Moroccan Journal of Algebra and Geometry with Applications
Supported by Sidi Mohamed Ben Abdellah University, Fez,

Volume 1, Issue 1 (2022), pp 18-48

ISSN: 2820-7114

Title :

On constructing angles with prescribed vertex and measure in the upper half-plane model of hyperbolic geometry

Author(s):

David E. Dobbs

Moroccan Journal of Algebra
and Geometry with Applications
Vol. 1(1) (2022), 18–48
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco
ISSN: 2820-7114

On constructing angles with prescribed vertex and measure in the upper half-plane model of hyperbolic geometry

David E. Dobbs
Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1320, USA
e-mail: *ddobbs1@utk.edu*

Communicated by Najib Mahdou
(Received 11 April 2021, Revised 26 May 2021, Accepted 30 May 2021)

Abstract. It is proved that if $k, m \in \mathbb{R}$ and $P_0(x_0, y_0)$ is a point in the Euclidean plane $\mathbb{R}^2$ with $y_0 \neq k$ and with $\mathcal{X}$ denoting the (horizontal) line with Cartesian equation $y = k$, then there exists a unique circle, say $\mathcal{K}$, such that the center of $\mathcal{K}$ is on $\mathcal{X}$, P_0 lies on $\mathcal{K}$, and the tangent to $\mathcal{K}$ at P_0 has slope m. An ensuing multi-step algorithmic result that is proved here for the Euclidean upper half-plane determines the angular measure of any directed angle that is formed by counterclockwise rotation from a designated initial side to a designated terminal side, in case each of those "sides" is a hyperbolic line segment (that is, either a vertical (Euclidean) line segment or an arc of a (Euclidean) circle centered on the x-axis). One consequence (for the Euclidean upper half-plane) is the construction of (the unique hyperbolic line segment playing the role of terminal (resp., initial) side of) a unique directed angle having a prescribed vertex, a prescribed measure between 0 and π, and a prescribed hyperbolic line segment as initial (resp., terminal) side. As the only prerequisites assumed here are related topics in analytic geometry and trigonometry that can be covered in a precalculus course, this paper could be used as enrichment material for a precalculus course, a calculus course, or a course on the classical geometries that features the upper half-plane model of hyperbolic plane geometry.

Key Words: Hyperbolic plane geometry, upper half-plane model, directed angle, vertex, Euclidean geometry, slope, tangent line, inverse tangent function, angle of inclination, inverse cosine function, bowed geodesic, straight geodesic.
2010 MSC: Primary 51-02; Secondary 33B10, 51N20, 51M04.

1 Introduction

In the mid-to-late 20$^{\text{th}}$ century, undergraduate courses on classical (Euclidean or hyperbolic) geometry, based on multi-edition textbooks such as [10] and [7], emphasized an axiomatic approach. Since 1993, a popular alternative for such courses has featured a model-based approach in multi-edition textbooks by Millman-Parker [9] and Stahl [12]. The viability of studying hyperbolic (plane) geometry via its upper half-plane model was popularized in 1980 by Millman [8]. Such studies are especially attractive to many undergraduates (and even to some high school students), as the required background consists only of some high school material (basic algebra, Euclidean plane geometry and trigonometry) and occasionally basic calculus. Augmenting the verifications of axioms in [8], a rigorous proof was given in [4] to show that the upper half-plane model satisfies the axiom concerning parallel lines that is appropriate for hyperbolic geometry. In addition, the increasing availability of calculators and computer programs to compute certain definite/line integrals led to a paper [3] which established that much of [12] could be carried out via a (hyperbolic) distance formula that is readily implemented on graphing calculators or computers. However, despite all the recent attention that has been given to carrying out the program spearheaded by Millman, Parker and Stahl, little attention has been given to proving and implementing methodology to measure angles in the upper half-plane model. The main purpose of this paper is to present, prove and illustrate such algorithmic methodology. As this work is done in the upper half of the Euclidean plane $\mathbb{R}^2$, much of this paper could serve as enrichment material for a precalculus course, some of this paper could be used to enrich a differential calculus course, and all of this paper could be used to enrich a course that studies hyperbolic geometry via the upper half-plane model.

In hyperbolic geometry (from now on, that will mean "in the upper half-plane model"), when two curves $\mathcal{F}$ and $\mathcal{G}$ intersect at a point P, typically four angles having vertex P are thereby created, and the (hyperbolic) measure of any of these angles is defined to be the usual (Euclidean) measure of the Euclidean angle having vertex P formed by the corresponding (Euclidean) tangential half-lines to $\mathcal{F}$ and $\mathcal{G}$. In practice, attention is focused on $\mathcal{F}$ and $\mathcal{G}$ being the "lines" (that is, the hyperbolic geodesics) of hyperbolic geometry. As is well known (cf. [8, Proposition 1], [12, Theorem 4.2.1]), a "line" in hyperbolic geometry is of one of two types: a so-called "straight geodesic," which is the intersection of the upper half-plane with a vertical (Euclidean) line; or a so-called "bowed geodesic," which is the intersection of the upper half-plane with a (Euclidean) circle whose center is on the x-axis. Of course, if $\mathcal{F}$ and $\mathcal{G}$ are (hyperbolic) "lines," one can work with their (Euclidean) tangential (half-)lines at the precalculus level, where the most relevant concept is that of "slope." For that reason, we begin with some slope-centered results that could be placed into a unit on the analytic geometry of circles and lines in a precalculus course. Their upshot for hyperbolic geometry is this: if $m \in \mathbb{R}$ and P is a point in the upper half-plane of $\mathbb{R}^2$, then there exists a unique bowed geodesic $\mathcal{K}$ such that P lies on $\mathcal{K}$ and the tangent to $\mathcal{K}$ at P has slope m. Thus, the measurement of angles in hyperbolic geometry (whose "sides" are portions of straight or bowed geodesics) comes down to measuring the angles formed at the intersection of a (Euclidean) line with slope m_1 with either a (Euclidean) line with slope m_2 or a (Euclidean) vertical line. Formulas to accomplish that, in turn, were given in [5, Theorem 2.2], which is restated below for convenience as Lemma 2.6.

For the most part, our interest here will be in angles whose measure is between 0 and π (because of their usefulness in studying triangles), but angles with measures between π and 2π will arise naturally in Corollary 2.18. As right angles are generally easy to detect (we will say more about this later), it may seem that the desired methodology for measuring angles would end with an application of Lemma 2.6. However, in order to apply Lemma 2.6, a user first needs to know (or should be reasonably confident as to) whether the non-right angle that is being measured is acute or obtuse. (As usual, if an angle has measure strictly between 0 and π, we say that angle is *acute* (resp., *obtuse*) if its measure is less than (resp., greater than) $\pi/2$.) If a user wishes to be certain whether a given non-right angle (with measure between 0 and π) that is being measured is acute or obtuse, Corollary 2.13 presents a computational 12-step algorithm to find that angle's measure without the need to have an opinion as to whether the angle is acute or obtuse. The "acute/obtuse" part of this algorithm is handled by the usual dot product of (bound) vectors. Familiarity with this concept, which is covered in only some precalculus courses, is not being assumed here. Indeed, in Example 2.15, the worked examples (illustrating uses of the algorithm from Corollary 2.13) are sufficiently detailed as to make the calculations transparent.

In working with an interior angle of a (Euclidean or hyperbolic) triangle, one often does not designate an "initial side" or a "terminal side" of the angle, perhaps out of a desire that each such angle have measure strictly between 0 and π. We will definitely *not* adopt that point of view here. For us, each "angle" being considered (regardless of whether its study involves a related triangle) will be understood to be a *directed angle* (that is, an angle that can be viewed as having arisen via a counterclockwise rotation from a designated initial side to a designated terminal side). We believe that the literature on angles (in both Euclidean and hyperbolic geometry) is often unclear on such matters. For that reason, we will often include "directed" (and, less often, "counterclockwise") in describing angles that figure in the statements of results. For more about the time-honored role of (directed) angles in the teaching of geometry and trigonometry, see Remark 2.4 (a) and Remark 2.11 (a).

Hyperbolic geometry is decidedly non-Euclidean, inasmuch as the open neighborhoods (relative to the hyperbolic metric) "near" two given points in the upper half-plane may not "look alike" (especially if the given points have different y-coordinates). Fortunately, our slope-based results that lead to the angle-measuring algorithm in Corollary 2.13 can also be used to establish Theorem 2.17 and Corollary 2.18, which give some ways (involving angles) in which hyperbolic geometry *does* behave like Euclidean geometry. For brevity, we state only Theorem 2.17 next. Let $\mathcal{G}$ be a given hyperbolic half-line emanating from a point P in the upper half-plane; let T be the tangential half-line of $\mathcal{G}$ emanating from P; let $\xi \in \mathbb{R}$ such that $0 \leq \xi \leq \pi$; then there exists a unique hyperbolic half-line $\mathcal{H}$ emanating from P such that the directed angle with initial (resp., terminal) side $\mathcal{G}$ and terminal (resp., initial) side $\mathcal{H}$ has measure ξ.

We close with some comments about terminology and notation. Since there is no such thing as a "vertical angle," students are often uncomfortable with the term "vertical angles." In sympathy with their unease, we will not use that term, preferring instead the older, more suggestive, term "vertically opposite angles." Also, given two distinct points P and Q in the upper half-plane, we will use the notation $\overrightarrow{PQ}$ in two different (but time honored) ways: to mean either the (Euclidean or hyperbolic) half-line that has initial point P and passes through Q; or the (bound) vector (also known as a directed line segment) with initial point P and terminal point Q, that is likely familiar from geometry, physics and calculus. We trust that context will always allow the reader to know in which sense this notation is being used here.

2 Results

We begin with an elementary but useful result in the analytic geometry of the Euclidean plane. Corollary 2.2 will state its application to a context that is suited to the upper half-plane model of hyperbolic geometry. While Theorem 2.1 is set in a more general context, we will assume that its line $\mathcal{X}$ is horizontal, in order to avoid unnecessary complications in dealing with the associated analytic geometry. The interested reader/instructor/student is invited to develop a generalization of Theorem 2.1 in case $\mathcal{X}$ is not assumed to be horizontal.

We offer two proofs of Theorem 2.1 which share a common beginning. The conclusion for the first proof would be accessible early in a precalculus course, whereas the conclusion for the second proof assumes a familiarity with the basics of differential calculus.

Theorem 2.1. Let $k, m \in \mathbb{R}$ with $m \neq 0$, and let $P_0(x_0, y_0)$ be a point in the Euclidean plane $\mathbb{R}^2$ such that $y_0 \neq k$. Let $\mathcal{X}$ denote the (horizontal) line with Cartesian equation $y = k$. Then there exists a unique circle, say $\mathcal{K}$, such that the center of $\mathcal{K}$ is on $\mathcal{X}$, P_0 lies on $\mathcal{K}$, and the tangent to $\mathcal{K}$ at P_0 has slope m. A Cartesian equation for this circle $\mathcal{K}$ is

$$x^2 - 2[x_0 + m(y_0 - k)]x + y^2 - 2ky = y_0^2 - x_0^2 - 2mx_0y_0 + 2k(mx_0 - y_0).$$

Proof. The circle centered at a typical point $C(c, k)$ on $\mathcal{X}$ and having radius r (> 0) has Cartesian equation

$$(x - c)^2 + (y - k)^2 = r^2;$$

and, by the distance formula, this circle passes through P_0 if and only if

$$(x_0 - c)^2 + (y_0 - k)^2 = r^2.$$

Thus, to prove the first assertion, it suffices to show that there is a unique $c \in \mathbb{R}$ such that the tangent to the graph of

$$(x - c)^2 + (y - k)^2 = (x_0 - c)^2 + (y_0 - k)^2$$

at the point P_0 has slope m.

Let T denote the tangent (line) to the above circle at P_0 and let S denote the line segment (radius vector) from C to P_0. By an elementary result about circles in Euclidean plane geometry, T and S are perpendicular. As we are given that $m \neq 0$, we need only address the case where T is not horizontal; equivalently, where S is not vertical; equivalently, where the slope of S is defined; equivalently, where $x_0 - c \neq 0$. Of course, $x_0 - c \neq 0 \Leftrightarrow x_0 \neq c \Leftrightarrow y_0 - k \neq \pm r$. Also, the hypotheses now ensure that, for each circle that we are considering, T does have a slope, that is, T is not vertical; equivalently, S is not horizontal; equivalently, the slope of S is nonzero; equivalently, $(y_0 - k)/(x_0 - c) \neq 0$; equivalently, $y_0 \neq k$ (which holds by assumption) and $x_0 \neq c$ (which holds by the above restriction).

Continuation of the first proof: By the reduction in the preceding paragraph, S is not vertical. Then the slope of S is $(y_0 - k)/(x_0 - c)$, which is well defined (and nonzero, as noted above). Let μ denote the slope of T. Recall the elementary result in analytic geometry that two non-vertical lines are perpendicular if and only of the product of their slopes is -1. Therefore, since T and S are perpendicular, it follows that $\mu \cdot [(y_0 - k)/(x_0 - c)] = -1$; that is, $\mu = [(k - y_0)/(x_0 - c)]^{-1}$. As the above circle has the desired properties if and only if $\mu = m$, the first assertion will follow by showing that c is uniquely determined by the condition that $(x_0 - c)/(k - y_0) = m$. This is, in fact, the case, with $c = x_0 + m(y_0 - k)$. Observe that this value of c also satisfies the above condition that $x_0 \neq c$ (since $m \neq 0$ and $y_0 \neq k$).

It remains only to obtain the asserted Cartesian equation for $\mathcal{K}$. This can be shown by substituting the above formula for c into the third displayed equation in this proof and then doing some routine algebraic simplifications. Those details can safely be left to the reader. This completes the first proof.

Continuation of the second proof: Recall from the first paragraph that a Cartesian equation for $\mathcal{K}$ is of the form

$$(x - c)^2 + (y - k)^2 = (x_0 - c)^2 + (y_0 - k)^2,$$

and we must find a unique c such that the tangent to the graph of $\mathcal{K}$ at the point P_0 has slope m. Let y be one of the functions of x that are implicitly defined by the last displayed equation. (Formulas for these functions could be found by using the quadratic formula, but we will not need to do so.) Let y' denote the derivative of y with respect to x. Apply implicit differentiation to the displayed equation; that is, equate the derivatives (with respect to x) of the left- and right-hand sides of that displayed equation. Standard differential calculus leads to $2(x - c) + 2(y - k)y' = 0$. It follows that $y' = (c - x)/(y - k)$ for any point on $\mathcal{K}$ such that $y \neq k$. Basic differential calculus tells us that the slope of the tangent (line) to the graph of $\mathcal{K}$ at the point (x, y) is $y'(x)$ provided that this tangent line is not vertical. Recall from the second paragraph of this proof that we are assuming that T is not vertical (and $x_0 \neq c$). Thus, applying the above reasoning to $(x, y) := (x_0, y_0)$, we see that the slope of T is $y'(x_0)$ and, therefore, that the first assertion will follow by showing that c is uniquely determined by the condition that $m = (c - x_0)/(y_0 - k)$. This is, in fact, the case, with $c = x_0 + m(y_0 - k)$. Observe that this value of c also satisfies the above condition that $x_0 \neq c$ (since $m \neq 0$ and $y_0 \neq k$). This completes the (second) proof of the first assertion. The asserted Cartesian equation for $\mathcal{K}$ is then obtained by repeating the final paragraph of the first proof. This completes the second proof. □

Corollary 2.2 states the case of Theorem 2.1 for $k := 0$ and $y_0 > 0$.

Corollary 2.2. Let $m \in \mathbb{R}$ with $m \neq 0$, and let $P_0(x_0, y_0)$ be a point in the Euclidean plane $\mathbb{R}^2$ such that $y_0 > 0$. Let $\mathcal{X}$ denote the x-axis. Then there exists a unique circle, say $\mathcal{K}$, such that the center of $\mathcal{K}$ is on $\mathcal{X}$, P_0 lies on $\mathcal{K}$, and the tangent to $\mathcal{K}$ at P_0 has slope m. A Cartesian equation for this circle $\mathcal{K}$ is

$$x^2 - 2[x_0 + my_0]x + y^2 = y_0^2 - x_0^2 - 2mx_0y_0.$$

It will be useful to record that the assertions in Theorem 2.1 and Corollary 2.2 are also valid if one allows the possibility that $m = 0$. We do so next for the analogue of Theorem 2.1. Of course, the case $k = 0$ of Proposition 2.3 gives the corresponding analogue of Corollary 2.2.

Proposition 2.3. Let $k \in \mathbb{R}$ and let $P_0(x_0, y_0)$ be a point in the Euclidean plane $\mathbb{R}^2$ such that $y_0 \neq k$. Let $\mathcal{X}$ denote the (horizontal) line with Cartesian equation $y = k$. Then there exists a unique circle, say $\mathcal{K}$, such that the center of $\mathcal{K}$ is on $\mathcal{X}$, P_0 lies on $\mathcal{K}$, and the tangent to $\mathcal{K}$ at P_0 is horizontal (that is, has slope 0). A Cartesian equation for this circle $\mathcal{K}$ is

$$x^2 - 2x_0x + y^2 - 2ky = y_0^2 - x_0^2 - 2ky_0.$$

Proof. Let T and S be as in the proof of Theorem 2.1. As noted in the proof of Theorem 2.1, T and S are perpendicular. Since $m = 0$, T is horizontal. Hence, S is vertical. It follows that $x_0 = c$, and so P_0 has coordinates either $(x_0, k + r)$ or $(x_0, k - r)$. Moreover, C has coordinates (x_0, k). A Cartesian equation for $\mathcal{K}$ is

$$(x - x_0)^2 + (y - k)^2 = [(x_0 - x_0)^2 + (y_0 - k)^2 =](y_0 - k)^2,$$

which simplifies to the asserted equation. Finally, note that in the present context (where $m = 0$), it is also true that $c = x_0 + m(y_0 - k)$. The proof is complete. □

Parts (b) and (c) of the next remark will collect some background about the upper half-plane model of hyperbolic geometry, by slightly elaborating on a fragment of the Introduction. Remark 2.4 (c) will also explain the role of two results (Lemmas 2.5 and 2.6) in our approach to some significant results (Theorems 2.8 and 2.9).

Remark 2.4. (a) In the mid-20th century, it was common for precalculus courses (which were then often called courses on "algebra, trigonometry and analytical geometry") to introduce the concept of an angle of inclination and to cover its relationship to the slope of a non-vertical line. Such coverage has declined markedly for several decades, as the teaching of trigonometry has transitioned to an approach featuring the unit circle rather than the classical approach featuring angles in standard position. That classical approach viewed an angle as an object obtained by a counterclockwise rotation starting from the initial side of the angle. That point of view naturally led to four kinds of angles (or angular measures) determined, in case the initial side of the angle has the direction of the positive x-axis, by the quadrant into which the terminal side of the angle is ultimately pointing. In addressing many problems, the classical approach consequently leads to analyses that involve many cases.

One's first experience with the approach using the unit circle may seem quite different and more "unified." However, in our opinion, deeper use of the unit circle approach often makes tacit (or overt) use of the triangles that are classically associated with angles in standard position and their associated reference/related angles/numbers. (In the proofs and worked examples later in this paper, we will occasionally mention this auxiliary tool; for its definition and use, the reader may consult works such as [1, pages 231-233] or [6, pages 276-278].) It is precisely the multitude of cases that the classical approach identifies which permits the discovery and proof of the lists of answers given in Theorems 2.8 and 2.9. With those long lists of answers in hand, one may well ask if the unit circle approach could lead naturally to those answers *without* tacitly employing the classical approach. As that set of answers (more precisely, its extension in Corollary 2.13) is clearly related to the titular object of study in this paper, the case has been made for using the classical approach here (including the concept of angle of inclination) in studying the question at hand.

(b) We next recall some basic facts about the upper half-plane model of hyperbolic geometry. Its "points" are the points in the upper-half plane of Euclidean geometry, that is, the points $(x, y) \in \mathbb{R}^2$ such that $y > 0$. Its "line segments," which are of two types, are actually the geodesics according to a certain metric. (We will not need to explicate that metric here, but the interested reader can find such explanations – and the other facts being recalled in this paragraph – in [8, pages 48-50] or [12, pages 53-58].) Consider any two "points" $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. If $x_1 \neq x_2$, then the "line segment" connecting P_1 and P_2 is the arc, with endpoints P_1 and P_2, of the (Euclidean) semicircle in the upper half plane that is centered on the x-axis and passes through P_1 and P_2; that "line segment" is called a "bowed geodesic" (for obvious reasons) and the semicircle containing it is one of the two types of "lines" in the model. If $x_1 = x_2$, then the "line segment" connecting P_1 and P_2 is the (Euclidean vertical) line segment with endpoints P_1 and P_2; that "line segment" is called a "straight geodesic" (for obvious reasons) and the vertical line containing it, when intersected with the upper half-plane, gives the second type of "line" in the model.

(c) One application of our results here is in measuring the angles that arise as interior angles in a hyperbolic triangle (embodied in the upper half-plane model of hyperbolic geometry). That leads naturally to an interest in measuring the angles formed at a point where a bowed geodesic intersects either another bowed geodesic or a straight geodesic. By definition, the angles thus formed are the Euclidean angles (intersected with the upper half-plane) that are formed by the tangent lines (in the sense of Euclidean geometry) of the intersecting figures at the point in question; and, also by definition, the (hyperbolic) measure of such an angle (in the model that we are studying) is the same as the usual Euclidean measure of the angle formed by those tangent lines. Lemma 2.6 will state two formulas which can be used to calculate that measure when the (tangent) lines in question are not perpendicular to one another. As in part (a), we will be interested in directed angles (that is, angles that can be viewed as having arisen via a counterclockwise rotation from an initial side to a terminal side). However, in order to reduce the number of relevant cases, we will often be concerned with measuring only angles with measures less than π. This focused concern is appropriate here because of the following fundamental result in hyperbolic geometry (which actually characterizes hyperbolic plane geometry up to isomorphism in the universe of so-called neutral or absolute geometries): the sum of the (radian) measures of the (interior) angles of any hyperbolic triangle is less than π (cf. [12, Corollary 10.1.4 and Examples 6.1.2 and 6.1.3]).

The proof of Lemma 2.6, as well as the implementation of Lemma 2.6 in proving Theorems 2.8 and 2.9 and Corollary 2.13, depends heavily on the notion of "angle of inclination" that was mentioned in (a). This notion will be defined next and its relevance will be established in Lemma 2.5. Also relevant in the proofs of items 2.5-2.9 is the notion of reference/related angle/number mentioned in (a). This completes the remark.

We next recall the definition of a key tool that was mentioned in Remark 2.4 (a), (c). If L is a line (in the Euclidean plane), then the *angle of inclination* of L is defined to be the angle φ between L and the positive x-axis such that $0 \leq \varphi < \pi$. (As the preceding sentence illustrates, we will occasionally find it convenient to engage in the time-honored practice of conflating an angle with its (radian) measure.) If L has positive slope, then $0 < \varphi < \pi/2$. If L has negative slope, then $\pi/2 < \varphi < \pi$. If L has slope equal to zero, then L is horizontal and it is then conventional to take the angle of inclination of L to be 0. If (as the last possibility) the slope of L is undefined (that is, if L is vertical), then $\varphi = \pi/2$.

One may wish to compare the above notion of "angle of incidence" with what Millman calls the "horizon angle" [8, pages 50-51] of what we would call a hyperbolic half-line. Although Millman refers to an angle formed by a given line and the x-axis, it is clear from the figures on [8, page 51] that he means the *positive* direction of the x-axis. That stipulation of direction was part of the above definition of "angle of inclination." One should note the following difference between our treatment of hyperbolic angles and that in [8]. The measures of horizon angles in [8] lie between −90 and 90 (degrees), whereas our measures of angles of inclination lie between 0 and π (radians).

Lemma 2.5. ([5, Proposition 2.1]) Let L be a non-vertical line having slope m and angle of inclination φ. Then:

(a) $\tan(\varphi) = m$.

(b) If φ is an acute angle, then $\varphi = \tan^{-1}(m)$.

(c) If φ is an obtuse angle, then $\varphi = \pi - \tan^{-1}(-m) = \pi + \tan^{-1}(m)$.

Lemma 2.6. ([5, Theorem 2.2]) Let L_1 and L_2 be two intersecting non-perpendicular lines in the Euclidean plane. Then:

(a) Suppose that L_1 and L_2 are each non-vertical, having slopes m_1 and m_2, respectively. Then the (measures of the) two acute angles formed by L_1 and L_2 at their point of intersection are each given by $\tan^{-1}(|\frac{m_1-m_2}{1+m_1m_2}|)$, and the (measures of the) two obtuse angles formed by L_1 and L_2 at their point of intersection are each given by $\pi-\tan^{-1}(|\frac{m_1-m_2}{1+m_1m_2}|)$.

(b) Suppose that L_1 is vertical and that L_2 has slope m_2. If $m_2>0$, then the (measures of the) two acute angles formed by L_1 and L_2 at their point of intersection are each given by $\frac{\pi}{2}-\tan^{-1}(m_2)$, and the (measures of the) two obtuse angles formed by L_1 and L_2 at their point of intersection are each given by $\frac{\pi}{2}+\tan^{-1}(m_2)$. If $m_2<0$, then the (measures of the) two acute angles formed by L_1 and L_2 at their point of intersection are each given by $\frac{\pi}{2}+\tan^{-1}(m_2)$, and the (measures of the) two obtuse angles formed by L_1 and L_2 at their point of intersection are each given by $\frac{\pi}{2}-\tan^{-1}(m_2)$.

In applying Lemma 2.6 (a), it will often be helpful to know the algebraic sign of an expression of the form $1+m_1m_2$. Accordingly, it will also be useful to isolate the following lemma. Its easy proof involves only high school algebra, and so it is left to the reader.

Lemma 2.7. Let $m\in\mathbb{R}$. Then:

(a) If $m>0$, then

$$1+mx\begin{cases}>0 & \text{if } x>-1/m\\ =0 & \text{if } x=-1/m\\ <0 & \text{if } x<-1/m.\end{cases}$$

(b) If $m<0$, then

$$1+mx\begin{cases}<0 & \text{if } x>-1/m\\ =0 & \text{if } x=-1/m\\ >0 & \text{if } x<-1/m.\end{cases}$$

We now give two results that can be of use in measuring a (directed) angle (of some hyperbolic triangle) in the upper half-plane model of hyperbolic geometry. Indeed, applications of Theorems 2.8 and 2.9 can address situations where the initial side of the angle in question lies along a prescribed half-line (also known as a "ray") determined by a (bowed or straight) geodesic.

Theorem 2.8. Let $P_0(x_0,y_0)$ be a point in the upper half-plane. Let L_1 be a non-vertical Euclidean half-line with initial point P_0, slope m_1 and angle of inclination α. Let L_2 be a non-vertical Euclidean half-line with initial point P_0, such that L_1 and L_2 are not perpendicular and L_2 can be obtained by a counterclockwise rotation of L_1 through an angle of radian measure ξ, with $0<\xi<\pi$ (and $\xi\neq\pi/2$). Let m_2 denote the slope of L_2. Then:

(a) Suppose that $\alpha<\pi/2$. Then m_2 is the solution for x in the equation

$$\begin{cases}\tan^{-1}(\frac{x-m_1}{1+m_1x})=\xi & \text{if } \xi<\pi/2-\alpha;\\ \tan^{-1}(\frac{x-m_1}{1+m_1x})=\xi & \text{if } \pi/2-\alpha<\xi<\pi/2 \text{ and } \xi\neq\pi-\alpha;\\ \tan^{-1}(\frac{m_1-x}{1+m_1x})=\pi-\xi & \text{if } \pi/2<\xi<\pi-\alpha;\\ \tan^{-1}(\frac{m_1-x}{1+m_1x})=\pi-\xi & \text{if } \xi>\pi-\alpha;\\ x=0 & \text{if } \xi=\pi-\alpha.\end{cases}$$

(b) Suppose that $\pi/2 < \alpha$ $(< \pi)$. Then m_2 is the solution for x in the equation

$$\begin{cases} \tan^{-1}(\frac{x-m_1}{1+m_1x}) = \xi & \text{if } \xi < \pi - \alpha; \\ \tan^{-1}(\frac{x-m_1}{1+m_1x}) = \xi & \text{if } \pi - \alpha < \xi < 3\pi/2 - \alpha \text{ and } \xi < \pi/2; \\ \tan^{-1}(\frac{m_1-x}{1+m_1x}) = \pi - \xi & \text{if } \pi - \alpha < \xi < 3\pi/2 - \alpha \text{ and } \xi > \pi/2; \\ \tan^{-1}(\frac{m_1-x}{1+m_1x}) = \pi - \xi & \text{if } \xi > 3\pi/2 - \alpha; \\ x = 0 & \text{if } \xi = \pi - \alpha. \end{cases}$$

Proof. (a) Since $\alpha < \pi/2$, Lemma 2.5 (a) gives $m_1 = \tan(\alpha) \geq 0$ (with equality if and only if L_1 is horizontal). If $\xi = \pi - \alpha$, then $\alpha + \xi = \pi$, and so L_2 is horizontal, which means that its slope is $m_2 = 0$. It remains to prove the first four assertions in (a). For each of these, we will first determine the algebraic signs of $m_1 - m_2$ and $1 + m_1m_2$, and the proof will end by applying one of the two conclusions in Lemma 2.6 (a). By the way, (a) does not address the "possibility" that $\xi = \pi/2 - \alpha$ (resp., $\xi = \pi/2$) because that situation cannot arise, in view of the hypothesis that L_2 is non-vertical (resp., the hypothesis that L_1 and L_2 are not perpendicular).

We next address the first assertion in (a); that is, suppose that $\xi < \pi/2 - \alpha$. Then Lemma 2.5 (a) gives $m_2 = \tan(\alpha + \xi) > 0$. Consequently, $1 + m_1m_2 > 0$. As $\alpha + \xi > \alpha$ and tan is an increasing function over the domain $(0, \pi/2)$, it follows that $m_2 = \tan(\alpha + \xi) > \tan(\alpha) = m_1$, whence $m_1 - m_2 < 0$. Thus, for this case, $|(m_1 - m_2)/(1 + m_1m_2)| = (m_2 - m_1)/(1 + m_1m_2)$, and so the assertion follows from the first conclusion in Lemma 2.6 (a).

Next, suppose that $\pi/2 - \alpha < \xi < \pi/2$ and $\xi \neq \pi - \alpha$. As in the proof of the first assertion, $m_1 = \tan(\alpha) \geq 0$. Also, $\pi/2 < \alpha + \xi < \pi/2 + \pi/2 = \pi$. Then Lemma 2.5 (a) gives $m_2 = \tan(\alpha + \xi) < 0$. It follows that $m_1 - m_2 > 0$. Moreover, $1 + m_1m_2 < 0$. (This can be seen by carefully applying Lemma 2.7 (a), the underlying point being that $1 + m_1x$ is a continuous function of x which, in case $m_1 > 0$, is strictly increasing with limit $-\infty$ as $x \to -\infty$ and limit 0 as $x \to 0^-$.) Therefore, $|(m_1 - m_2)/(1 + m_1m_2)| = (m_2 - m_1)/(1 + m_1m_2)$, and so the assertion follows from the first conclusion in Lemma 2.6 (a).

Next, suppose that $\pi/2 < \xi < \pi - \alpha$. Then, as above, $m_1 = \tan(\alpha) \geq 0$. Also, $\pi/2 < \xi \leq \xi + \alpha < \pi$, and so by Lemma 2.5 (a), $m_2 = \tan(\alpha + \xi) < 0$. Hence, $m_1 - m_2 > 0$. In addition, since $\xi > \pi/2$, we see, by tweaking the reasoning in the preceding paragraph, that $1 + m_1m_2 > 0$. Therefore, $|(m_1 - m_2)/(1 + m_1m_2)| = (m_1 - m_2)/(1 + m_1m_2)$, and so the assertion follows from the second conclusion in Lemma 2.6 (a).

Lastly, suppose that $\xi > \pi - \alpha$; that is, $\alpha + \xi > \pi$. Then $\xi > \pi - \pi/2 = \pi/2$. Therefore, since $\xi < \pi$ and $\alpha + \xi < \pi/2 + \pi = 3\pi/2$, the angle of inclination of L_2 is an acute angle (say θ_r) whose measure is less than that of α. As above, Lemma 2.5 (a) combines with the fact that tan is increasing on $(0, \pi/2)$ to show that $0 < \tan(\theta_r) = m_2 < \tan(\alpha) = m_1$. It follows that $m_1 - m_2 > 0$ and $1 + m_1m_2 > 0$, and so the assertion follows from the second conclusion in Lemma 2.6 (a).

(b) Since $\pi/2 < \alpha < \pi$, Lemma 2.5 (a) gives $m_1 = \tan(\alpha) < 0$. If $\xi = \pi - \alpha$, then one can repeat the proof from (a) that $m_2 = 0$. It remains to prove the first four assertions in (b). For each of these, we will determine the algebraic signs of $m_1 - m_2$ and $1 + m_1m_2$, after which the assertion will follow by applying one of the two conclusions in Lemma 2.6 (a).

We next address the first assertion in (b); that is, suppose that $\xi < \pi - \alpha$. Then $\xi < \pi - \pi/2 = \pi/2$. Also, since $\pi/2 < \alpha < \alpha + \xi < \pi$ and $\alpha + \xi$ is the angle of inclination of L_2, Lemma 2.5 (a) gives $m_2 = \tan(\alpha + \xi) < 0$. Consequently, $1 + m_1m_2 > 0$. Moreover, since tan is an increasing function over the domain $(\pi/2, \pi)$, it follows that $m_2 = \tan(\alpha + \xi) > \tan(\alpha) = m_1$, whence $m_1 - m_2 < 0$. Thus, for this case, $|(m_1 - m_2)/(1 + m_1m_2)| = (m_2 - m_1)/(1 + m_1m_2)$. Therefore, the assertion follows from the first conclusion in Lemma 2.6 (a).

We will prove the next two assertions together. Assume that $\pi - \alpha < \xi < 3\pi/2 - \alpha$. As $\pi < \alpha + \xi < 3\pi/2$, it follows from Lemma 2.5 (a) that $m_2 = \tan(\alpha + \xi) > 0$. Consequently, $m_1 < 0 < m_2$, whence $m_1 - m_2 < 0$. If $\xi < \pi/2$ (resp., $\xi > \pi/2$), then a continuity argument (involving $1 + m_1x$, as in the proof of the second and third assertions in (a)) shows that $1+m_1m_2>0$ (resp., $1+m_1m_2<0$). Therefore, the second (resp., third) assertion in (b) follows from the first (resp., second) conclusion in Lemma 2.6 (a).

Lastly, suppose that $\xi > 3\pi/2-\alpha$. Then $\xi > 3\pi/2-\pi = \pi/2$. Also, since $\alpha+\xi > 3\pi/2$, Lemma 2.5 (a) gives $m_2 = \tan(\alpha+\xi) < 0$. Consequently, $1+m_1m_2 > 0$. It remains only to prove that $m_1-m_2>0$ (for the assertion will then follow by applying the second conclusion in Lemma 2.6 (a)). To that end, note that the angle of inclination of L_2 is an obtuse angle (say θ) whose measure is less than that of α (which is the angle of inclination of L_1). Since tan is an increasing function on the domain $(\pi/2,\pi)$, it follows from Lemma 2.5 (a) that $m_2=\tan(\theta)<\tan(\alpha)=m_1$; that is, $m_1-m_2>0$, as desired. The proof is complete. □

Our choice of the contexts studied in Theorem 2.8 was motivated by our desire to measure the angles formed at the point of intersection of two bowed (hyperbolic) geodesics in the upper half-plane. We assumed in Theorem 2.8 that L_1 and L_2 are not perpendicular because the "$m_1m_2 = -1$" criterion is the quickest way to detect perpendicularity for a pair of non-vertical (Euclidean) lines. As for vertical lines (in particular, in regard to measuring the angles formed at the point of intersection of a straight geodesic and a bowed geodesic), Theorem 2.9 will fully address all such contexts where the lines in question are not manifestly perpendicular. The proof of Theorem 2.9 will be self-contained and will use a number of familiar results from analytic trigonometry, including its basic identities (cofunction, ratio) and the facts that tan and cot are odd functions.

Theorem 2.9. Let $P_0(x_0,y_0)$ be a point in the upper half-plane. Let L_1 and L_2 be distinct Euclidean half-lines that each have initial point P_0, such that L_2 can be obtained by a counterclockwise rotation of L_1 through an angle of radian measure ξ, with $0<\xi<\pi$. Then:

(a) Suppose that L_1 is an upwardly directed vertical (Euclidean) half-line and that L_2 is not vertical. Let m_2 denote the slope of L_2. Then

$$m_2 = \begin{cases} -\cot(\xi) & \text{if } \xi < \pi/2; \\ 0 & \text{if } \xi = \pi/2; \\ -\cot(\xi) & \text{if } \xi > \pi/2. \end{cases}$$

(b) Suppose that L_1 is a downwardly directed vertical (Euclidean) half-line (and so L_2 is not vertical). Let m_2 denote the slope of L_2. Then

$$m_2 = \begin{cases} -\cot(\xi) & \text{if } \xi < \pi/2; \\ 0 & \text{if } \xi = \pi/2; \\ -\cot(\xi) & \text{if } \xi > \pi/2. \end{cases}$$

(c) Suppose that L_2 is an upwardly directed vertical (Euclidean) half-line (and so L_1 is not vertical). Let m_1 denote the slope of L_1. Then

$$m_1 = \begin{cases} \cot(\xi) & \text{if } \xi < \pi/2; \\ 0 & \text{if } \xi = \pi/2; \\ \cot(\xi) & \text{if } \xi > \pi/2. \end{cases}$$

(d) Suppose that L_2 is a downwardly directed vertical (Euclidean) half-line (and so L_1 is not vertical). Let m_1 denote the slope of L_1. Then

$$m_1 = \begin{cases} \cot(\xi) & \text{if } \xi < \pi/2; \\ 0 & \text{if } \xi = \pi/2; \\ \cot(\xi) & \text{if } \xi > \pi/2. \end{cases}$$

Proof. The parenthetical assertions in (a)-(d) all follow because of the assumption that $0 < \xi < \pi$. Also, if $\xi = \pi/2$, then in parts (a) and (b) (resp., in parts (c) and (d)), L_2 (resp., L_1) is horizontal and so its slope m_2 (resp., m_1) is 0.

(a) Suppose first that (L_1 is vertical and upwardly directed and) $\xi < \pi/2$. Then the angle of inclination of L_2 is (that is, has measure) $\xi + \pi/2$. Hence, by Lemma 2.5 (a),

$$m_2 = \tan(\xi + \pi/2) = \tan(\pi/2 - (-\xi)) = \cot(-\xi) = -\cot(\xi).$$

Next, suppose that $\xi > \pi/2$. Then, since "vertically opposite" angles are congruent, the angle of inclination of L_2 is $\xi - \pi/2$. So, by Lemma 2.5 (a),

$$m_2 = \tan(\xi - \pi/2) = \tan(-(\pi/2 - \xi)) = -\tan(\pi/2 - \xi) = -\cot(\xi).$$

(b) Suppose first that (L_1 is vertical and downwardly directed and) $\xi < \pi/2$. Then, by again invoking the fact that "vertically opposite" angles are congruent, the angle of inclination of L_2 is $\pi/2 + \xi$. So, by Lemma 2.5 (a), $m_2 = \tan(\pi/2 + \xi)$. As we saw above, this simplifies to $-\cot(\xi)$.

Next, suppose that $\xi > \pi/2$. Observe that the angle of inclination of L_2 is $\xi - \pi/2$. Then, as in the proof of the final assertion of (a), it follows that $m_2 = -\cot(\xi)$.

(c) Suppose first that (L_2 is vertical and upwardly directed and) $\xi < \pi/2$. Then the angle of inclination of L_1 is the complement of ξ, namely, $\pi/2-\xi$. Hence, by Lemma 2.5 (a), $m_1 = \tan(\pi/2-\xi) = \cot(\xi)$.

Next, suppose that $\xi > \pi/2$. Then, once again invoking the congruence of vertically opposite angles, we see that the angle of inclination of L_1 is $\pi/2 + (\pi - \xi) = 3\pi/2 - \xi$. Hence, by Lemma 2.5 (a), $m_1 = \tan(3\pi/2 - \xi)$. By applying several cofunction identities, coupled with the fact that both tan and cot are odd functions, one can establish the identity $\tan(3\pi/2 - \xi) = \cot(\xi)$. That would finish the proof of (c) and, as was the case for the above arguments in this proof, would be a good fit for readers/classes whose introduction to trigonometry has been via the unit circle. Perhaps some readers have noticed that the above arguments could also have been established by more classical methods. That is also true of the present, probably less familiar, identity. In that spirit, we present the following sketch of an alternate proof of the identity $\tan(3\pi/2 - \xi) = \cot(\xi)$: use the expansion formulas,

$$\sin(u + v) = (\sin u)(\cos v) + (\cos u)(\sin v) \text{ and}$$

$\cos(u+v) = (\cos u)(\cos(v)-(\sin u)(\sin v)$, together with the facts that $\tan = \sin/\cos$, $\sin(3\pi/2) = -1$ and $\cos(3\pi/2) = 0$. (Of course, one cannot use the corresponding expansion formula for $\tan(u + v)$ here because $\tan(3\pi/2)$ is undefined.)

(d) Suppose first that (L_2 is vertical and downwardly directed and) $\xi < \pi/2$. Then $m_1 > 0$ and we see, by considering vertically opposite angles, that the angle of inclination of L_1 is (congruent to) a complement of ξ. Hence, by Lemma 2.5 (a), $m_1 = \tan(\pi/2 - \xi) = \cot(\xi)$.

Lastly, suppose that $\xi > \pi/2$. Then $m_1 < 0$ and the angle of inclination of L_1 is $\pi - (\xi - \pi/2) = 3\pi/2 - \xi$. Hence, by Lemma 2.5 (a), $m_1 = \tan(3\pi/2 - \xi)$ which, as we showed above, simplifies to $\cot(\xi)$. The proof is complete. □

This paragraph identifies some computational skills that will be assumed from Example 2.10 onward, and the next three paragraphs summarize some methods for carrying out those procedures. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be distinct points in the upper half-plane (viewed as a subset of the Euclidean plane $\mathbb{R}^2$). Let $\mathcal{G}$ denote the (straight or bowed) geodesic connecting P_1 and P_2. Let T denote the tangent line to $\mathcal{G}$ at P_1. The first skill is twofold: to determine whether T is vertical; and, if T is not vertical, to calculate the slope, say m, of T. The second skill is to find a Cartesian equation for $\mathcal{G}$ (more precisely, for either the vertical line passing through P_1 and P_2 or the circle that is centered on the x-axis and passes through P_1 and P_2).

Suppose first that $x_1 = x_2$. Then $\mathcal{G}$ is a straight geodesic. Since $\mathcal{G}$ is a (subset of a Euclidean) line L in this case, T is that line, and so any Cartesian equation of L is also a Cartesian equation of $\mathcal{G}$. Hence, we can carry out both of the desired skills if $x_1 = x_2$, for T is then vertical and a Cartesian equation for $\mathcal{G}$ is $x = x_1$.

Assume for this paragraph and the next paragraph that $x_1 \neq x_2$. Then $\mathcal{G}$ is a subset of a (unique, Euclidean) circle $\mathfrak{C}$ with center $C(c,0)$ for some $c \in \mathbb{R}$. The only (two) points on $\mathfrak{C}$ where there exists a vertical tangent are the intersection points of $\mathfrak{C}$ with the horizontal diameter of $\mathfrak{C}$. Of course, P_1 cannot be either of these points, since they lie on the x-axis while P_1 lies in the upper half-plane. Therefore, T is not vertical, and so its slope m exists. Observe, by combining either proof of Theorem 2.1 with the proof of Proposition 2.3 (and bearing in mind that $k = 0$ in the present context), that $c = x_1 + my_1$ (regardless of whether $m = 0$), whence $m = (c - x_1)/y_1$. Thus, since we are given P_1, *knowing m is equivalent to knowing c*. Next, observe that the (Euclidean) line segment connecting P_1 and P_2 is not vertical (since $x_1 \neq x_2$), so the (Euclidean) perpendicular bisector of that line segment is not horizontal, so that perpendicular bisector meets the x-axis at some point, which is $C(c,0)$ by a fundamental fact about Euclidean geometry (cf. the first proof of Theorem 2.1 or [8, page 49] or [12, page 58]). With c thus in hand, one Cartesian equation for $\mathcal{G}$ is

$$(x-c)^2 + y^2 = (x_0 - c)^2 + y_0^2,$$

thereby completing both of the desired skills if $x_1 \neq x_2$.

In the spirit of [3], we wish to point out an alternate (more algebraic) way to complete both of the desired skills if $x_1 \neq x_2$. First, we find a Cartesian equation for $\mathcal{G}$, this time in the form

$$x^2 + y^2 + bx = d.$$

To do so, substitute the coordinates of P_1 and P_2 into the last-displayed equation, thus obtaining a system of two linear equations in the unknowns b and d, solve that system for (the unique solutions for) b and d, and substitute those values for b and d into the last-displayed equation to obtain the desired Cartesian equation. Then, by completing squares, we obtain $c = -b/2$ (and then, as explained in the preceding paragraphs, we also obtain, thanks to the proofs of Theorem 2.1 and Proposition 2.3, the value of $m = (c - x_1)/y_1$).

Next, to make the statement of some of the remaining results more manageable, it will be convenient to agree on relaxing some usages of terminology. Suppose that P and Q are distinct points in the upper half-plane (viewed as a subset of the Euclidean plane $\mathbb{R}^2$). We know that there is a unique (hyperbolic) geodesic connecting P and Q, and that geodesic is either bowed or straight. If we want to stress an interest in one of the two directions of that geodesic (because of an interest in either the hyperbolic “line" segment that goes from P to Q or the hyperbolic half-line with initial point P and passing through Q), we will refer to “the geodesic from P to Q." Occasionally, if the context is clear, we may say either “the geodesic connecting P and Q" or “the geodesic from P to Q" and mean any of the following three items: the segment (that stays within the upper half-plane) of the geodesic passing through P and Q; a preferred ordering of the just-named segment, viewed as having initial point P and terminal point Q; or the intersection of the upper half-plane with either the circle (in the case of a bowed geodesic) or the vertical line (in the case of a straight geodesic) that contains the given geodesic passing through P and Q as a subset. Moreover, a Cartesian equation of the just-mentioned circle or vertical line will be what is meant by a “(Cartesian) equation of the geodesic passing through P and Q."

Part of our concern in Corollary 2.13 will be (as possibly suggested by the title of this paper) the construction of a hyperbolic (directed) angle having as its measure a preassigned number from the interval $(0, 2\pi)$. We know that no Euclidean triangle can have an interior angle with measure exceeding π. That is, in fact, the case for hyperbolic triangles as well. The next example illustrates this strikingly and is intended to enhance the reader's intuitive understanding of what is meant by an "interior angle" of a hyperbolic triangle and its measure (and, for that matter, possibly of what is meant by the similarly named concepts in Euclidean geometry).

Example 2.10. Consider the points $A(0,5)$, $B(0,6)$ and $C(3,4)$ in the upper half-plane. Let η be the "directed" angle $\angle BAC$ whose initial side is the upwardly directed straight geodesic $\mathcal{G}_1$ from A to B and whose terminal side is the bowed geodesic $\mathcal{G}_2$ from A to C. (By "directed," we mean that α is generated by rotating counterclockwise about the vertex A so that the tangential half-line to $\mathcal{G}_1$ at A is carried to the tangential half-line to $\mathcal{G}_2$ at A.) Then:

(a) (The measure of) $\eta = 3\pi/2$.

(b) η is not an interior angle of any hyperbolic triangle (in the upper half-plane model of hyperbolic geometry).

(c) Consider the hyperbolic triangle $\triangle := \triangle ABC$ and its interior angle $\alpha := \angle BAC$. Then $\mathcal{G}_1$ has Cartesian equation $x = 0$, $\mathcal{G}_2$ has Cartesian equation $x^2 + y^2 = 25$, and the bowed geodesic (say $\mathcal{G}$) connecting B and C has Cartesian equation $x^2 + y^2 + (11/3)x = 36$. If one views the side BC as having been generated by traveling along $\mathcal{G}$ counterclockwise from C to B (resp., clockwise from B to C), then the interior angle α of $\triangle$ is viewed as having initial (resp., terminal) side $\mathcal{G}_1$ and terminal (resp., initial) side $\mathcal{G}_2$. Regardless of whether one views the generation of $\triangle$ as having arisen in a counterclockwise or clockwise manner, the measure of α is $\pi/2$. In particular, α is not η.

Proof. (a) The tangent line to $\mathcal{G}_1$ at A is the y-axis (with Cartesian equation $x = 0$). Since $\mathcal{G}_1$ goes *from A to B* (and $5 < 6$), it follows that the tangential half-line of $\mathcal{G}_1$ at A is the *upward* directed vertical ray $\mathcal{R}_1$ with initial point A. Next, a Cartesian equation for $\mathcal{G}_2$ is $x^2 + y^2 = 25$, so the tangent line (say T) to $\mathcal{G}_2$ at A has slope $-0/5 = 0$, and so T is the horizontal line with Cartesian equation $y = 5$. Since $\mathcal{G}_2$ goes *from A to C* (and $0 < 3$), it follows that the tangential half-line of $\mathcal{G}_2$ at A is the *rightward* directed horizontal ray $\mathcal{R}_2$ with initial point A. By definition, η is the minimum (positive) number such that a counterclockwise rotation of η radians about the vertex A carries $\mathcal{R}_1$ to $\mathcal{R}_2$. It is evident that $\eta = 3\pi/2$.

(b) Since each interior angle of a hyperbolic triangle has measure strictly between 0 and π, the assertion follows from (a).

(c) In view of the above comments, it remains only to prove that the measure of α is $\pi/2$. This measure has two significant properties: it is the measure of an interior angle of a (hyperbolic) triangle; and it is the measure of an angle formed at the point of intersection of two perpendicular Euclidean half-lines (namely, $\mathcal{R}_1$ and $\mathcal{R}_2$). The first of these properties tells us that the measure of α is strictly between 0 and π. When combined with the second property, this tells us that α is a right angle and necessarily has measure $\pi/2$. The proof is complete. □

Remark 2.11. (a) One may wish an explanation for the formulation/placement of the third sentence in the statement of Example 2.10 (c). In that regard, we would first point to the classical treatment, especially for less mature audiences, of the notion of "directed" angles (also known as positive angles and negative angles) in pre-calculus texts such as [1, pages 197-199] and [6, pages 248-249]. Those treatments relied on their readers having an intuitive understanding of what is meant by a counterclockwise (or, for that matter, a clockwise) rotation in the Euclidean plane. This kind of pedagogic attitude has served its intended audiences well by emphasizing practical applications, while admittedly avoiding a deeper study of some related logical foundations. Such texts offered a similarly intuitive approach in introducing the notion of a "directed line segment" (cf. [1, pages 11 and 21], [6, page 403]). The statement of Example 2.10 was formulated in keeping with such a pedagogic approach.

However, we wish to stress here that, in our opinion, a rigorous development of much of classical Euclidean or hyperbolic geometry (that is to say, of neutral/absolute geometry) requires attention to the notion of a half-line and its related notions, such as that of a “tangential half-line" that was mentioned in the statement of Example 2.10. The literature suggests that this concept may have been termed a “tangential ray" in the first edition (which we have not been able to access) of [7], but it does not seem to be in [7]. There seems to be no explicit mention of this concept in [11], [12] or [8]. To be fair, [8] does rigorously verify the axioms of the approach to Euclidean geometry of G. D. Birkhoff (dating from 1940), as revisited in the various editions of Moise's classic text such as [10]. It should be noted that a “Warning" was issued by Greenberg [7, page 68] advising readers to guard against allowing their intuitive beliefs that are based on a familiarity with Euclidean geometry to subconsciously affect their understanding of hyperbolic geometry. As a segue to the next two paragraphs and Lemma 2.12, we note that Greenberg also offered a particularly daunting exercise relating to ordering the points on a line that are “to the left/right of" a given point on the line [7, Exercise 7, page 86].

A rigorous approach to the concept of a “half-line" and its related concepts must recognize (as Euclid did not explicitly do) that Euclidean geometry (as well as the study of hyperbolic geometry within the upper half-plane model) requires some order-theoretic axioms. In axiomatizing Euclidean (and hyperbolic) geometry, Moise [10] made use of Hilbert's “plane separation axiom." (For a statement of this axiom and of the related axiom of Pasch, see [7, pages 63 and 66].) Related to such foundational studies was Dobbs' first contribution to mathematical research [2]. That thesis produced what was essentially a one-to-one correspondence between certain three dimensional Desarguesian geometries and ordered division rings. In [2], Dobbs found an order-theoretic axiom [2, Axiom 5.12] that is (at least for the kind of geometries that were studied in [2]) equivalent to Hilbert's plane separation axiom (as well as Pasch's Axiom): cf. [2, Theorem 5.12 and Corollary 5.12.3]. That theoretical development allowed for precise definitions of a “direction" (for the set of points) on a line and of a “half-line" (see [2, Definitions 5.6 and 5.8]) which, in turn, led to additional results having the flavor of classical Euclidean geometry. For an approach to the definitions of “direction" and “half-line" that is more aligned with that of Hilbert, see [7, Definitions 5.6 and 5.8].

As Example 2.10 may have suggested (and as we will see below), a comprehensive method to effectively calculate the measure of an angle, regardless of whether the angle is Euclidean or hyperbolic (but analyzed within the half-plane model), depends in part on the notion of a “tangential half-line." As noted above, one can see from the cited texts that related notions have often been assumed to be self-evident, especially for less mature audiences. While recognizing that an adherence to this pedagogic attitude has helped to explain some of the formulation of Example 2.10, we should, in the interest of a more rigorous approach here, make clearer the notion of a “tangential half-line" insofar as it relates to the angles formed at points of intersection of geodesics in the upper half-plane model of hyperbolic geometry. While some instructors may prefer to let one's intuition lead to a determination of a relevant tangential half-line, we address this skill more rigorously in Lemma 2.12. That result will also address one related, more practical, skill, which has importance in carrying out certain calculations effectively. That skill concerns the determination of the coordinates of a point on a specified half-line of a hyperbolic geodesic.

(b) Example 2.10 illustrated, in part, the fact that a “directed” angle in hyperbolic geometry can have as its (radian) measure any real number ξ such that $0 < \xi < 2\pi$. Of course, the same is true of angles in Euclidean geometry; and, in fact, for both geometries, one can allow any real number to be the measure of some directed angle. Knowing this, one may well be led to ask for an explanation of the emphasis here (and, to some extent, in [5]) on constructing a directed angle $\angle_1$ having a preassigned vertex A, a presassigned geodesic half-line $\mathcal{G}_1$ as the initial (resp., terminal) side of $\angle_1$ (in regard to a counterclockwise rotation about A), and a preassigned measure ξ such that $0 < \xi < \pi$. With respect to “constructing,” we will next indicate how, in case $\pi < \xi < 2\pi$, to reduce that problem to constructing a directed angle $\angle_2$ with vertex A, such that the terminal (resp., initial) side of $\angle_2$ is $\mathcal{G}_1$, the measure of $\angle_2$ is $\eta := 2\pi - \xi$ and, of course, the initial (resp., terminal) side of $\angle_2$ is one of the geodesic half-lines determined by a to-be-determined geodesic $\mathcal{H}$ that passes through A.

As we have assumed that $\pi < \xi < 2\pi$ and $\eta := 2\pi - \xi$, we have $0 < \eta < \pi$. By assumption, we can construct an angle $\angle_2$ with initial (resp., terminal) side a half-line $\mathcal{H}_1$ (which is one of the half-lines emanating from A that are determined by some geodesic $\mathcal{H}$ passing through A), the measure of $\angle_2$ is η, and the terminal (resp., initial) side of $\angle_2$ is $\mathcal{G}_1$. Let $\mathcal{G}_2$ and $\mathcal{H}_2$ be the half-lines of $\mathcal{G}$ and $\mathcal{H}$ that are respectively opposite to $\mathcal{G}_1$ and $\mathcal{H}_1$, in the sense that $\mathcal{G}_2$ and $\mathcal{H}_2$ are respectively determined by $\mathcal{G}$ and $\mathcal{H}$ while being respectively unequal to $\mathcal{G}_1$ and $\mathcal{H}_1$. By considering supplementary angles, we see that the measure of the directed angle with initial (resp., terminal) side $\mathcal{G}_2$ and terminal (resp., initial) side $\mathcal{H}_1$ is $\pi - \eta$. Also, it is clear that the measure of the directed angle with initial (resp., terminal) side $\mathcal{G}_1$ and terminal (resp., initial) side $\mathcal{G}_2$ is π. It follows that the measure of the directed angle with initial (resp., terminal) side $\mathcal{G}_1$ and terminal (resp., initial) side $\mathcal{H}_1$ is $\pi + (\pi - \eta) = 2\pi - \eta = \xi$, as desired. This completes the remark.

We now have almost all the tools that will be needed to state an algorithm that solves a computational question which was mentioned in the Introduction. In doing so, Corollary 2.13 will use relaxed terminology, in the manner that was indicated above. It will also use, without specific references, the skills that were identified and summarized in the four paragraphs that followed the proof of Theorem 2.9. First, we pause to give a lemma that addresses how/which data are available when we are "given" a (hyperbolic) geodesic half-line that is emanating from a given point P_0.

Lemma 2.12. Let $\mathcal{G}$ be a "given" (hyperbolic) geodesic half-line emanating from a point $P_0\,(x_0, y_0)$ in the Euclidean upper half-plane. Then:

(a) Suppose that $\mathcal{G}$ is known to be a straight geodesic (necessarily with Cartesian equation $x = x_0$). If $\mathcal{G}$ is upwardly directed, then $P_1\,(x_0, 2y_0)$ is another point of $\mathcal{G}$ (that is distinct from P_0). If $\mathcal{G}$ is downwardly directed, then $P_1\,(x_0, y_0/2)$ is another point of $\mathcal{G}$ (that is distinct from P_0). In each case, the directed line segment (also known as a bound vector, or more simply, a vector) $\overrightarrow{P_0P_1}$ can play the role of (an initial part of) the tangential half-line of $\mathcal{G}$ at P_0.

(b) Suppose that $\mathcal{G}$ is known to be a (portion of a) bowed geodesic. Let $\mathcal{K}$ denote the circle (with center on the x-axis) whose graph contains $\mathcal{G}$ as a subset. Obtain a Cartesian equation for $\mathcal{K}$ in the form $x^2 + y^2 + bx = d$. Let $r_1 < r_2$ be the x-coordinates of the points of intersection of $\mathcal{K}$ and the x-axis. Let $(r_i, 0)$ be the coordinates of the point of $\mathbb{R}^2$ that is being approached as the limit (in the Euclidean sense, that is, in the sense of calculus of real-valued functions of several real variables) as the y-coordinate of a point on $\mathcal{G}$ approaches 0 (in the Euclidean sense). Put $c := -b/2$. Let m denote the slope of the tangent to $\mathcal{K}$ at P_0 (that is, $m = (c - x_0)/y_0$). Consider the points

$$P_1\,(\lambda, \sqrt{d - \lambda^2 - b\lambda}) \text{ and } P_2\,(\lambda, m(\lambda - x_0) + y_0), \text{ with } \lambda := \frac{x_0 + r_i}{2}.$$

Then P_1 is another point (distinct from P_0) that is on the geodesic half-line $\mathcal{G}$. Moreover, P_2 is a point of the upper half-plane that is distinct from P_0 and the vector $\overrightarrow{P_0P_2}$ can play the role of (an initial part of) the tangential half-line of $\mathcal{G}$ at P_0.

Proof. (a) The assertions follow since $y_0 > 0$ implies that $2y_0, y_0/2 > 0$.

(b) One way to obtain a Cartesian equation for $\mathcal{K}$ would be to begin with the tangent line to $\mathcal{K}$ at P_0 (and then proceed as in the first proof of Theorem 2.1 or apply Proposition 2.3 with $k := 0$). Another way would be to assume as given the coordinates of a point that is distinct from P_0 and is situated on the *opposite* hyperbolic half-line to $\mathcal{G}$ (and then proceed as in either the third or fourth paragraph that followed the proof of Theorem 2.9). Absent such data, one should not consider $\mathcal{G}$ as "given," in our opinion.

Note that either $x_0 < \lambda < r_i$ or $r_i < \lambda < x_0$. It follows that λ is the x-coordinate of some point P_1 on the geodesic half-line $\mathcal{G}$. Since P_1 is necessarily in the upper half-plane, its y coordinate is positive.

That y coordinate can be found by solving for y in the equation

$$\lambda^2 + y^2 + b\lambda = d.$$

It is evident that the positive solution for y in the displayed equation is $\sqrt{d - \lambda^2 - b\lambda}$. This proves the assertions concerning P_1.

We turn next to the determination of a suitable vector serving as an initial portion of a tangential half-line for the hyperbolic half-line $\mathcal{G}$ at P_0. By the reasoning in the first sentence of the preceding paragraph, λ is the x-coordinate of some point, say P, on the tangential half-line of $\mathcal{G}$ emanating from P_0. As that tangent(ial) (half-)line has Cartesian equation $y = m(x - x_0) + y_0$, it is clear that P_2 is that point P. Hence, P_2 lies on the appropriate tangential half-line emanating from P_0. The proof is complete. □

Corollary 2.13 presents the algorithm that was promised in the Introduction. The statement of Corollary 2.13 gives step-by-step instructions for performing the task in question. The interested reader is invited to convert those instructions into what would generally be considered an "algorithm." We believe that the instructions given in Corollary 2.13 are complete and unambiguous. A possible *caveat* may concern the role of the x-intercept r_i in our determination of the tangential half-line in Lemma 2.12 (b). This author would dismiss such a worry. In our opinion, any hyperbolic half-line that is properly "given" should allow its recipient to determine which x-intercept of the associated circle $\mathcal{K}$ plays the role of r_i. The proof of Corollary 2.13 begins with three paragraphs, which are followed by annotations for some of the steps of instructions in the statement of Corollary 2.13. Those annotations address situations where some relevant justifications are not already in the statements of those steps. Corollary 2.13 comprehensively addresses (possibly more than) one question raised by the title of this paper. We wish to alert the reader that in any effective application of Corollary 2.13, it must be assumed that we know *a priori*, for each of the given hyperbolic half-lines, whether it is a subset of a straight geodesic (and, of course, if the answer is in the negative, then the hyperbolic half-line in question must be a subset of a bowed geodesic).

For any hyperbolic half-line $\mathcal{G}$ emanating from a point $P_0 \in \mathbb{R}^2$, it will be convenient to let $-\mathcal{G}$ denote the hyperbolic half-line emanating from P_0 such that $\mathcal{G} \cup -\mathcal{G}$ is a hyperbolic line (that is, a hyperbolic geodesic) and to call $-\mathcal{G}$ the *opposite* (*half-line*) of $\mathcal{G}$. It is clear that $-(-\mathcal{G}) = \mathcal{G}$, the angle with initial (resp., terminal) side (an initial portion of) $\mathcal{G}$ and terminal (resp., initial) side (an initial portion of) $-\mathcal{G}$ has measure π (when the angle is viewed as a directed angle arising via a counterclockwise rotation about P_0), and $\mathcal{G} \cap -\mathcal{G} = \{P_0\}$.

Corollary 2.13. Let $\mathcal{G}$ and $\mathcal{H}$ be given distinct hyperbolic half-lines emanating from a point $P_0\,(x_0, y_0) \in \mathbb{R}^2$ such that $\mathcal{H} \neq -\mathcal{G}$. Let ξ be the measure of the angle with initial side (an initial portion of) $\mathcal{G}$ and terminal side (an initial portion of) $\mathcal{H}$ (where this angle is viewed as a directed angle arising via a counterclockwise rotation about P_0). Assume that $\xi \le \pi$ (that is, $0 < \xi < \pi$). Let T (resp., U) be the tangential half-line of $\mathcal{G}$ (resp., $\mathcal{H}$) at P_0 (that is, emanating from P_0). It cannot be the case that both T and U are vertical. In addition, ξ can be determined by performing the following steps in the listed order.

Step 1: Determine whether T is vertical. Also determine whether U is vertical.

Step 2: For $\mathcal{G}$ (resp., $\mathcal{H}$), determine a Cartesian equation for either a vertical Euclidean line or a Euclidean circle that contains $\mathcal{G}$ (resp., $\mathcal{H}$). In the case of a Cartesian equation of a Euclidean circle, obtain the associated parameters b, d, c and m.

Step 3: If T is not vertical, let m_1 denote its slope and determine m_1. If U is not vertical, let m_2 denote its slope and determine m_2.

Step 4: Find the coordinates of a point P_1 on T (resp., P_2 on U) that is distinct from P_0.

Step 5: Compute the components of the (bound) vectors $\mathbf{u} := \overrightarrow{P_0P_1}$ and $\mathbf{v} := \overrightarrow{P_0P_2}$. Then compute the magnitudes (also known as absolute values) $|\mathbf{u}|$ and $|\mathbf{v}|$.

Step 6: Compute the dot product $\delta := \mathbf{u} \cdot \mathbf{v}$.

Step 7: If $\delta = 0$, then $\xi = \pi/2$. If $\delta > 0$, then $0 < \xi < \pi/2$. If $\delta < 0$, then $\pi/2 < \xi < \pi$.

Step 8: Assume henceforth that $\xi \neq \pi/2$ (and $\delta \neq 0$).

Step 9: Suppose that neither T nor U is vertical (that is, suppose that $\mathcal{G}$ and $\mathcal{H}$ are each a subset of a Euclidean circle in $\mathbb{R}^2$). If $\delta > 0$, then $\xi = \tan^{-1}(|\frac{m_1 - m_2}{1+m_1 m_2}|)$. If $\delta < 0$, then $\xi = \pi - \tan^{-1}(|\frac{m_1 - m_2}{1+m_1 m_2}|)$. Alternatively, if neither T nor U is vertical, then $\xi = \cos^{-1}(\delta/(|\mathbf{u}||\mathbf{v}|))$.

Step 10: Assume henceforth that (exactly) one of T, U is vertical.

Step 11: Suppose that T is vertical (and so the slope of U is $m_2 \neq 0$). If $\delta m_2 > 0$, then $\xi = \pi/2 - \tan^{-1}(m_2)$. If $\delta m_2 < 0$, then $\xi = \pi/2 + \tan^{-1}(m_2)$.

Step 12: Suppose that U is vertical (and so the slope of T is $m_1 \neq 0$). If $\delta m_1 > 0$, then $\xi = \pi/2 - \tan^{-1}(m_1)$. If $\delta m_1 < 0$, then $\xi = \pi/2 + \tan^{-1}(m_1)$.

Proof. We will show first that $\xi \neq 0$. Suppose, on the contrary, that $\xi = 0$. Then $T = U$. Either T is vertical or T is the intersection of the upper half-plane with a (uniquely determined) Euclidean half-line having slope $m \in \mathbb{R}$. Note that the tangent line to a bowed geodesic is never vertical (at any point in the upper half-plane). Consequently, if T $(= U)$ is vertical, then there exists a Euclidean line L in $\mathbb{R}^2$ such that $\mathcal{G}$ and $\mathcal{H}$ are each subsets of L having the same direction, whence $\mathcal{G} = \mathcal{H}$, which contradicts the hypothesis that $\mathcal{G}$ and $\mathcal{H}$ are distinct. It remains to consider the possibility that T has slope m. Then, by either Theorem 2.1 (if $m \neq 0$) or Proposition 2.3 (if $m = 0$), there exists a unique Euclidean circle $\mathcal{K}$ in $\mathbb{R}^2$ such that $\mathcal{K}$ has its center on the x-axis and the tangent to $\mathcal{K}$ at P_0 has slope m. Necessarily, $\mathcal{G}$ and $\mathcal{H}$ are each equal to the intersection of $\mathcal{K}$ with the upper half-plane, whence $\mathcal{G} = \mathcal{H}$, again contradicting the hypothesis that $\mathcal{G}$ and $\mathcal{H}$ are distinct. This completes the proof that $\xi \neq 0$.

We will show next that $\xi \neq \pi$. Suppose, on the contrary, that $\xi = \pi$. If T were vertical, then U would also be vertical but with the opposite direction from that of T, so that the reasoning in the above paragraph would show that $\mathcal{G}$ (resp., $\mathcal{H}$) would coincide with T (resp., U), whence $\mathcal{H} = -\mathcal{G}$, which would contradict a hypothesis. It remains to remains to consider the possibility that neither T nor U is vertical. Then T and U each have the same slope, say m. It follows from Theorem 2.1 and Proposition 2.3, as in the above paragraph, that there exists a unique Euclidean circle $\mathcal{K}$ in $\mathbb{R}^2$ such that $\mathcal{K}$ has its center on the x-axis and the tangent to $\mathcal{K}$ at P_0 has slope m, so that $\mathcal{G}$ and $\mathcal{H}$ are each a hyperbolic half-line arising from the intersection of $\mathcal{K}$ with the upper half-plane. This, however, gives a contradiction since we have assumed that $\mathcal{H} \neq \mathcal{G}$ and $\mathcal{H} \neq -\mathcal{G}$. This completes the proof that $\xi \neq \pi$.

We will prove next that it cannot be the case that both T and U are vertical. Suppose, on the contrary, that T and U are each vertical. As T is vertical, then the above reasoning shows that $\mathcal{G}$ is a portion of a uniquely determined straight geodesic, say $\mathfrak{G}$. Similarly, as U is vertical, we see that $\mathcal{H}$ is a portion of a uniquely determined straight geodesic, say $\mathfrak{H}$. Then, since $\mathcal{G}$ and $\mathcal{H}$ have the point P_0 in common, so do $\mathfrak{G}$ and $\mathfrak{H}$. Consequently, $\mathfrak{G}$ and $\mathfrak{H}$ are each subsets of the same vertical Euclidean line, say L. Thus, $\mathcal{G}$ and $\mathcal{H}$ are each subsets of L. As $\mathcal{G}$ and $\mathcal{H}$ each emanate from P_0, it follows that either $\mathcal{H} = \mathcal{G}$ or $\mathcal{H} = -\mathcal{G}$, which is the desired contradiction. This completes the proof that at least one of T, U is nonvertical.

About step 1: An equivalent task is to determine whether $\mathcal{G}$ (resp., $\mathcal{H}$) is a subset of some vertical Euclidean line. It is fair to assume that step 1 can be carried out if $\mathcal{G}$ and $\mathcal{H}$ each deserve to be considered “given."

About step 2: Note that T (resp., U) is vertical if and only if $\mathcal{G}$ (resp., $\mathcal{H}$) is a subset of a straight geodesic. When this condition holds, the required Cartesian equation is $x = x_0$. When this condition does not hold, it is fair to assume that we know the coordinates of a point on $\mathcal{G}$ (resp., $\mathcal{H}$) that is distinct from P_0. One can then find a Cartesian equation of the Euclidean circle in question by reasoning as in the fourth paragraph that followed the proof of Lemma 2.9. That paragraph also contains the required formulas for b, d, c and m.

About step 3: If T (resp., U) is not vertical, its slope is the corresponding m that was found in step 2.

About step 4: Apply Lemma 2.12, using the parameters that were found in step 3, together with the parameters r_i and λ that were defined in Lemma 2.12 (b).

About steps 5 and 6: Readers should have the vectorial skills that are assumed in these two steps. Background for those skills, written at the level of precalculus, can be found in [6, pages 406-407 and page 417].

About steps 7 and 8: A standard fact states that $\delta = |\mathbf{u}||\mathbf{v}|\cos(\xi)$ (cf. [6, page 418]). Note that $\mathbf{u}$ and $\mathbf{v}$ are each nonzero vectors (since $P_0 \neq P_1$ and $P_0 \neq P_2$), and so $|\mathbf{u}|$ and $|\mathbf{v}|$ are positive real numbers. It follows that $\delta = 0 \Leftrightarrow \cos(\xi) = 0 \Leftrightarrow \xi = \pi/2$. (Remember that $0 < \xi < \pi$.) It also follows that if $\delta \neq 0$, then δ has the same algebraic sign as $\cos(\xi)$. So, $\delta > 0$ if and only if $0 < \xi < \pi/2$; and $\delta < 0$ if and only if $\pi/2 < \xi < \pi$.

About step 9: For the first and second statements, see step 7 and Lemma 2.6 (a). For the final statement, recall the formula for δ in the preceding paragraph and use step 7.

About step 11: As $\xi \neq \pi/2$ (by step 8) and T is assumed vertical, it follows that $m_2 \neq 0$. Also as noted in step 8, we have $\delta \neq 0$. Thus, δm_2 is either positive or negative. Four cases arise naturally, according as to the algebraic signs of the factors (the point being that δ and m_2 may have the same sign or different signs). The impact of the algebraic sign of δ was summarized in the final sentence in “About steps 7 and 8," while the impact of the algebraic sign of m_2 was summarized in the two statements in Lemma 2.6 (b). The assertions in step 11 follow by using those impacts to examine the above-mentioned four cases.

About step 12: Repeat the reasoning in “About step 11," with U playing the former role of T and m_1 playing the former role of m_2. The proof is complete. □

Example 2.15 will collect several examples to illustrate the methodology in Corollary 2.13. First, in the spirit of step 9 of Corollary 2.13, we pause to collect some additional comments about alternate methods.

Remark 2.14. (a) One is free to choose from among the methods that were noted in step 9 of Corollary 2.13. Furthermore, in working some examples, many readers will find it possible to omit certain steps in the list of instructions given in the statement of Corollary 2.13. For instance, in the case of a bowed geodesic for which one has a Cartesian equation, we saw in the second proof of Theorem 2.1 that the corresponding slope m can be quickly found, using implicit differentiation, as $y'(x_0)$ if one has a Cartesian equation for the Euclidean circle containing the given bowed geodesic (as a subset). Thus, if both $\mathcal{G}$ and $\mathcal{H}$ are (subsets of) bowed geodesics, one may be able to find m_1 and m_2 quickly. In that situation, if one believes from graphical evidence that it is clear whether ξ is acute or obtuse, it would be sensible to go directly to step 9 and use the (first) method that appeals to Lemma 2.6 (a). However, for an example where the reader has carried out all the steps in the instructions, including the steps with vectorial aspects (if only to be certain whether ξ is acute or obtuse), the reader may find it more pleasant to use the second, alternate method (that requires one to apply $\cos^{-1}$ to vectorially obtained data) rather than the first method (which requires one to apply $\tan^{-1}$ to data that involves absolute values, m_1 and m_2).

(b) In view of the multitude of cases that were considered in “About step 11" and “About step 12" in the proof of Corollary 2.13, the reader may have wondered why we did not mention an alternate method, possibly appealing to Theorem 2.9, for those steps. In particular, by parts (a) and (b) of Theorem 2.9, we see that if T is vertical, then $\cot(\xi) = -m_2$, and so $\tan(\xi) = -1/m_2$; and by parts (c) and (d) of Theorem 2.9, if U is vertical, then $\cot(\xi) = m_2$, and so $\tan(\xi) = 1/m_2$. So, one can fairly ask if it would be possible to approach steps 11 and 12 of Corollary 2.13 in an alternate way that uses $\tan^{-1}(1/m_2)$ and $\tan^{-1}(-1/m_2)$; and, if so, why such an alternate method was not mentioned earlier.

My answer is threefold. First, it comes down to asking about a role for $\tan^{-1}(1/m_2)$, since $\tan^{-1}$ is an odd function. (Indeed, $\tan^{-1}(-x) = -\tan^{-1}(x)$ for any real number x. An elementary proof of this fact at the level of precalculus can be given by using reference angles and considering four cases, determined by the quadrant containing the terminal side of an angle in standard position that has measure ξ. An accessible proof that assumes calculus uses "integration by substitution" and the fact that $\int_0^x 1/(1+t^2)\,dt = \tan^{-1}(x)$ for any real number x.) Second, the above rendering of steps 9 and 12 could have been reformulated in terms of $\tan^{-1}(1/m_2)$ and $\tan^{-1}(1/m_1)$, by using the identity $\tan^{-1}(m)+\tan^{-1}(1/m)$ equals $\pi/2$ (resp., equals $-\pi/2$) for all $m > 0$ (resp., for all $m < 0$). (This identity was not familiar to the author, so a quick sketch of some proofs of it would seem to be in order here. For a proof at the precalculus level, let $u := \tan^{-1}(x)$ and $v := \tan^{-1}(1/x)$ for some $x \neq 0$. It follows easily from the definition of the inverse tangent function that $0 < u + v < \pi$ if $x > 0$; and that $-\pi < u + v < 0$ if $x < 0$. Then the conclusion follows because an attempt to express $\tan(u + v)$ by the familiar expansion formula would be illegitimate (since it would involve division by 0.) For a proof using calculus, consider the function f given by $f(x) := \tan^{-1}(x) + \tan^{-1}(1/x)$ for all real nonzero numbers x. It is easy to use differential calculus to check that $f'(x) = 0$ for all $x \neq 0$. So, by the Mean Value Theorem, f is a constant function when restricted to the interval $(-\pi/2, 0)$ or the interval $(0, \pi/2)$. That constant is $\pm\pi/2$, since $\tan^{-1}(1) = \pi/4$ and $\tan^{-1}(-1) = -\pi/4$, and so the asserted identity has been proved.) Third, when you use the identity that was just established, that identity *can* be used to reformulate the statements of steps 11 and 12 in terms of $\tan^{-1}(1/m_2)$ and $\tan^{-1}(1/m_2)$ *and* that formulation can be proved directly by a case analysis that uses Theorem 2.9, *but* we found that case analysis to be approximately as tedious as the case analysis that was given in the above proof for steps 11 and 12, *so* we decided not to provide the full details here for what turned out to be an equally difficult/easy proof of a logically equivalent statement. The remark is complete.

The next result gives examples illustrating how to use the method(s) and results of this paper, especially as organized in Corollary 2.13, to calculate the measures of six (different kinds of) angles in hyperbolic triangles (in the upper half-plane model of hyperbolic geometry). Five of those six angles were also measured (via other methods and, for one of the angles, with a different result) in two worked examples appearing in [12, Corollary 6.1.3] and [8, page 52]. The calculations, in both [12] and [8], were done for the purpose of measuring the interior angles of (hyperbolic) triangles.

Before determining the measures of some angles, it is timely to devote this paragraph and the next three paragraphs to some remarks about notation, its meaning and its context. These comments apply to both Euclidean geometry and hyperbolic geometry. Consider the generic nontrivial angle $\angle := \angle XYZ$. We take this to be the same as $\angle ZYX$. It is "nontrivial" in the sense that X, Y and Z are not collinear; in other words, X, Y and Z do not all lie on the same "line" in the geometry. The vertex of $\angle$ is the point Y. The "sides" of $\angle$ can be described as the half-"lines" $\overrightarrow{YZ}$ and $\overrightarrow{YX}$; or as the "line" segments YZ (from Y to Z) and YX (from Y to X). The former interpretation of "sides" is especially helpful in our study of hyperbolic geometry, as it facilitates the calculation of the parameter r_i that was introduced in Lemma 2.12 (b) and played an important role in step 4 (and implicitly, for some vectorial considerations, also in steps 5-9 and 11-12) of Corollary 2.13. The latter interpretation of "sides" is typical in case one's interest in $\angle$ is part of a study of the (Euclidean or hyperbolic) triangle $\triangle := \triangle XYZ$.

Our point of view here is to consider $\angle$ and its measure without regard to the existence or relevance of $\triangle$. For that reason, (any) $\angle$ (under consideration here) must have a designated initial side and a designated terminal side. Denoting $\angle$ as $\angle XYZ$ *does not* automatically identify which side is to designated as the initial side of $\angle$. (It could be the "line segment" or half-"line" from Y to/through X; or it could be the "line segment" or half-"line" from Y to/through Z.) We will always measure $\angle$ by determining the number of radians needed for a counterclockwise rotation about the vertex Y to carry the (tangent half-"line" at Y of) the initial side of $\angle$ to the (tangent half-"line" at Y of) the terminal side of $\angle$.

Since $\angle$ is nontrivial, there is exactly one way to decide which of its sides is to be designated as its initial side *if* one wants the measure of $\angle$ to be strictly between 0 and π. Such a desire is often evinced in studies of $\triangle$, where there is universal agreement that (for both Euclidean geometry and hyperbolic geometry) the three interior angles of a triangle must be viewed in a way that ensures that the sum of their measures does not exceed π. However, as we saw in Example 2.10 (a), it is easy to produce a (Euclidean or hyperbolic) angle whose measure exceeds π *if* one uses a "directed angle" approach that involves counterclockwise rotation and that specifically designates which side of $\angle$ is its designated side (without regard to any concerns having to do with the study of a related triangle). Nevertheless, we will always use that "directed angle" approach here. One consequence is that the measure of a (directed) angle here will be between 0 and 2π (but not necessarily between 0 and π).

When dealing with the calculation of the measure of an angle in the literature, a reader/user must use context to determine whether the author of that piece of the literature has specified (as we have) that the angles are to be measured via counterclockwise rotations from (the tangential half-line, at the vertex, of) the designated initial side of the angle toward the designated terminal side of the angle. In the event of an apparent discrepancy (and we will see one in Example 2.15 (f)), the only possible explanations are the following: someone has made a calculational mistake; someone has not used the conventions of this paper and/or someone has misapplied our Corollary 2.13; or the author of that piece of the literature and I may be using different definitions of the measure of an angle. In regard to the last-mentioned possibility, we believe that our definition of measure, focusing as it does on tangential half-lines, is fully in the spirit of Stahl's comment that "The *hyperbolic measure of an angle* [whose sides are hyperbolic half-lines] is identical to its Euclidean measure" [12, page 59]. We would also remind the reader that, as noted in the final sentence before the statement of Lemma 2.5, Millman's rigorous definition of angular measure seems to differ from ours (possibly in unimportant ways); and, as noted in the second paragraph of Remark 2.11 (a) (and amplified in the first paragraph of [8]), Millman's rigorous definition seems to have been designed, at least in part, with the specific goal of verifying certain axioms.

Example 2.15. Consider the following seven points in the upper half-plane of $\mathbb{R}^2$: $A(0,1)$, $B(2,1)$, $C(4,1)$, $D(3,1)$, $E(6,1)$, $F(6,4)$ and $G(2,4\sqrt{2})$. Then:

(a) Let $\angle_1$ be the (directed) angle whose initial side is $\overrightarrow{BC}$ and whose terminal side is $\overrightarrow{BA}$. Then, as determined using a counterclockwise rotation, the measure of $\angle_1$ is $\pi/2$.

(b) Let $\angle_2$ be the (directed) angle whose initial side is $\overrightarrow{AB}$ and whose terminal side is $\overrightarrow{AC}$. Then, as determined using a counterclockwise rotation, the measure of $\angle_2$ is $\tan^{-1}(1/3)$.

(c) Let $\angle_3$ be the (directed) angle whose initial side is $\overrightarrow{CA}$ and whose terminal side is $\overrightarrow{CB}$. Then, as determined using a counterclockwise rotation, the measure of $\angle_3$ is $\tan^{-1}(1/3)$.

(d) Let $\angle_4$ be the (directed) angle whose initial side is $\overrightarrow{EF}$ and whose terminal side is $\overrightarrow{ED}$. Then, as determined using a counterclockwise rotation, the measure of $\angle_4$ is $\pi/2+\tan^{-1}(-3/2)$.

(e) Let $\angle_5$ be the (directed) angle whose initial side is $\overrightarrow{FG}$ and whose terminal side is $\overrightarrow{FE}$. Then, as determined using a counterclockwise rotation, the measure of $\angle_5$ is $3\pi/4$.

(f) Let $\angle_6$ be the (directed) angle whose initial side is $\overrightarrow{FD}$ and whose terminal side is $\overrightarrow{FE}$. Then, as determined using a counterclockwise rotation, the measure of $\angle_6$ is $\pi/2-\tan^{-1}(1/4)$.

.

Proof. Recall the following comment from the paragraph that preceded Theorem 2.9: the "$m_1m_2 = -1$" criterion is the quickest way to detect perpendicularity for a pair of non-vertical (Euclidean) lines. One reasonable inference is that, whenever there seems to be a good chance that an angle in question is a right angle, one should determine if $m_1m_2=-1$. If so, we are done. Suppose next that (m_1 and m_2 exist but) $m_1m_2 \neq -1$. Also suppose that the measure of the angle in question is less than

π. If one has reliable evidence (for instance, from a graph) as to whether the angle is acute or obtuse, one can use Lemma 2.6 (a) to determine the measure of the angle. If one is not sure whether the angle is acute or obtuse, it would be appropriate to apply the instructions from Corollary 2.13 step by step.

The preceding overview suggests that it would be helpful to be ready to calculate all the slopes that will be relevant in any of the parts (a)-(f). Without additional explanation, we will be determining the slope m of the tangent, at a point (x_0, y_0), to a bowed geodesic having center $(c, 0)$ by the formula $m = (c - x_0)/y_0$, which is familiar from the proofs of Theorem 2.1 and Corollary 2.3. To that end, the next six paragraphs provide a name and a Cartesian equation for each bowed geodesic that figures in any of (a)-(f) and, for each such geodesic, we also provide the value of its parameter c.

Let $\mathcal{F}$ denote the (bowed) geodesic that passes through the points A and C. Then a Cartesian equation for $\mathcal{F}$ is $x^2 + y^2 - 4x = 1$, with parameter $c = 2$.

Let $\mathcal{G}$ denote the (bowed) geodesic that passes through the points A and B. Then a Cartesian equation for $\mathcal{G}$ is $x^2 + y^2 - 2x = 1$, with parameter $c = 1$.

Let $\mathcal{H}$ denote the (bowed) geodesic that passes through the points B and C. Then a Cartesian equation for $\mathcal{H}$ is $x^2 + y^2 - 6x = -7$, with parameter $c = 3$.

Let $\mathcal{M}$ denote the (bowed) geodesic that passes through the points D and E. Then a Cartesian equation for $\mathcal{M}$ is $x^2 + y^2 - 9x = -17$, with parameter $c = 9/2$.

Let $\mathcal{N}$ denote the (bowed) geodesic that passes through the points D and F. Then a Cartesian equation for $\mathcal{N}$ is $x^2 + y^2 - 14x = -32$, with parameter $c = 7$.

Let $\mathcal{C}$ denote the (bowed) geodesic that passes through the points F and G. Then a Cartesian equation for $\mathcal{C}$ is $x^2 + y^2 - 4x = 28$, with parameter $c = 2$.

(a) A graph suggests that there is a good chance that $\angle_1$ is a right angle. So, we proceed to calculate the product of the relevant slopes. The slope of the tangent line to $\mathcal{H}$ at B is $(3 - 2)/1 = 1$. The slope of the tangent line to $\mathcal{G}$ at B is $(1 - 2)/1 = -1$. The product of these two slopes is -1, and so these two tangent lines are perpendicular. Thus, the measure of $\angle_1$ is $\pi/2$ (equivalently, $90°$).

To be complete, we should note that Corollary 2.13 assumed that the angle in question at hand has measure at most π. Whenever an assertion of perpendicularity has been confirmed (as it was in the preceding paragraph) then, before one can conclude that the relevant measure is $\pi/2$ (and not $3\pi/2$, as in Example 2.10 (a)), one must be certain that the initial and terminal sides of the angle have been chosen so that the measure of the angle is at most π. We trust that the reader recognizes that to be the case for the data in (a).

(b) The slope of the tangent to $\mathcal{G}$ at A is $(1 - 0)/1 = 1$. The slope of the tangent to $\mathcal{F}$ at A is $(2 - 0)/1 = 2$. So, if $\angle_2$ is acute (as is strongly suggested by the diagram in [12, page 82]), then an appeal to Lemma 2.6 (a) would show that the measure of $\angle_2$ is

$$\tan^{-1}(|\frac{2-1}{1+2\cdot 1}|) = \tan^{-1}(\frac{1}{3}) \approx 0.3217505544(\text{radians}) \approx 18.43494882°$$

which agrees nicely with Stahl's answer of $\cos^{-1}(3/\sqrt{10})$ (which Stahl reports as being approximately $18.4°$) [12, page 81]. To confirm the displayed value, it suffices to verify that $\angle_2$ is acute. That verification is done in the next paragraph, by applying steps 4-7 from Corollary 2.13. In the paragraph following that, we will confirm that Stahl's answer for the measure of $\angle_2$ is exactly correct.

We will first find the coordinates of a point P on the tangent half-line to $\mathcal{G}$ at A (resp., a point Q on the tangent half-line to $\mathcal{F}$ at A) that is distinct from A. By Lemma 2.12, we find these points to be

$$P(\frac{1+\sqrt{2}}{2}, \frac{3+\sqrt{2}}{2}) \text{ and } Q(\frac{2+\sqrt{5}}{2}, 3+\sqrt{5}).$$

(In detail, the coordinates of P are found via Lemma 2.12 (b), using the fact that $\mathcal{G}$ has parameter $c = 1$ and, for $\overrightarrow{AB}$, $r_i = r_2 = 1 + \sqrt{2}$, so that $\lambda = (0 + r_i)/2$, and the y-coordinate for P is found by using

$m = 1$ and $(x_0, y_0) = (0, 1)$. Similarly, the coordinates of Q are found via Lemma 2.12 (b), using the fact that $\mathcal{F}$ has parameter $c = 2$ and, for $\overrightarrow{AC}$, $r_i = r_2 = 2 + \sqrt{5}$, so that $\lambda = (0 + r_i)/2$, and the y-coordinate for Q is found by using $m = 2$ and $(x_0, y_0) = (0, 1)$.) Consider the vectors $\mathbf{u} := \overrightarrow{AP}$ and $\mathbf{v} := \overrightarrow{AQ}$. The dot product $\delta := \mathbf{u} \cdot \mathbf{v} =$

$$(\frac{1+\sqrt{2}}{2})(\frac{2+\sqrt{5}}{2}) + (\frac{1+\sqrt{2}}{2})(2+\sqrt{5}) = \frac{6+3\sqrt{5}+6\sqrt{2}+3\sqrt{10}}{4},$$

which is greater than 0. This confirms that $\angle_2$ is acute and thus completes a proof that the measure of $\angle_2$ is $\tan^{-1}(1/3)$.

We will give a second proof to calculate the measure of $\angle_2$ (this time, without mentioning Lemma 2.6). Using the above notation, it is straightforward to calculate that

$$|\mathbf{u}| = \sqrt{(\frac{1+\sqrt{2}}{2})^2 + (\frac{1+\sqrt{2}}{2})^2} = \frac{2+\sqrt{2}}{2} \text{ and}$$

$$|\mathbf{v}| = \sqrt{(\frac{2+\sqrt{5}}{2})^2 + (2+\sqrt{5})^2} = \frac{5+2\sqrt{5}}{2},$$

so that

$$\cos(\angle_2) = \frac{\delta}{|\mathbf{u}| \cdot |\mathbf{v}|} = \frac{6+3\sqrt{5}+6\sqrt{2}+3\sqrt{10}}{(2+\sqrt{2})(5+2\sqrt{5})}.$$

Fortunately, one can use high school algebra to check that the last-displayed expression equals $3/\sqrt{10}$. This completes a proof that the measure of $\angle_2$ is $\cos^{-1}(3/\sqrt{10})$. The interested reader is invited to find a proof (using "right triangle trigonometry" at the precalculus level) to show directly that

$$\tan^{-1}(1/3) = \cos^{-1}(3/\sqrt{10}).$$

That confirms the earlier result of our "acute" intuition.

(c) The slope of the tangent to $\mathcal{F}$ at C is $(2-4)/1 = -2$. The slope of the tangent to $\mathcal{H}$ at C is $(3-4)/1 = -1$. So, if $\angle_3$ is acute (as is strongly suggested by the diagram in [12, page 82]), then an appeal to Lemma 2.6 (a) would show that the measure of $\angle_3$ is

$$\tan^{-1}(|\frac{-1-(-2)}{1+(-2)(-1)}|) = \tan^{-1}(\frac{1}{3}) \approx 0.3217505544 \approx 18.43494882^\circ$$

which agrees nicely with Stahl's answer of $\cos^{-1}(3/\sqrt{10})$ (which Stahl reports as being approximately 18.4°) [12, page 81]. To confirm the displayed value, it suffices to verify that $\angle_2$ is acute. That verification is done in the next paragraph.

We will first find the coordinates of a point P on the tangent half-line to $\mathcal{F}$ at C (resp., a point Q on the tangent half-line to $\mathcal{H}$ at C) that is distinct from C. By Lemma 2.12, we find these points to be

$$P(\frac{6-\sqrt{5}}{2}, 3+\sqrt{5}) \text{ and } Q(\frac{7-\sqrt{2}}{2}, \frac{3+\sqrt{2}}{2}).$$

(In detail, the coordinates of P are found via Lemma 2.12 (b), using the fact that $\mathcal{F}$ has parameter $c = 2$ and, for $\overrightarrow{CA}$, $r_i = r_1 = 2 - \sqrt{5}$, so that $\lambda = (4 + r_i)/2$, and the y-coordinate for P is found by using $m = -2$ and $(x_0, y_0) = (4, 1)$. Similarly, the coordinates of Q are found via Lemma 2.12 (b), using the fact that $\mathcal{H}$ has parameter $c = 3$ and, for $\overrightarrow{CB}$, $r_i = r_1 = 3 - \sqrt{2}$, so that $\lambda = (4 + r_i)/2$, and the y-coordinate for Q is found by using $m = -1$ and $(x_0, y_0) = (4, 1)$.) Consider the vectors $\mathbf{u} := \overrightarrow{CP}$ and $\mathbf{v} := \overrightarrow{CQ}$.

Observe that $\mathbf{u}$ (resp., $\mathbf{v}$) has the negative of the first component and the same second component as the vector that was denoted by $\mathbf{v}$ (resp., $\mathbf{u}$) in the proof of (b). Consequently, the values of δ and of $|\mathbf{u}|\cdot|\mathbf{v}|$ are the same as the correspondingly denoted quantities in the proof of (b). In particular, $\delta > 0$, whence $\angle_3$ is acute, and this completes the proof that the measure of $\angle_3$ is $\tan^{-1}(1/3)$. Perhaps more elegantly, one should simply note now that

$$\cos(\angle_3) = \frac{\delta}{|\mathbf{u}|\cdot|\mathbf{v}|} = \cos(\angle_2)$$

and repeat the last part of the proof of (b) *verbatim*, thus completing the proof that the measure of $\angle_3$ is the same as Stahl's answer of $\cos^{-1}(3/\sqrt{10})$.

(d) The slope of the tangent to $\mathcal{M}$ at E is $(9/2-6)/1 = -3/2 < 0$. So, if $\angle_4$ is acute (as is strongly suggested by the diagram in [8, page 52]), then an appeal to Lemma 2.6 (b) would show that the measure of $\angle_4$ is

$$\pi/2 + \tan^{-1}(-3/2) \approx 0.5880026035\,(\text{radians}) \approx 33.69006753°.$$

The just-displayed value agrees well with Millman's finding that the measure of $\angle_4$ is 33.7°. To confirm the displayed value, it suffices to verify that $\angle_4$ is acute. That verification is done in the next paragraph, by applying steps 4-7 from Corollary 2.13.

We will first find the coordinates of a point P on $\overrightarrow{EF}$ (resp., Q on the tangent half-line to $\mathcal{M}$ at E) that is distinct from E. By Lemma 2.12, we can take P to be $(6,2)$ and Q to be $((21-\sqrt{13})/4, (17+3\sqrt{13})/8)$. (In detail, the coordinates of Q are found via Lemma 2.12 (b), using the fact that $\mathcal{M}$ has parameter $c = 9/2$ and, for $\overrightarrow{ED}$, $r_i = r_1 = (9-\sqrt{13})/2$, so that $\lambda = (6+r_i)/2$, and the y-coordinate for Q is found by using $m = -3/2$ and $(x_0, y_0) = (6,1)$.) Consider the vectors $\mathbf{u} := \overrightarrow{EP}$ and $\mathbf{v} := \overrightarrow{EQ}$. The dot product

$$\delta := \mathbf{u}\cdot\mathbf{v} = 0\cdot\left(\frac{-3-\sqrt{13}}{4}\right) + 1\cdot\left(\frac{9+3\sqrt{13}}{8}\right) = \frac{9+3\sqrt{13}}{8},$$

which is greater than 0. This confirms that $\angle_4$ is acute and thus completes a proof that the measure of $\angle_4$ is $\pi/2 + \tan^{-1}(-3/2)$.

We will give a second proof to calculate the measure of $\angle_4$ (this time, without mentioning Lemma 2.6). Using the above notation, it is straightforward to calculate that $|\mathbf{u}| = 1$ and

$$|\mathbf{u}| = 1 \text{ and } |\mathbf{v}| = \frac{\sqrt{286+78\sqrt{13}}}{8},$$

so that

$$\cos(\angle_4) = \frac{\delta}{|\mathbf{u}|\cdot|\mathbf{v}|} = \frac{9+3\sqrt{13}}{\sqrt{286+78\sqrt{13}}}.$$

Fortunately, one can use high school algebra to (tediously) check that the last-displayed expression equals $3/\sqrt{13}$. Hence, the measure of $\angle_4$ is $\cos^{-1}(3/\sqrt{13})$. This completes a second proof calculating that measure. The interested reader is invited to find (at least two) proofs at the precalculus level to show directly that

$$\pi/2 + \tan^{-1}(-3/2) = \cos^{-1}(3/\sqrt{13}).$$

That confirms the earlier result of our "acute" intuition.

(e) The slope of the tangent to $\mathcal{C}$ at F is $(2-6)/4 = -1$ (< 0). So, if $\angle_4$ is obtuse (as is strongly suggested by a diagram), then an appeal to Lemma 2.6 (b) would show that the measure of $\angle_5$ is

$$\pi/2 - \tan^{-1}(-1) = \pi/2 - (-\pi/4) = 3\pi/4 = 135°.$$

To confirm the displayed value, it suffices to verify that $\angle_5$ is obtuse. That verification is done in the next paragraph, by applying steps 4-7 from Corollary 2.13.

We will first find the coordinates of a point P on the tangent half-line to $\mathcal{C}$ at F (resp., Q on $\overrightarrow{FE}$) that is distinct from F. By Lemma 2.12, we can take Q to be $(6,2)$ and P to be $(4-2\sqrt{2}, 6+2\sqrt{2})$. (In detail, the coordinates of P are found via Lemma 2.12 (b), using the fact that $\mathcal{C}$ has parameter $c = 2$ and, for $\overrightarrow{FG}$, $r_i = r_1 = 2-4\sqrt{2}$, so that $\lambda = (6+r_i)/2$, and the y-coordinate for Q is found by using $m = -1$ and $(x_0, y_0) = (6,4)$.) Consider the vectors $\mathbf{u} := \overrightarrow{FP}$ and $\mathbf{v} := \overrightarrow{FQ}$. The dot product

$$\delta := \mathbf{u}\cdot\mathbf{v} = (-2-2\sqrt{2})\cdot 0 + (2+2\sqrt{2})\cdot(-2) = -4-4\sqrt{2},$$

which is less than 0. This confirms that $\angle_5$ is obtuse and thus completes a proof that the measure of $\angle_5$ is $3\pi/4$ (radians) $= 135°$.

We will give a second proof to calculate the measure of $\angle_5$ (for a class/reader that chose to not use Lemma 2.6). Using the above notation, it is straightforward to calculate that

$$|\mathbf{u}| = 4+2\sqrt{2} \text{ and } |\mathbf{v}| = 2,$$

so that

$$\cos(\angle_5) = \frac{\delta}{|\mathbf{u}|\cdot|\mathbf{v}|} = \frac{-4-4\sqrt{2}}{2(4+2\sqrt{2})} = -\frac{1}{\sqrt{2}}.$$

Hence, the measure of $\angle_5$ is $\cos^{-1}(-1/\sqrt{2}) = 3\pi/4$. This completes a second proof calculating that measure and confirms the earlier result of our "obtuse" intuition.

(f) The slope of the tangent to $\mathcal{N}$ at F is $(7-6)/4 = 1/4$ (> 0). So, if $\angle_6$ is acute (as is strongly suggested by the diagram in [8, page 52]), an appeal to Lemma 2.6 (b) would show that the measure of $\angle_6$ is

$$\pi/2 - \tan^{-1}(1/4) \approx 1.325817664 \,(\text{radians}) \approx 75.96375653°,$$

which appears to be incompatible with Millman's finding that the measure of $\angle_6$ is 104°. So, we must address (f) by applying steps 4-7 from Corollary 2.13.

We will first find the coordinates of a point P on the tangent half-line to $\mathcal{N}$ at F (resp., Q on $\overrightarrow{FE}$) that is distinct from E. By Lemma 2.12, we can take Q to be $(6,2)$ and P to be $((13-\sqrt{17})/2, (33-\sqrt{17})/8)$. (In detail, the coordinates of Q are found via Lemma 2.12 (b), using the fact that $\mathcal{N}$ has parameter $c = 7$ and, for $\overrightarrow{FD}$, $r_i = r_1 = 7-\sqrt{17}$, so that $\lambda = (6+r_i)/2$, and the y-coordinate for Q is found by using $m = 1/4$ and $(x_0, y_0) = (6,4)$.) Consider the vectors $\mathbf{u} := \overrightarrow{FP}$ and $\mathbf{v} := \overrightarrow{FQ}$. The dot product

$$\delta := \mathbf{u}\cdot\mathbf{v} = (\frac{1-\sqrt{17}}{2})\cdot 0 + (\frac{1-\sqrt{17}}{8})(-2) = \frac{-1+\sqrt{17}}{4},$$

which is greater than 0. This confirms that $\angle_6$ is acute and thus completes a proof that the measure of $\angle_6$ is $\pi/2 - \tan^{-1}(1/4)$.

Using the above notation, it is straightforward to calculate that

$$|\mathbf{u}| = \frac{17-\sqrt{17}}{2} \text{ and } |\mathbf{v}| = 2,$$

so that

$$\cos(\angle_6) = \frac{\delta}{|\mathbf{u}|\cdot|\mathbf{v}|} = \frac{\frac{-1+\sqrt{17}}{4}}{17-\sqrt{17}} = 1/\sqrt{17}.$$

Hence, the measure of $\angle_6$ is $\cos^{-1}(1/\sqrt{17})$. This completes a second proof calculating that measure. The interested reader is invited to find (at least two) proofs at the precalculus level to show directly that

$$\pi/2 - \tan^{-1}(1/4) = \cos^{-1}(1/\sqrt{17}).$$

That completes a rigorous proof of (f) and confirms the earlier result of our "acute" intuition. Yet another proof of our assertion in (f) will be given in Remark 2.16 (d). The proof is complete. □

The reader may have noticed that only one of the angles that were measured in Example 2.15 was obtuse. Our analysis of some related data in Example 2.19 will continue to use the methodology from Corollary 2.13 and that work will naturally include the determination of the measure of another obtuse angle.

Remark 2.16 collects some comments about alternate methods.

Remark 2.16. (a) For each of parts (b)-(f) of Corollary 2.15, we offered more than one proof. This illustrates the fact that alternate methods abound in analytic geometry and trigonometry. In that regard, it is interesting to compare Lemma 2.6 with one of the lesser results in this paper, Theorem 2.9. Let us mention here only the fact that, while these results do not have the same focus, they are compatible. For instance, suppose, in the context of Theorem 2.9 (a), that $\xi < \pi/2$. Then Theorem 2.9 (a) asserts that $m_2 = -\cot(\xi)$. On the other hand, for this context (when $\xi < \pi/2$ and $m_2 < 0$), Lemma 2.6 (b) asserts that $\xi = \pi/2 + \tan^{-1}(m_2)$. These assertions are compatible, since the fact that tan is an odd function gives that

$$-\cot(\pi/2 + \tan^{-1}(m_2)) = -\tan(-\tan^{-1}(m_2)) = \tan(\tan^{-1}(m_2)) = m_2.$$

The interested reader is encouraged to check the compatibility of Theorem 2.9 with Lemma 2.6 in all the other contexts.

(b) In view of the different roles that $\tan^{-1}$ and $\cos^{-1}$ played in Corollary 2.13, one may ask if one of these functions is more useful than the other. We believe that a fair answer is that each of them is best suited to certain purposes. We have often found the functions tan and $\tan^{-1}$ to be helpful in giving modern analytic proofs of results from classical Euclidean plane geometry. On the other hand, cos and $\cos^{-1}$ are more directly helpful when one is using the dot product of two vectors to study the *undirected* angle between those vectors. This is to be contrasted with the fact that it was quicker to apply $\tan^{-1}$ in calculating the measure of a (directed) angle that was known to be either acute or obtuse. No doubt, the above choices/uses are related to the identities

$$\cos(2\pi - \theta) = \cos(-\theta) = \cos(\theta) \text{and} \tan(2\pi - \theta) = \tan(-\theta) = -\tan(\theta).$$

(c) While we have striven to use familiar and standard tools from Euclidean analytic geometry in developing the formulas and methods in this paper, one should note that in [12, Proposition 6.1.1], Stahl gives some elegant formulas that can be used to calculate the measure of angles that can appear as interior angles of hyperbolic triangles. To be frank, the author and his students have found those formulas difficult to memorize, and we are not aware of their widespread use in other contexts. Nevertheless, we wish to point out that an application of [12, Proposition 6.1.1] would lead to the same answer for the measure of $\angle_6$ in Example 2.15 (f) as we gave in the statement of Example 2.15 (f).

In detail, we use the notation from Example 2.15 (f) and adapt the notation from [12] accordingly. Consider the tangent to the bowed geodesic $\mathcal{N}$ at F. The (Euclidean) perpendicular to that tangent line meets the x-axis at the center $\mathfrak{C}(7,0)$ of $\mathcal{N}$. By [12, Proposition 6.1.1], $\angle_6$ is congruent to the acute (undirected) angle, say $\angle_S$, between the (Euclidean) line segment $F\mathfrak{C}$ and (the negative direction of) the x-axis. Hence, $\tan(\angle_6) = \tan(\angle_S)$. Consider the point $\mathfrak{D}(6,0)$. Applying "right triangle trigonometry" to $\triangle F\mathfrak{C}\mathfrak{D}$, we get that $\tan(\angle_S) = DF/D\mathfrak{C} = (4-0)/(7-6) = 4$. Thus $\tan(\angle_6) = 4$ and, since $\angle_6$ is acute, it follows that the measure of $\angle_6$ is $\tan^{-1}(4)$. This finding is compatible with the value that we calculated for that measure in Corollary 2.15 (f), since

$$\tan^{-1}(4) + \tan^{-1}(1/4) = \pi/2.$$

This ends our third proof that the measure of $\angle_6$ is $\pi/2 - \tan^{-1}(1/4)$.

(d) Recall that in Example 2.15 (f), we gave two proofs that the measure of $\angle_6$ is $\pi/2 - \tan^{-1}(1/4)$ (radians) $\approx 75.96375653°$. A third proof of this fact is now available, thanks to (c) above. In regard to Millman's having calculated that measure as 104°, it seems relevant to note that

$$\pi/2 + \tan^{-1}(1/4) \approx 1.81577499\,(\text{radians}) \approx 104.0362435°.$$

We will not comment further on the fact that Millman and we seem to have made different uses of the same reference/related angle with measure $\tan^{-1}(1/4)$. As to possible reasons that a different answer for the measure of $\angle_6$ was given in [8], I would direct the reader to the final paragraph that preceded the statement of Example 2.15.

(e) For a class that has emphasized neutral geometry, the calculations in [12, page 81] of the measures of what we have called $\angle_2$ and $\angle_3$ in Example 2.15 are noteworthy. While our proof observed that our vectorial approach gave the same answer(s) as in [12] for those (equal) measures, Stahl?s approach used different tools, specifically, the above-mentioned [12, Proposition 6.1.1] and the (Euclidean) Law of Cosines. We believe that Stahl's proof calculating the measures of $\angle_2$ and $\angle_3$ is slightly less accessible to most undergraduate classes than the approach given above in Example 2.15.

Still another proof for the measure of $\angle_3$ is available, once the measure of $\angle_2$ has been found. To wit, it suffices to show that $\angle_2$ is congruent to $\angle_3$ (for then these angles would have the same measure). That congruence, in turn, holds because these angles are the "base angles" of the hyperbolically isosceles triangle $\triangle BAC$. (This invocation of the classical *Pons Asinorum* for hyperbolic geometry is valid because the SAS (Side-Angle-Side) criterion for congruence of triangles holds in neutral geometry.) To establish that "isosceles" assertion (specifically that the hyperbolic line segments AB and BC have the same hyperbolic lengths), one need only apply the formulas for calculating hyperbolic distance along a bowed geodesic in [12] or [3].

(f) The final comment in (b) underscores the fact that $2\pi - \theta$ should not be confused with θ when applying the function tan. Put differently, if one knows only that $\angle$ is a (directed) angle whose measure is "close to" but not equal to π, then the algebraic sign of $\tan(\angle)$ is unknown. If Corollary 2.13 is to be viewed as the key tool in a comprehensive algorithm to calculate the measure of a "typical" directed angle (whose measure, for all practical purposes, is "typically" between 0 and 2π), one needs to also answer the following question. If $\angle$ is a directed angle whose initial side is the hyperbolic geodesic $\mathcal{F}$, whose terminal side is the hyperbolic geodesic $\mathcal{G}$, and whose measure ξ is such that $|\xi - \pi|$ is a "small" positive number, how can one determine whether $\xi < \pi$? An examination of cases shows that the answer to this question can be summarized as follows (and, happily, the answer is slope-theoretic).

Let the angle $\angle$ be as above; let P be the vertex of $\angle$; and for the generic bowed geodesic $\mathcal{C}$ passing through P, let $\mu_{\mathcal{C}}$ denote the slope of the tangent to $\mathcal{C}$ at P. Then:
(i) If both $\mathcal{F}$ and $\mathcal{G}$ are bowed, then $\xi < \pi$ if and only if $\mu_{\mathcal{G}} < \mu_{\mathcal{F}}$.
(ii) If $\mathcal{F}$ is bowed and $\mathcal{G}$ is straight, then $\xi < \pi$ if and only if $\mu_{\mathcal{F}} < 0$.
(iii) If $\mathcal{F}$ is straight and $\mathcal{G}$ is bowed, then $\xi < \pi$ if and only if $\mu_{\mathcal{G}} > 0$.
This completes the remark.

The next result was suggested by – and generalizes – some material in [12, page 81]. Indeed, one upshot of Theorem 2.17 and Corollary 2.18 is that if P_0 is any point of the upper half-plane, with $\mathcal{G}$ a given hyperbolic half-line emanating from P_0 and $\eta \in \mathbb{R}$ such that $0 \le \eta \le 2\pi$, then there exists a hyperbolic angle having $\mathcal{G}$ as one of its sides, vertex P_0, and measure η. As was the case with Corollary 2.13, the proofs of Theorem 2.17 and Corollary 2.18 depend in part on Theorem 2.1 and Corollary 2.3. It would be straightforward to convert the proofs of Theorem 2.17 and Corollary 2.18 into algorithms, but they are written in a rather conversational tone because these results are easier than Corollary 2.13. Nevertheless, one motivation for Theorem 2.17 and Corollary 2.18 is that they solve a natural question that is analogous to the question that was answered in Corollary 2.13. In fact, the title of this paper was chosen in the hope that it would lead some readers to expect results such as Corollary 2.13 and Theorem 2.17. In Example 2.19 (some of whose details depend on parts of Example 2.15 which, in turn, depended on Corollary 2.13) and Remark 2.21, we show how to use the methodology in these three results to expeditiously answer some other natural questions.

Theorem 2.17. Let $\mathcal{G}$ be a given hyperbolic half-line emanating from a point $P_0(x_0, y_0) \in \mathbb{R}^2$. Let T be the tangential half-line of $\mathcal{G}$ at P_0 (that is, emanating from P_0). Let $\xi \in \mathbb{R}$ such that $0 \leq \xi \leq \pi$. Then there exists a unique hyperbolic half-line $\mathcal{H}$ emanating from P_0 such that the directed angle with initial (resp., terminal) side $\mathcal{G}$ and terminal (resp., initial) side $\mathcal{H}$ has measure ξ.

Proof. Any hyperbolic half-line emanating from P_0 is uniquely determined by its tangential half-line at P_0. (Indeed, this is evident for straight geodesics; and for bowed geodesics, it follows from Theorem 2.1 and Corollary 2.3.) Consequently, the assertion is clear in case $\xi = 0$ (resp., $\xi = \pi$), with $\mathcal{H} := \mathcal{G}$ (resp., $\mathcal{H} := -\mathcal{G}$). Hence, without loss of generality, $0 < \xi < \pi$.

Let U be the Euclidean half-line obtained by rotating T counterclockwise about P_0 through an angle with radian measure ξ (resp., $2\pi - \xi$). It is enough to show that there exists a (necessarily unique) hyperbolic half-line $\mathcal{H}$ emanating from P_0 such that the tangential half-line of $\mathcal{H}$ at P_0 is U. This, in turn, is clear if U is vertical, for $\mathcal{H}$ can then be taken to be U itself. Hence, without loss of generality, U is not vertical. Let m denote the slope of U. By Theorem 2.1 and Corollary 2.3, there exists a unique bowed geodesic (that is, a unique nonvertical hyperbolic "line") $\mathfrak{H}$ passing through P_0 such that the tangent line to $\mathfrak{H}$ at P_0 has slope m. It is clear that exactly one of the (two) hyperbolic half-lines that are determined by $\mathfrak{H}$ and emanate from P_0 is such that its tangential half-line at P_0 is U, and *that* hyperbolic half-line is the desired $\mathcal{H}$. The proof is complete. □

Corollary 2.18. Let $\mathcal{G}$ be a given hyperbolic half-line emanating from a point $P_0(x_0, y_0) \in \mathbb{R}^2$. Let $\xi \in \mathbb{R}$ such that $0 \leq \xi \leq \pi$. Then there exists a unique hyperbolic geodesic (that is, a unique hyperbolic "line;" that is, a unique entity that is either a straight geodesic or a bowed geodesic) $\mathfrak{H}$ which passes through P_0 and for which the (two) hyperbolic half-lines that are determined by $\mathfrak{H}$ and emanate from P_0 can be labeled as $\mathcal{H}_1$ and $\mathcal{H}_2$ in such a way that a counterclockwise rotation of ξ radians about P_0 carries the tangential half-line of $\mathcal{G}$ at P_0 to the tangential half-line of $\mathcal{H}_1$ at P_0 and a counterclockwise rotation of $\pi - \xi$ radians about P_0 carries the tangential half-line of $\mathcal{H}_2$ at P_0 to the tangential half-line of $\mathcal{G}$ at P_0; that is, in such a way that the directed angle with initial side $\mathcal{G}$ and terminal side $\mathcal{H}_1$ has measure ξ and the directed angle with initial side $\mathcal{H}_2$ and terminal side $\mathcal{G}$ has measure $\pi - \xi$.

Proof. By Theorem 2.17, there exists a unique hyperbolic half-line, say $\mathcal{H}_1$, such that a counterclockwise rotation of ξ radians about P_0 carries the tangential half-line of $\mathcal{G}$ at P_0 to the tangential half-line of $\mathcal{H}_1$ at P_0. Put $\mathcal{H}_2 := -\mathcal{H}_1$; and let $\mathfrak{H}$ denote the union of $\mathcal{H}_1$ and $\mathcal{H}_2$. It is clear that $\mathfrak{H}$ is a hyperbolic geodesic that passes through P_0 and that $\mathcal{H}_1$ and $\mathcal{H}_2$ are the hyperbolic half-lines that emanate from P_0 and are determined by $\mathfrak{H}$. We have that a counterclockwise rotation of ξ radians about P_0 carries the tangential half-line of $\mathcal{G}$ at P_0 to the tangential half-line of $\mathcal{H}_1$ at P_0. It remains only to show that a counterclockwise rotation of $\pi - \xi$ radians about P_0 carries the tangential half-line of $\mathcal{H}_2$ at P_0 to the tangential half-line of $\mathcal{G}$ at P_0. These conclusions can, in turn, be easily established by using the following two standard facts about Euclidean (in fact, neutral) geometry: if two angles are supplementary, the sum of their measures is π; and vertically opposite angles are congruent (and hence have the same measure). The proof is complete. □

It would be desirable to have a construction of bowed geodesics $\mathfrak{F}$ and $\mathfrak{G}$ that meet, but are not perpendicular, at a point A (in the upper half-plane) and of points P, Q and R such that all the following conditions hold: P, Q and R are each distinct from A; the hyperbolic half-lines $\overrightarrow{AP}$ and $\overrightarrow{AQ}$ are the (two distinct) hyperbolic half-lines that are determined by $\mathfrak{F}$ (and emanate from A); the hyperbolic half-line $\overrightarrow{AR}$ is a specific one of the hyperbolic half-lines that are determined by $\mathfrak{G}$ (emanating from A); and, when measures of angles (with vertex A) are determined with respect to counterclockwise rotations (about A) of the designated initial side of the angle toward the designated terminal side of the angle, the angle with initial side $\overrightarrow{AR}$ and terminal side $\overrightarrow{AP}$ is acute and the angle with initial side $\overrightarrow{AQ}$ and terminal side $\overrightarrow{AR}$ is obtuse. By reexamining and augmenting some data from Example 2.15 (a), we present a construction of the desired kind in Example 2.19. Part (f) of this result essentially states that all the above conditions are satisfied by the assembled data.

The interested reader is encouraged to use the methods from this paper to produce a construction showing that the phenomenon exhibited in Example 2.19 can be illustrated for any given vertex in the upper half-plane and for any given bowed hyperbolic half-line as a “side" emanating from the given vertex. The proof of Example 2.19 makes rather efficient use of the proof of Example 2.15 (a), but one finds that for other data, additional methods from this paper may also be helpful. In that regard, some readers/classes will likely find uses for Theorem 2.8, especially in conjunction with the SOLVER function of a modern graphing calculator.

Example 2.19. Observe that the bowed geodesic $\mathfrak{G}$ with Cartesian equation $x^2 + y^2 - 2x = 1$ passes through the points $A(0,1)$ and $B(2,1)$. Let $\mathcal{G}_1$ be the hyperbolic half-line that is determined by $\mathfrak{G}$, emanates from A and passes through B. Put $\eta := \pi - \tan^{-1}(1/3)$. Observe that $\pi/2 < \eta < \pi$. By Theorem 2.17, there exists a unique hyperbolic geodesic $\mathfrak{F}$ which passes through A and for which the (two) hyperbolic half-lines that are determined by $\mathfrak{F}$ and emanate from A can be labeled as $\mathcal{F}_1$ and $\mathcal{F}_2$ in such a way that a counterclockwise rotation of η radians about A carries the tangential half-line of $\mathcal{F}_2$ at A to the tangential half-line of $\mathcal{G}_1$ at A; that is, such that the directed angle with initial side $\mathcal{F}_2$ and terminal side $\mathcal{G}_1$ has measure η. (Consequently, this angle is obtuse.) Consider the points $P(1,2)$, $Q(-0.1,\sqrt{0.599})$, and $R(1,\sqrt{2})$. Then:

(a) The directed angle with initial side $\mathcal{G}_1$ and terminal side $\mathcal{F}_1$ (when measured using a counterclockwise rotation about A from $\mathcal{G}_1$ toward $\mathcal{F}_1$) has measure $\tan^{-1}(1/3)$.

(b) A Cartesian equation for $\mathfrak{F}$ is $x^2 + y^2 - 4x = 1$.

(c) One point on $\mathcal{F}_1$ (which is distinct from A) is P.

(d) One point on $\mathcal{F}_2$ (which is distinct from A) is Q.

(e) One point on $\mathcal{G}_1$ (which is distinct from A) is R.

(f) The points P and Q are each distinct from A and lie on distinct hyperbolic half-lines emanating from A (namely, $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively) that are determined by $\mathfrak{F}$ and emanate from A. The point R is distinct from A and lies on the hyperbolic half-line $\mathcal{G}_1$ that emanates from A in a northeasterly direction. The directed angle with initial side $\mathcal{F}_2$ and terminal side $\mathcal{G}_1$ is obtuse, with measure η. The directed angle with initial side $\mathcal{G}_1$ and terminal side $\mathcal{F}_1$ is acute, with measure $\tan^{-1}(1/3)$.

Proof. (a) A supplement of the angle in question has measure η, supplementary angles have measures adding to π, and $\pi - \eta = \tan^{-1}(1/3)$.

(b), (c): As stated, Theorem 2.17 ensures that $\mathfrak{F}$ is uniquely determined. Therefore, since $2\pi - (\pi - \eta) \neq \eta$, it follows that $\mathcal{F}_1$ is uniquely determined, and hence so is $\mathcal{F}_2$. Thus, by Example 2.15 (a), it follows that a Cartesian equation for $\mathfrak{F}$ (and hence also for $\mathcal{F}_1$ and $\mathcal{F}_2$) is $x^2 + y^2 - 4x = 1$; and that $\mathcal{F}_1$ is the hyperbolic half-line which is determined by $\mathfrak{F}$ and emanates from A in a northeasterly direction. This completes the proof of (b). Moreover, as the coordinates of P satisfy $x^2 + y^2 - 4x = 1$, (c) is now clear as well.

(d) It follows from the preceding paragraph that $\mathcal{F}_2$ is the hyperbolic half-line which is determined by $\digamma$ and emanates from A in a southwesterly direction. Thus, to find the coordinates of a suitable point on $\mathcal{F}_2$, it suffices to choose a negative value of x for which there exists a positive value y such that $x^2 + y^2 - 4x = 1$. Setting $x := -0.1$ in this equation and solving for y, we get $y = \sqrt{0.599}$, thus completing the proof of (d).

(e) Recall from Example 2.15 (a) that $x^2 + y^2 - 2x = 1$ is a Cartesian equation for $\mathcal{G}_1$. As $A(0,1)$ and $B(2,1)$ are on the (hyperbolic) half-line $\mathcal{G}_1$, so is the point with coordinates $(1,\nu)$ such that $1^2 + \nu^2 - 2 \cdot 1 = 1$. Solving this equation for $\nu > 0$ gives $\nu = \sqrt{2}$, thus proving (e).

(f) By the second sentence in the proof of (b), $\mathcal{F}_1 \neq \mathcal{F}_2$. The "obtuse" and "acute" assertions hold because $\pi/2 < \pi - \tan^{-1}(1/3) = \eta$ and $0 < \tan^{-1}(1/3) < \pi/2$. The rest of the statement of (f) collects previous observations in the proof to this point. The purpose of the "summary" aspect of the statement of (f) is to record the fact that we have fulfilled a promise that was made prior to the statement of this example. The proof is complete. □

Although Corollary 2.13 was stated for application to the upper half-plane, it is clear that its methodology can be extended to the more general context in Theorem 2.1. Apart from such minor generalizations, one should not expect this paper's results to extend much further. This fact is due to the relatively uncomplicated nature of the graphs of circles and lines. The next remark gives an instance where such a putative slope-based extension would fail.

Remark 2.20. (a) The following example is inspired by a comment of Stahl [12, page 59]. For $n = 2,3$, define a real-valued function f_n by $f_n(x) = 1 + x^{2n}(1 + \sin(1/x))$ if $x \neq 0$ and $f_n(0) = 1$. Then $f_n(x) > 0$ for all $x \in \mathbb{R}$ and f_n is a differentiable function with $f_n'(0) = 0$ (for each n). Moreover, $f_2'(0) = 0 = f_3'(0)$. Also, consider the real-valued function h defined by $h(x) = 1$. Of course, h' is identically 0. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ denote the respective graphs of f_2, f_3 and h (all of which lie in the upper half-plane). For this set of data, one cannot use the slopes of the tangent lines at the point $P_0(0,1)$ to distinguish among the graphs of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$. In particular, one cannot use the slopes of the tangent lines at P_0 to distinguish between the angles $\angle_1$ and $\angle_2$, where $\angle_1$ and $\angle_2$ each have vertex P_0 and initial side the rightward-pointing half-line of $\mathcal{H}$ while $\angle_1$ (resp., $\angle_2$) has terminal side the northeasterly-pointing tangential half-line of $\mathcal{F}$ (resp., of $\mathcal{G}$). Note that, while h shares the property of being an analytic function with the functions whose graphs are bowed geodesics (of course, these functions do not have domain $\mathbb{R}$), the same cannot be said of the functions f_2 and f_3. Indeed, neither f_2 nor f_3 is a $C^{(2)}$ function. In fact, f_2' is not continuous at 0 and f_3'' is not continuous at 0.

(b) We next illustrate another kind of obstacle that would face a putative generalization of Corollary 2.13. Consider the (infinitely differentiable) real-valued functions f and g defined, for $0 \le x \le \pi$, by $f(x) = 1 + x\sin(x)$ and $g(x) = 1 + x$, with their graphs (which lie in the upper half-plane) being denoted by $\mathcal{F}$ and $\mathcal{G}$, respectively. These graphs intersect at $P_0(0,1)$ and have the same tangent line (with slope 1) at P_0. One may ask if the angles formed by $\mathcal{F}$ and $\mathcal{G}$ with vertex P_0 could be measured in the spirit of Corollary 2.13. Consider, for instance, the angle formed by the northeasterly- pointing tangent line(s). By definition (cf. [12, page 59], the measure of that angle is the measure of the angle between (in this case, identical) tangential half-lines, and so that angle has measure 0. However, in our opinion, the enterprise of measuring that angle by using the usual definition of angular measure lacks any relevance to the geometric nature of the relationship between $\mathcal{F}$ and $\mathcal{G}$ in any open (Euclidean or hyperbolic) neighborhood of P_0, because $\mathcal{F}$ and $\mathcal{G}$ intersect at $(1/(2n\pi), 0)$ for each positive integer n. This completes the remark.

Our closing result will point out a sense in which Corollary 2.18 is best possible.

Remark 2.21. Let $\mathcal{G}$ be a given nonvertical hyperbolic half-line emanating from a point $P_0(x_0, y_0) \in \mathbb{R}^2$. Let α be the measure of the angle of inclination of the tangent to $\mathcal{G}$ at P_0. Let $\xi \in \mathbb{R}$ such that $0 < \xi < \pi$ and

$$\xi \notin \{\pi/2, \pi/2 - \alpha, 3\pi/2 - \alpha, \alpha + \pi/2, \alpha - \pi/2\}.$$

Then by Corollary 2.18, there exists a unique hyperbolic geodesic $\mathfrak{H}$ which passes through P_0 and for which the (two) hyperbolic half-lines that are determined by $\mathfrak{H}$ and emanate from P_0 can be labeled as $\mathcal{H}_1$ and $\mathcal{H}_2$ in such a way that the (directed) angle with initial side $\mathcal{G}$ and terminal side $\mathcal{H}_1$ has measure ξ and the (directed) angle with initial side $\mathcal{H}_2$ and terminal side $\mathcal{G}$ has measure $\pi-\xi$; and there exists a unique hyperbolic geodesic $\mathfrak{K}$ which passes through P_0 and for which the (two) hyperbolic half-lines that are determined by $\mathfrak{K}$ and emanate from P_0 can be labeled as $\mathcal{K}_1$ and $\mathcal{K}_2$ in such a way that the (directed) angle with initial side $\mathcal{G}$ and terminal side $\mathcal{K}_1$ has measure $\pi - \xi$ and the (directed) angle with initial side $\mathcal{K}_2$ and terminal side $\mathcal{G}$ has measure ξ. Since $\xi \neq \pi/2$, we have $\xi \neq \pi - \xi$ (and conversely). Also, by the above-displayed restrictions on ξ, neither $\alpha + \xi$ nor $\alpha + (\pi - \xi)$ is a member of the set $\{\pi/2, 3\pi/2\}$; nor do $\alpha + \xi$ and $\alpha + (\pi - \xi)$ differ by π. It follows that the tangential half-lines of $\mathcal{H}_1$ and $\mathcal{K}_1$ (resp., of $\mathcal{H}_2$ and $\mathcal{K}_2$) at P_0 are distinct; and that neither $\mathfrak{H}$ nor $\mathfrak{K}$ has a vertical tangent at P_0. Hence, the tangents to $\mathfrak{H}$ and $\mathfrak{K}$ at P_0 are each nonvertical and have unequal slopes. Therefore, by the uniqueness assertions in Theorem 2.1 and Corollary 2.3, $\mathfrak{H} \neq \mathfrak{K}$. This reasoning identifies the following sense in which Corollary 2.18 is best possible: using the notation and hypotheses of Corollary 2.18, if one places the above-displayed five restrictions on ξ, then one cannot interchange the roles that are played by ξ and $\pi - \xi$ in the statement of Corollary 2.18. While this fact (and its analogue for Euclidean geometry) may seem intuitively obvious, it is pleasant to close by observing that a rigorous proof of it was facilitated in part by our attention to measuring *directed* angles and focusing on the counterclockwise rotations which produced them. This completes the remark.

3 Appendix

The rest of this paper was completed on April 8, 2021. The next day, the author was able to access a copy of [9] for the first time. (In fact, a used book vendor provided us with online access to a large portion of [9].) Having read that material, we are composing this appendix and submitting this manuscript on April 10, 2021.

The second sentence of the Introduction mentioned [9] because this textbook is famous for having aroused interest in a models-based approach to the undergraduate course on geometry that is taken by many majors in mathematics education and by some other mathematics majors. The purpose of this appendix is to record our views on some parts of [9] and on the present paper's relationship to [9].

As was the case with [8], a main purpose of [9] was to verify that the upper half-plane model satisfies the axioms of hyperbolic (plane) geometry. However, unlike [8], [9] is a textbook and, as such, it slowly introduces the concept of a neutral geometry by introducing a succession of increasingly complicated axiomatic structures. Reading the process from end to (nearly the) start, we see on [9, page 127] that a neutral geometry is a protractor geometry that satisfies SAS; on [9, page 91] that a protractor geometry is a Pasch geometry with an angle measure; on [9, page 76] that a Pasch geometry is a metric geometry that satisfies Pasch's Axiom; on [9, pages 90-91] that an angle measure (on a Pasch geometry) is a function from the set of all angles in the geometry to $\mathbb{R}$ satisfying three certain axioms (one of which stipulates that if one is using radian measure, then the measure of each angle is strictly between 0 and π); on [9, page 30] that a metric geometry is an incidence geometry, together with a distance function, such that every line in the geometry has a ruler; etc. We stop the list here because the concepts of "incidence geometry," "distance function" and "ruler" are likely known or easily accessible to most readers.

It is clear from the above explication that the attention that is paid to "angle measure" in [9], as it applies to the upper half-plane model, is largely restricted to whatever may be needed to verify the relevant axioms of hyperbolic geometry. In particular, [9] does not consider the possibility of angles with measures of 0, π, or a (real) number strictly between π and 2π. This observation is not intended as negative criticism, but it is simply intended to indicate that the decision of Millman and Parker to focus in [9] on angles having measures strictly between 0 and π is perfectly in accord with the goal in [9] to verify that a certain model satisfies certain celebrated axioms. It is generally a fool's errand to complain that a book or paper which is being reviewed was not written in order to satisfy a goal, or the taste, of a reviewer. In addition, the author has too much respect for Millman and Parker to suggest that their choice of material was inappropriate or lacking in some regard. We do hope, however, that our comments are starting to make clear to the reader that substantial differences exist between the goals/coverage in [9] and the goals/coverage in this paper.

Since one goal in [9] is to construct something (that is, *some one thing*) that would satisfy the axioms of an angle measure on the upper half-plane model, Miller and Parker, quite sensibly, give only one way to measure angles. Their formula for doing so, on [9, pages 94-95], is equivalent to the last formula (involving $\cos^{-1}$) in step 9 in our Corollary 2.13. Corresponding to the vectors $\mathbf{u}$ and $\mathbf{v}$ in that step, one finds certain positive scalar multiples of $\mathbf{u}$ and $\mathbf{v}$ in the angle-measuring formula of Millman and Parker. Different readers may decide differently as to whose formulation is more "geometrically intuitive." While [9] does not mention a role for either $\tan^{-1}$ or the "$m_1 m_2 = -1$" criterion in regard to measuring angles, that is perfectly in accord with their main goals.

One should note that [9, Proposition 5.4.16 "Angle Construction"] has essentially the same statement as our Theorem 2.17. Although [9] does not explicitly discuss *directed* angles, Millman and Parker get around that drawback by stipulating that the required geodesic half-line (in [9, Proposition 5.4.16]) must point into a specified one of the (two) half-planes determined by the given geodesic half-line. We find it regrettable that, in our opinion, the proof of [9, Proposition 5.4.16] is lacking in rigor (or, at least, in detail) to some extent. Specifically, the proof of [9, Proposition 4.16] depends on [9, Proposition 4.15], but the proof of the latter result is left to an exercise [9, Exercise A6, page 123]. What is, perhaps, worse is that the statement of [9, Proposition 4.15] is vague. Our opinion about this is supported by the inclusion of "(See figure 5-26.)" in the statement of [9, Proposition 4.15]. That figure is intended to convey what is meant by a particular "side" of a bowed geodesic. To clarify what is meant by the "top" and "bottom" sides of a bowed geodesic, Millman and Parker write that these notions "have intuitive meaning. This terminology could be made formal if needed." As one who endeavored to make Theorem 2.17 clear, I had hoped for a clear and more rigorous presentation of [9, Proposition 5.4.16]. We hope that it will not be seen as mean-spirited for us to note that [9] does not seem to have followed up on [9, Proposition 5.4.16] with anything having the flavor of any of our Corollary 2.18, Example 2.19 or Remark 2.21.

Perhaps the greatest strength of this paper, as compared with [9], is our emphasis on directed angles having measures between 0 and 2π. While angles with measures between π and 2π are irrelevant to the main goals in [9], they are very much the stuff of real mathematics that is used and studied every day. So are the specification of sides and the directions of circulation of fluids. Specifying the direction of an (outer normal) vector is part of the study of the Divergence Theorem in advanced calculus. Traversing bounding curves consistently in a counterclockwise direction is fundamental in working with Cauchy's integral formula in a complex analysis course. It is important to take care in addressing such concepts rigorously. By addressing angular measures from 0 to 2π, our work here makes available a broader range of examples and activities for classes, whether they be at the precalculus or the "beyond" level.

Just as the authors of [9] progressively worked toward their goals, so have we. After beginning with a slope-focused program in results 2.1-2.3, we used those results in the proofs of our main results (items 2.13, 2.17 and 2.18). The algorithm in Corollary 2.13 is comprehensive and unambiguous. In cases where it is clear whether the (directed) angle being measured is acute or obtuse, the algorithm couples slopes with $\tan^{-1}$ to make short work of calculating the measure of the angle. In cases where one is unsure if the angle is acute or obtuse, following the 12 steps in Corollary 2.13 is decisive, leaving nothing to intuition or guesswork. When one couples Corollary 2.13 with Remark 2.16 (f), one effectively has an algorithm to measure any directed angle (with measure between 0 and 2π).

Finally, we would hope that students and instructors alike will welcome Examples 2.15 and 2.19 as useful additions to the literature.

References

[1] R. W. Brink, A first year of college mathematics, second edition, Appleton-Century-Crofts, New York, 1954.

[2] D. E. Dobbs, Three dimensional vector geometry, M. A. thesis, University of Manitoba, Fort Garry, Manitoba, Canada, 1965.

[3] D. E. Dobbs, Applications of a distance formula in hyperbolic plane geometry, Inter. J. Math. Educ. Sci. Technol., 32 (1) (2001), 148–156.

[4] D. E. Dobbs, Verification of the hyperbolic parallel postulate in a half-plane model, Math. and Computer Educ. J., 35 (2) (2001), 140–146.

[5] D. E. Dobbs, Determining the angles between two lines, delta-K Math. J., 41 (2) (2004), 4–9.

[6] D. E. Dobbs and J. C. Peterson, Precalculus, Wm. C. Brown, Dubuque, 1993.

[7] M. Greenberg, Euclidean and non-Euclidean geometries: development and history, second edition, W. H. Freeman, San Francisco, 1974.

[8] R. S. Millman, The upper half plane model for hyperbolic geometry, Amer. Math. Monthly, 87 (1) (1980), 48–53.

[9] R. S. Millman and G. D. Parker, Geometry: a metric approach with models, second edition, Undergraduate Texts in Math., Springer-Verlag, New York, 1991.

[10] E. Moise, Elementary geometry from an advanced standpoint, second edition, Addison-Wesley, Reading (MA), 1974.

[11] S. Stahl, The Poincaré half-plane: a gateway to modern geometry, Jones and Bartlett, Boston, 1993.

[12] S. Stahl, A gateway to modern geometry: the Poincaré half-plane, second edition, Jones and Bartlett, Boston, 2008.