[X]]$. Then R is a Noetherian TP domain which is not Gorenstein.

(2) Let K be an algebraically closed field, X an indeterminate over K , $T = K[X]$ and $I = \bigcap_{i=1}^{i=n} (X - a_i)$, $a_i \in K$.

Let $u : K \longrightarrow T/I = \prod_{i=1}^{i=n} K$ and R be the pullback in the diagram of Theorem 3.4.

In [14, Lemma 3.3], it was proved that for a Noetherian TP domain R , the localization R_P at any prime ideal P of R is a TP domain. The next proposition shows that in fact this is a characterization of Noetherian RTP domains.

Proposition 3.6. ([21, Proposition 2.1]) Let R be a Noetherian domain. Then R is an RTP domain if and only if R_P is a TP domain for each prime ideal P of R .

Combining results from [14] and [21], we have the following characterization of Noetherian domains with the radical trace property.

Theorem 3.7. ([31, Theorem 2.14]) The following are equivalent for a Noetherian domain R that is not a field.

1. R is an RTP domain.
2. R_M is a TP domain for each maximal ideal M .

3. R is one-dimensional and for each maximal ideal M , either R_M is a discrete rank one valuation domain or $MR_M = (R_M : R'_M)$.
4. R is one-dimensional and for each maximal ideal M , MR_M is an ideal of R'_M .

Recall that a Mori domain is a domain satisfying the ascending chain conditions on divisorial ideals. Noetherian and Krull domains are Mori domains. The next theorem, due to Gabelli, extends to Mori domains the characterization of RTP Noetherian domains.

Theorem 3.8. ([15, Theorem 2.14]) Let R be a Mori domain. The following are equivalent:

- (1) R is an RTP domain.
- (2) R_P is an RTP domain for each non-zero prime ideal P of R .
- (3) R_P is a TP domain for each non-zero prime ideal P of R .
- (4) R has dimension one and either PR_P is principal or $\bar{R}_P = (R_P : PR_P)$ for each non-zero prime ideal P of R .

The complete statement in Theorem 2.6 does not extend to all domains, not even those with Prüfer integral closure, including one-dimensional Noetherian domains. On the other hand, if R is either a Noetherian domain or a Mori domain, then condition (2) in Theorem 2.6 is enough to get the radical trace property.

Example 3.9. Let $R = k[[X^2, X^5]]$ where k is a field. Then R is a one-dimensional local Noetherian domain with maximal ideal $M = (X^2, X^5)$ with $M^{-1} = k[[X^2, X^3]]$. So R is not an RTP domain (in fact R is not a TP domain). However, R has the property that for each trace ideal I of R , $IJ(R : I) = I$ for each ideal J of R that contains I .

Theorem 3.10. ([30, Theorem 3.4]) Let R be a Mori domain. Then R is an RTP domain if and only if $IB(R : IB) = I$ for each trace ideal I of R and each ideal B of $(R : I)$ that contains I .

Recall that an ideal I of an integral domain R is said to be SV-stable (short for "Sally-Vasconcelos stable") if I is invertible as an ideal of $(I : I)$ ([1]). The next theorem establishes a link between Mori domains being RTP domains and SV-stability of maximal ideals.

Theorem 3.11. ([24, Theorem 18]) Let R be a Mori domain, which is not a field. Then the following are equivalent.

- (1) R is an RTP domain.
- (2) R is a TTP domain.
- (3) R is an LTP domain.
- (4) For each maximal ideal M and each M -primary ideal Q , M is SV-stable and QQ^{-1} contains M .
- (5) For each maximal ideal M , M is SV-stable and each maximal ideal of $(M : M)$ that contains M is invertible as an ideal of $(M : M)$.
- (6) For each non-zero radical ideal I of R , I is SV-stable and each maximal ideal of $(I : I)$ that contains I is invertible as an ideal of $(I : I)$.

Next, we focus on what occurs when the dual of a trace ideal of an RTP domain or LTP domain is either an RTP domain or an LTP domain.

Theorem 3.12. ([30, Theorem 5.1]) Let R be an RTP domain and let I be a trace ideal of R . If $(R : I)$ is either an LTP domain or an RTP domain, then each maximal ideal of $(R : I)$ that contains I is either idempotent (with trivial dual) or invertible, and each of these ideals is minimal over I .

The next theorem establishes the converse for Theorem 3.12 under the additional assumption that I is also a radical ideal of $(R : I)$.

Theorem 3.13. ([30, Theorem 5.3]) Let R be an LTP (resp. an RTP) domain and let I be a trace ideal of R . If I is a radical ideal of $(R : I)$, then the following are equivalent.

- (i) $(R : I)$ is an LTP (resp. RTP) domain.
- (ii) $R \subseteq (R : I)$ satisfies INC.
- (iii) Each minimal prime of I in $(R : I)$ is a maximal ideal of $(R : I)$.

4 Pullbacks

Let T be an integral domain, M a maximal ideal of T , K its residue field, $\phi : T \rightarrow K$ the canonical surjection, D a proper subring of K , and $k := \text{qf}(D)$. Let $R := \phi^{-1}(D)$ be the pullback issued from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & K = T/M \end{array}$$

We assume that $R \subsetneq T$ and we refer to this diagram as a diagram of type (\square) .

Theorem 4.1. ([25, Theorems 11, 12, 13 and 15]) For the diagram of type (\square) , R is an LTP (resp. TPP, resp. RTP) domain if and only if both T and D are LTP (resp. TPP, resp. RTP) domains. Further assume that T is quasilocal. Then R is a TP domain if and only if both T and D are TP domains.

Recall from Hedstrom and Houston (1978) that a domain R is pseudo-valuation domain if it is quasilocal and shares its maximal ideal with a valuation domain which necessarily must contain R and be unique. In terms of pullbacks, according to [2, Proposition 2.6], R is a pseudo-valuation domain if and only if there is a valuation domain V with maximal ideal M and a subfield k of $V/M = K$ such that R is the pullback in the following diagram

$$\begin{array}{ccc} R & \longrightarrow & k \\ \downarrow & & \downarrow \\ V & \xrightarrow{\phi} & K = V/M \end{array}$$

Corollary 4.2. *Every pseudo-valuation domain is a TP domain.*

The next example shows that R may be not a TP domain when T is not local.

Example 4.3. ([25, Example 33]) Let k be a field and let X and Y be indeterminates over k . Set $T = k[Y] + Xk(Y)[X]$, $M = (X + 1)k(Y)[X] \cap T$ and $Q = Xk(Y)[X]$. Let R be the pullback in the following diagram:

$$\begin{array}{ccc} R & \longrightarrow & D = k[Y] \\ \downarrow & & \downarrow \\ V & \longrightarrow & K = V/M \end{array}$$

Then: (a) Both T and D are TP domains.

(b) $J = M \cap Q$ is a trace ideal of R that is not a prime ideal.

(c) R is not a TP domain.

The next theorem characterizes different types of pullbacks where the ideal M is not necessarily a maximal ideal of T . We shall consider the following cases:

Case 1. “Radical ideal”: M is a radical ideal of T and T/M contains a field F , D a subdomain of F with $\text{qf}(D) = F$ and each minimal prime of M is a maximal ideal of T . We refer to this diagram as a diagram of type (\square_1) .

Case 2. “An irredundant intersection” M is an irredundant intersection of its minimal primes and for which each such minimal prime is a maximal ideal of T . T/M contains a field F , D a subdomain of F with $\text{qf}(D) = F$. We refer to this diagram as a diagram of type (\square_2) .

Case 3. “ \sqrt{M} is invertible”: M is an ideal of T such that \sqrt{M} is invertible, each minimal prime of M in T is a maximal ideal of T and that T/M contains a field F , D a subdomain of F with $\text{qf}(D) = F$. We refer to this diagram as a diagram of type (\square_3) .

Theorem 4.4. (1) For the diagram (\square_1) , R is an LTP domain if and only if both T and D are LTP domains, ([25, Corollary 18]).

(2) For the diagram (\square_2) , R is an RTP (resp. a TPP) domain if and only if both T and D are RTP (resp. TPP) domains, ([25, Corollary 22]).

(3) For the diagram (\square_3) R is an LTP (resp. RTP, resp. TPP) domain if and only if T and D are LTP (resp. RTP, resp. TPP) domains, ([25, Corollary 29]).

(4) For the diagram (\square_3) assume further that T is a Dedekind domain. Then R is a TP domain if and only if D is a TP domain, ([25, Theorem 30]).

Example 4.5. ([25, Example 31]) Let $T = k[X^2, X^3]$ and $R = k[X^2, X^5]$ with $M = (X^2, X^5)R$. Then:

(i) T is an RTP domain and $M = X^2T$ is an invertible ideal of T , but the radical of M in T is the maximal ideal $N = (X^2, X^3)T$ which is not invertible (as an ideal of T , but is invertible in $k[x] = (T : N)$).

(ii) The ring R is not even an LTP domain. The ideal $I = (X^4, X^5)R$ is a proper M -primary trace ideal of R .

The next example shows that T can have a trace property while R does not when we only have that M , and not the radical of M in T , is invertible as an ideal of T even if the radical of M in T is a maximal ideal.

Example 4.6. ([25, example 32]) Let T be a one-dimensional valuation ring of the form $F + N$ which is not discrete and let x be a nonzero nonunit of T . Let $M = xT$ and $R = F + M$. Since T is a valuation domain, it has the trace property. Obviously, M is an invertible ideal of T , but its radical is not. The ideal $I = xN$ is a proper M -primary trace ideal of R . Thus R is not even an LTP domain.

5 The rings $R(x)$ and $R\langle x \rangle$

Recall that the content of a polynomial $f(x) \in K[x]$ with respect to the domain R is the fractional ideal of R generated by the coefficients of $f(x)$. If the coefficients generate R as an ideal, then $f(x)$ is said to have unit content. We let $\mathcal{U}(R)$ denote the polynomials in $R[x]$ with unit content and let $\mathcal{M}(R)$ denote the set of monic polynomials of $R[x]$. The Nagata ring is the ring $R(x) = R[x]_{\mathcal{U}(R)}$, and the ring $R\langle x \rangle$ (also called the ring of Serre’s conjecture) is the ring defined by $R\langle x \rangle = R[x]_{\mathcal{M}(R)}$.

The first main theorem characterizes when the rings $R(x)$ and $R\langle x \rangle$ are LTP (resp. RTP) domains over an integrally closed domain R .

Theorem 5.1. ([31, Theorem 3.6]) The following are equivalent for an integrally closed domain R that is not a field.

1. $R\langle x \rangle$ is an LTP domain.
2. $R\langle x \rangle$ is an RTP domain.

3. $R(x)$ is an LTP domain and R is one-dimensional.
4. $R(x)$ is an RTP domain and R is one-dimensional.
5. $R(x)$ is an LTP domain and R is a one-dimensional Prüfer domain.
6. $R(x)$ is an RTP domain and R is a one-dimensional Prüfer domain.
7. R is both a one-dimensional Prüfer domain and an LTP domain.
8. R is both a one-dimensional Prüfer domain and an RTP domain.

Recall that a domain R (with quotient field K) is seminormal if for every $x \in K$, such that $x^2, x^3 \in R$ we must have that $x \in R$. In [31, Theorem 2.12], it was proved that if R is seminormal and $R(x)$ is an LTP domain, then the integral closure R' of R is a Prüfer domain and both R and $R(x)$ are RTP domains. As a pseudo-valuation domain is seminormal, this yields that if R is a pseudo-valuation domain such that $R(x)$ is an LTP domain, then $R(x)$ is a pseudo-valuation domain and both R' and $R'(x)$ are valuation domains (corresponding to R and $R(x)$, respectively). The next theorem deals with a pseudo-valuation domain R such that $R(x)$ satisfies one of the trace properties.

Theorem 5.2. ([31, Theorem 2.15]) Let R be a pseudo-valuation domain with maximal ideal M and corresponding valuation domain V . Then the following are equivalent.

1. $R(x)$ is a TP domain.
2. $R(x)$ is an RTP (LTP) domain.
3. $R(x)$ is a pseudo-valuation domain.
4. V is the integral closure of R .
5. $R'(x)$ is a TP domain.
6. $R'(x)$ is an RTP (LTP) domain.

The next two theorems examine trace properties of $R(x)$ over Noetherian and Mori domains.

Theorem 5.3. ([31, Theorem 2.17]) The following are equivalent for a Noetherian domain R .

1. R is an LTP domain.
2. R is an RTP domain.
3. $R(x)$ is an LTP domain.
4. $R(x)$ is an RTP domain.

Notice that the analogous result does not hold for Mori domains as is shown by the following example. In fact, what allows the equivalence to hold for Noetherian domains is that the integral closure of a one-dimensional Noetherian domain is a Dedekind domain.

Example 5.4. Let k be a field, y and z indeterminates over k and consider the domain $R = k + yk(y)[[z]]$. The complete integral closure of R is the valuation domain $V = k(y)[[z]]$ and $M = yk(y)[[z]]$ is the conductor of V into R . As R is the pullback of a field over the maximal ideal of a discrete rank one valuation, it is both a PVD and a Mori domain (see, for example, [20] and [6], respectively). Since R is a PVD, it has the trace property [25, Page 1098]. But R is integrally closed (since k is algebraically closed in $k(y)$) and not a Prüfer domain. Thus $R(x)$ does not have the radical trace property.

Theorem 5.5. ([31] Theorem 2.18) The following are equivalent for a Mori domain R .

1. R is an LTP domain and R' is a Prüfer domain.
2. R is an RTP domain and R' is Prüfer domain.
3. R is an RTP domain and R' is a Dedekind domain.
4. $R\langle x \rangle$ is an RTP domain.
5. $R\langle x \rangle$ is an LTP domain.

Example 5.6. Let \mathbb{Q} be the field of rational numbers and consider the ring $R = \mathbb{Q} + \gamma L[[Y]]$ where L is an infinite algebraic extension of the rational numbers. Then R is both a Mori domain and a PVD but not Noetherian. On the other hand, the integral closure of R is the valuation domain $V = L[[Y]]$ with which R shares its maximal ideal $M = \gamma L[[Y]]$. Moreover, $MV\langle x \rangle$ is the maximal ideal of both $R\langle x \rangle$ and the valuation domain $V\langle x \rangle$. Hence $R\langle x \rangle$ is a PVD and therefore $R\langle x \rangle$ is a TP domain.

The next three theorems establish characterizations for $R\langle x \rangle$ being an RTP/LTP domain for Noetherian domains, Mori domains and PVDs.

Theorem 5.7. ([31] Theorem 3.7) If R is a Noetherian domain that is not a field, then the following are equivalent.

1. R is an RTP domain.
2. $R\langle x \rangle$ is an RTP domain.
3. $R\langle x \rangle$ is an RTP domain.
4. $R\langle x \rangle$ is an LTP domain.

Theorem 5.8. ([31] Theorem 3.11) If R is a Mori domain that is not a field, then the following are equivalent.

1. R is an RTP domain and R' is a Prüfer domain.
2. $R\langle x \rangle$ is an RTP domain.
3. $R\langle x \rangle$ is an RTP domain.
4. $R\langle x \rangle$ is an LTP domain.

Theorem 5.9. ([31] Theorem 3.12) The following are equivalent for a pseudo-valuation domain R with nonzero maximal ideal M and corresponding valuation domain V .

1. $R\langle x \rangle$ is an LTP domain.
2. $R\langle x \rangle$ is an RTP domain.
3. $R\langle x \rangle$ is a TP domain.
4. R is one-dimensional and V is the integral closure of R .

Recall that a domain R is said to be h -local if each nonzero nonunit is contained in only finitely many maximal ideals and each nonzero prime ideal is contained in a unique maximal ideal. A domain for which each nonzero nonunit is contained in only finitely many maximal ideals is said to have finite character. Thus a one-dimensional domain is h -local if and only if it has finite character.

Theorem 5.10. ([31] Theorem 3.9) Let R be an h -local domain. Then R is an RTP domain if and only if R_M is an RTP domain for each maximal ideal M .

Let R be a one-dimensional domain. Then the maximal ideals of $R\langle x \rangle$ are of two types, $MR\langle x \rangle$ for some maximal ideal M of R and $PR\langle x \rangle$ for some upper to zero P (i.e., $P \cap R = 0$) of $R[x]$ that does not contain a monic polynomial, but does contain one with unit content in R . If, in addition, the integral closure of R is a Prüfer domain, then $R\langle x \rangle$ is one-dimensional. We conclude with the following result on one-dimensional domains.

Theorem 5.11. ([31] Theorem 3.16) The following are equivalent for a one-dimensional domain R .

1. $R\langle x \rangle$ is an RTP domain.
2. $R\langle x \rangle$ is an LTP domain.
3. $R(x)$ is an LTP domain.
4. $R(x)$ is an RTP domain.

We close this survey with some open questions.

6 A few open questions

Open questions/problems with regard to RTP/TPP/LTP domains include the following.

Q1: If $R(x)$ is an LTP domain, is it an RTP domain? The answer is "Yes" if and only if $R(x)$ LTP implies $R[x]$ is an almost principal ideal domain. Recall that a nonzero ideal J of $R[x]$ is said to be *almost principal* if there is a nonzero element $r \in R$ and a polynomial $g(x) \in J$ of positive degree such that $rJ \subseteq g(x)R[x]$. The polynomial ring $R[x]$ is said to be an almost principal ideal domain if each nonzero ideal of $R[x]$ with proper extension to $K[x]$ is almost principal [19, Page 65].

Q2: If R' is a Prüfer domain and R is an RTP domain, is $R(x)$ an RTP domain?

Q3: If R has finite character, is R an RTP domain if and only if R_M is an RTP domain for each maximal ideal M ?

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References

- [1] D. D. Anderson, J. Huckaba and I. Papick, A note on stable domains, *Houston J. Math.* 13 (1987), 13–17.
- [2] D. F. Anderson and D. E. Dobbs, Pairs of rings with the same prime ideals, *Canad. J. Math.* 32 (1980), 362–384.
- [3] D. F. Anderson, D. E. Dobbs and M. Fontana, On treed Nagata rings, *J. Pure Appl. Algebra* 61 (1989), 107–122.
- [4] J. Arnold, On the ideal theory of the Kronecker function ring and the ring $D(X)$, *Canad. J. Math.* 21 (1969), 558–563.

- [5] A. Ayache, P.-J. Cahen, and O. Echi, Anneaux quasi-préfériens un P-anneaux, *Boll. Un. Mat. It.* 10 (1996), 1–24.
- [6] V. Barucci, On a class of Mori domains, *Comm. Algebra* 11 (1983), 1989–2001.
- [7] H. Bass, On the ubiquity of Gorenstein rings, *Math. Z.* 82 (1963), 8–28.
- [8] S. Bazzoni and L. Salce, Warfield domains, *J. Algebra* 185 (1996), 836–868.
- [9] J. Brewer and D. L. Costa, Projective modules over some non-Noetherian polynomial rings. *J. Pure Appl. Algebra* 13 (1978), 157–163.
- [10] P.-J. Cahen and T. Lucas, The special trace property, in “Commutative Ring Theory” (Fès, Morocco, 1995), *Lecture Notes in Pure and Applied Mathematics* 185, 161–172, Marcel Dekker, New York, 1997.
- [11] G. W. Chang and M. Fontana, Uppers to zero in polynomial rings and Prüfer-like domains, *Comm. Algebra* 37 (2009), 164–192.
- [12] A. de Souza Doering and Y. Lequain, Chains of primes in polynomial rings, *J. Algebra* 78 (1982), 163–180.
- [13] D. E. Dobbs, E. Houston, T. Lucas, M. Roitman and M. Zafrullah, On t -linked overrings, *Comm. Algebra* 20 (1992), 1463–1488.
- [14] M. Fontana, J. Huckaba and I. Papick, Domains satisfying the trace property, *J. Algebra* 107 (1987), 169–182.
- [15] S. Gabelli, Domains with the radical trace property and their complete integral closure, *Comm. Algebra*, 20 (1992), 829–845.
- [16] R. Gilmer, *Multiplicative Ideal Theory*, *Queen’s Papers in Pure and Applied Mathematics*, 90, Queen’s Univ. Press, Kingston, 1992.
- [17] R. Gilmer and W. Heinzer, Overrings of Prüfer rings, II, *J. Algebra* 7 (1967), 281–302.
- [18] R. Gilmer and J. Hoffman, A characterization of Prüfer domains in terms of polynomials, *Pacific J. Math.* 60 (1975), 81–85.
- [19] E. Hamann, E. Houston and J. Johnson, Properties of uppers to zero in $R[x]$, *Pacific J. Math.* 135 (1988), 65–79.
- [20] J. Hedstrom and E. Houston, Pseudo-valuation domains, *Pacific J. Math.* 75 (1978), 137–147.
- [21] W. Heinzer and I. Papick, The radical trace property, *J. Algebra* 112 (1988), 110–121.
- [22] E. Houston, S. Kabbaj, T. Lucas and A. Mimouni, When is the dual of an ideal a ring?, *J. Algebra*, 225 (2000), 429–450.
- [23] J. Johnson, Three topological properties from Noetherian rings, *Canad. J. Math.* 34 (1982), 525–534.
- [24] S. Kabbaj, T. Lucas and A. Mimouni, Trace properties and integral domains, in *Advances in Commutative Ring Theory*, (Fez, Morocco 1997), 421–436, *Lecture Notes in Pure and Applied Mathematics*, 205, Marcel Dekker, New York, 1999.

- [25] S. Kabbaj, T. Lucas and A. Mimouni, Trace properties and pullbacks, *Comm. Algebra*, 31 (2003), 1085–1111.
- [26] L. le Riche, The ring $R\langle X \rangle$, *J. Algebra* 67 (1980), 327–341.
- [27] T. Lucas, The radical trace property and primary ideals, *J. Algebra* 184 (1996), 1093–1112.
- [28] T. Lucas, Almost principal ideals of $R[x]$, *Arabian J. Math.*, 1 (2012), 97–111.
- [29] T. Lucas and McNair, D. Trace properties in rings with zero divisors, *J. Algebra* 343 (2011), 201–223.
- [30] T. Lucas and A. Mimouni, Trace Properties and Integral Domains, II, *Comm. Algebra* 40 (2012), 497–513.
- [31] T. Lucas and A. Mimouni, Trace properties and the rings $R(X)$ and $R\langle x \rangle$, *Annali di Matematica Pura ed Applicata* 199 (2020), 2087–2104.
- [32] T. Lucas and A. Mimouni, Trace Properties and Integral Domains, III, submitted.
- [33] T. Nishimura, On the v -ideal of an integral domain, *Bull. Kyoto Gakugei Univ., Ser B* 17 (1961), 47–50.
- [34] I. Papick, Topologically defined classes of going-down domains. *Trans. Amer. Math. Soc.* 219 (1976), 1–37.
- [35] F. Richman, Generalized quotient rings, *Proc. Amer. Math. Soc.* 16 (1965), 794–799.