Moroccan Journal of Algebra and Geometry with Applications
Supported by Sidi Mohamed Ben Abdellah University, Fez,

Volume 1, Issue 1 (2022), pp 139-182

ISSN: 2820-7114

Title :

Amalgamation extension in commutative ring theory: a survey

Author(s):

Abdelhaq El Khalfi, Hwankoo Kim & Najib Mahdou

Moroccan Journal of Algebra
and Geometry with Applications
Vol. 1(1) (2022), 139–182
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco ISSN: 2820-7114

Amalgamation extension in commutative ring theory: a survey

Abdelhaq El Khalfi
Laboratory of Topology, Algebra, Geometry and Discrete Mathematics,
Faculty of Sciences Ain Chock, Hassan II University of Casablanca, Morocco
e-mail: *abdelhaqelkhalfi@gmail.com*

Hwankoo Kim
Division of Computer Engineering, Hoseo University, Asan 31499, Republic of Korea
e-mail: *hkkim@hoseo.edu*

Najib Mahdou
Laboratory of Modelling and Mathematical Structures. Department of Mathematics,
Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco
e-mail: *mahdou@hotmail.com*

Communicated by Mohammed Tamekkante
(Received 30 November 2021, Revised 30 January 2021 , Accepted 07 February 2021)

Abstract. Let A and B be two rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called the amalgamation of A with B along J with respect to f (introduced and studied by D'Anna, Finocchiaro, and Fontana). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana and denoted by $A \bowtie I$). In this paper, we survey known results concerning $A \bowtie^f J$.

Key Words: Amalgamated duplication, amalgamation of ring, pullback, trivial ring extension.
2010 MSC: Primary 16B50; Secondary 13B99, 18A05.

1 Introduction

Throughout this survey, all rings are commutative with identity and all modules are unitary. If R is a ring and E is an R-module, then we use the following notations:

- $\mathcal{I}(R)$, the set of ideals of R;
- $\sqrt{I}$, the radical of an ideal I of R, in the sense of [88, page 17];
- $\mathrm{Nil}(R) := \sqrt{0}$, the set (ideal) of all nilpotent elements of R;
- $\mathrm{Z}(R)$, the set of all zero-divisors of R;
- $\mathrm{Reg}(R) = R \setminus \mathrm{Z}(R)$;
- $\mathrm{Jac}(R)$, the Jacobson radical of R;
- $\mathrm{Spec}(R)$, the set of prime ideals of R;
- $\mathrm{Max}(R)$, the set of maximal ideals of R;

- $\mathrm{Min}(R)$, the set of minimal prime ideals of R;
- $\mathrm{Tot}(R) := R_{\mathrm{Reg}(R)}$, the total quotient ring of R;
- $\bar{R}$, the integral closure of R (in $\mathrm{Tot}(R)$);
- $(I : I) := \{x \in \mathrm{Tot}(R) \mid xI \subseteq I\}$ for any ideal I of R;
- $\mathrm{Ann}_R(E) := \mathrm{Ann}(E)$, the annihilator of E.
- $\mathrm{Idem}(R)$, the set of all idempotent elements of R.

If R is a (commutative integral) domain, we will usually denote its quotient field by $\mathrm{qf}(R)$ (rather than by $\mathrm{Tot}(R)$).

Let R be a ring and E an R-module. Then $R \ltimes E$, the *trivial (ring) extension of R by E*, is the ring whose additive structure is that of the external direct sum $R \oplus E$ and whose multiplication is defined by $(a,e)(b,f) := (ab, af + be)$ for all $a,b \in R$ and all $e,f \in E$. (This construction is also known by other terminology and other notation, such as the *idealization* $R(+)E$.) The basic properties of trivial ring extensions are summarized in the books [73] and [78]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (cf. [8], [24], [84]).

In [46], M. D'Anna considered a different type of construction, obtained involving a ring R and an ideal $I \subseteq R$, which is denoted by $R \bowtie I$ and defined as the following subring of $R \times R$.

$$R \bowtie I := \{(r, r+i) \mid r \in R, i \in I\}.$$

More generally this construction can be given starting with a ring R and an ideal E of an overring S of R (that is $S \subseteq \mathrm{Tot}(R)$). This extension has been studied, in the general case and from the different point of view of pullbacks, by D'Anna and Fontana in [48]. One main difference of this construction with respect to the idealization is that the ring $R \bowtie I$ is reduced whenever R is reduced.

Let A and B be rings, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In [49], D'Anna, Finocchiaro and Fontana introduced the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called the amalgamation of A with B along J with respect to f. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied in [56] and [57]). Moreover, other classical constructions (such as the $A + XB[X]$ construction, the $D + M$ construction and the Nagata's idealization) can be studied as particular cases of the amalgamation. On the other hand, the amalgamation $A \bowtie^f J$ is related to a construction proposed by D.D. Anderson in [15] and motivated by a classical construction due to Dorroh [56], concerning the embedding of a ring without identity in a ring with identity. The level of this generality is due to the fact that the amalgamation can be studied in the frame of pullback constructions. This point of view allows us to provide easily an ample description of the properties of $A \bowtie^f J$ in connection with the properties of A, J and f. The present survey is devoted to covering most results about the amalgamation of ring.

2 Definitions and basic results

This section is due to D'Anna, Finocchiaro, and Fontana and covers results from [49, 50, 51].

Definition 2.1. Let A and B be two rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\},$$

which is called the amalgamation of A with B along J with respect to f.

A particular case of the construction introduced above is the amalgamated duplication of a ring (introduced and studied by D'Anna and Fontana in [46, 47, 48]).

Example 2.2. [49, Example 2.4] Let A be a commutative ring with identity, and let E be an A-submodule of $\mathrm{Tot}(A)$ of A such that $E \cdot E \subseteq E$. In this case, E is an ideal in the subring $B := (E : E)(:= \{z \in \mathrm{Tot}(A) \mid zE \subseteq E\})$ of $\mathrm{Tot}(A)$. If $\iota : A \to B$ is the natural embedding, then $A \bowtie^\iota E$ coincides with $A \bowtie E$, the amalgamated duplication of A along E. A particular and relevant case is when $E := I$ is an ideal in A. In this case, we can take $B := A$, we can consider the identity map $\mathrm{id} : = \mathrm{id}_A : A \to A$ and we have that $A \bowtie I$, the amalgamated duplication of A along the ideal I, coincides with $A \bowtie^{\mathrm{id}_A} I$, that we will call also the simple amalgamation of A along I (instead of the amalgamation of A along I, with respect to id_A).

Example 2.3. [49, Example 2.5] Let $A \subset B$ be an extension of commutative rings and $X := \{X_1, X_2, \dots, X_n\}$ a finite set of indeterminates over B. In the polynomial ring $B[\mathrm{X}]$, we can consider the following subring

$$A + XB[X] := \{h \in B[X] \mid h(\mathbf{0}) \in A\},$$

where $\mathbf{0}$ is the n-tuple whose components are 0. This is a particular case of the general construction introduced above. In fact, if $\alpha' : A \hookrightarrow B[X]$ is the natural embedding and $J_1 := XB[X]$, then it is easy to check that $A \bowtie^{\alpha'} J_1$ is isomorphic to $A + XB[X]$.

Similarly, the subring $A + XB[[X]] := \{h \in B[[X]] \mid h(\mathbf{0}) \in A\}$ of the ring of power series $B[[X]]$ is isomorphic to $A \bowtie^{\alpha''} J_2$, where $\alpha'' : A \hookrightarrow B[[X]]$ is the natural embedding and $J_2 := XB[[X]]$.

Example 2.4. [49, Example 2.6] Let M be a maximal ideal of a ring (usually, an integral domain) T and let D be a subring of T such that $M \cap D = (0)$. The ring $D + M := \{x + m \mid x \in D, m \in M\}$ is canonically isomorphic to $D \bowtie^\iota M$, where $\iota : D \hookrightarrow T$ is the natural embedding.

More generally, let $\{M_\lambda \mid \lambda \in \Lambda\}$ be a subset of the set of the maximal ideals of T such that $M_\lambda \cap D = (0)$ for all $\lambda \in \Lambda$ and set $J := \bigcap_{\lambda \in \Lambda} M_\lambda$. The ring $D + J := \{x + j \mid x \in D, j \in J\}$ is canonically isomorphic to $D \bowtie^\iota J$. In particular, if $D := K$ is a field contained in T and $J := \mathrm{Jac}(T)$ is the Jacobson ideal of (the K-algebra) T, then $K + \mathrm{Jac}(T)$ is canonically isomorphic to $K \bowtie^\iota \mathrm{Jac}(T)$, where $\iota : K \hookrightarrow T$ is the natural embedding.

Example 2.5. [49, Example 2.7] Let A be a ring and P be a prime ideal of A. Let $k(P)$ be the residue field of the localization A_P and denote by ψ_P (or simply by ψ) the canonical surjective ring homomorphism $A_P \longrightarrow k(P)$. It is well-known that $k(P)$ is canonically isomorphic to the quotient field of A/P, and so we can identify A/P with its canonical image into $k(P)$. Then the subring $C(A, P) := \psi^{-1}(A/P)$ of A_P is called the CPI-extension of A with respect to P. It is immediately seen that, if we denote by λ_P (or, simply by λ) the localization homomorphism $A \longrightarrow A_P$, then $C(A, P)$ coincides with the ring $\lambda(A) + PA_P$. On the other hand, if $J := PA_P$, we can consider $A \bowtie^\lambda J$ and we have the canonical projection $A \bowtie^\lambda J \to \lambda(A) + PA_P$, defined by $(a, \lambda(a) + j) \mapsto \lambda(a) + j$ where $a \in A$ and $j \in PA_P$. It follows that $C(A, P)$ is canonically isomorphic to $\left(A \bowtie^\lambda PA_P\right)/(P \times \{0\})$.

More generally, let I be an ideal of A and let S_I be the set of the elements $s \in A$ such that $s + I$ is a regular element of A/I. Obviously S_I is a multiplicative subset of A and if $\overline{S_I}$ is its canonical projection onto A/I, then $\mathrm{Tot}(A/I) = \left(\overline{S_I}\right)^{-1}(A/I)$. Let $\varphi_I : S^{-1}A \longrightarrow \mathrm{Tot}(A/I)$ be the canonical surjective ring homomorphism defined by $\varphi_I\left(as^{-1}\right) := (a + I)(s + I)^{-1}$, for all $a \in A$ and $s \in S$. Then, the subring

$C(A,I) := \varphi_I^{-1}(A/I)$ of $S_I^{-1}A$ is called the CPI-extension of A with respect to I. If $\lambda_I : A \longrightarrow S_I^{-1}A$ is the localization homomorphism, then it is easy to see that $C(A,I)$ coincides with the ring $\lambda_I(A)+S_I^{-1}I$. It will follow by [49, Proposition 5.1(3)] that, if we consider the ideal $J := S_I^{-1}I$ of $S_I^{-1}A$, then $C(A,I)$ is canonically isomorphic to $(A \bowtie J)/\left(\lambda_I^{-1}(J)\times\{0\}\right)$.

Example 2.6. [49, Remark 2.8] The Nagata's idealization can be interpreted as a particular case of the general amalgamation construction. Let $B := A \ltimes E$ and $\iota : A \longrightarrow B$ be the canonical embedding. After identifying E with $J := 0 \ltimes E$, E becomes an ideal of B. It is now straightforward that $A \ltimes E$ coincides with the amalgamation $A \bowtie^{\iota} J$.

We recall that, if $\alpha : A \to C, \beta : B \to C$ are ring homomorphisms, the subring $D := \alpha \times_C \beta := \{(a,b) \in A\times B \mid \alpha(a) = \beta(b)\}$ of $A \times B$ is called the pullback (or fiber product) of α and β.

The fact that D is a pullback can also be described by saying that the triplet (D, p_A, p_B) is a solution of the universal problem of rendering commutative the diagram built on α and β

$$\begin{array}{ccc} D & \xrightarrow{p_A} & A \\ \downarrow{\scriptstyle p_B} & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow{\beta} & C \end{array}$$

where p_A (respectively, p_B) is the restriction to $\alpha\times_C\beta$ of the projection of $A\times B$ onto A (respectively, B).

Proposition 2.7. [49, Proposition 4.2] Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. If $\pi : B \to B/J$ is the canonical projection and $\check{f} := \pi\circ f$, then $A \bowtie^f J = \check{f} \times_{B/J} \pi$.

Remark 2.8. [49, Remark 4.3] Notice that we have many other ways to describe the ring $A \bowtie^f J$ as a pullback. In fact, if $C := A \times B/J$ and $u : A \to C, v : A\times B \to C$ are the canonical ring homomorphisms defined by $u(a) := (a, f(a)+J), v((a,b)) := (a, b+J)$, for every $(a,b) \in A\times B$, it is straightforward to show that $A \bowtie^f J$ is canonically isomorphic to $u \times_C v$. On the other hand, if $I := f^{-1}(J), \breve{u} : A/I \to A/I \times B/J$ and $\breve{v} : A\times B \to A/I\times B/J$ are the natural ring homomorphisms induced by u and v, respectively, then $A \bowtie^f J$ is also canonically isomorphic to the pullback of $\breve{u}$ and $\breve{v}$.

Proposition 2.9. [49, Proposition 5.1] Let $f : A \to B$ be a ring homomorphism, J an ideal of B and let $A \bowtie^f J$ be the amalgamation of A with B along J with respect to f.

(1) Let $\iota : A \to A \bowtie^f J$ be the natural ring homomorphism defined by $\iota(a) := (a, f(a))$, for all $a \in A$. Then ι is an embedding, making $A \bowtie^f J$ a ring extension of A.

(2) Let I be an ideal of A and set $I \bowtie^f J := \{(i, f(i)+j) \mid i \in I, j \in J\}$. Then $I \bowtie^f J$ is an ideal of $A \bowtie^f J$, the composition of canonical homomorphisms $A \to A \bowtie^f J \to A \bowtie^f J/I \bowtie^f J$ is a surjective ring homomorphism, and its kernel coincides with I. Hence we have the following canonical isomorphism:
$$\frac{A \bowtie^f J}{I \bowtie^f J} \cong \frac{A}{I}.$$

(3) Let $p_A : A \bowtie^f J \to A$ and $p_B : A \bowtie^f J \to B$ be the natural projections of $A \bowtie^f J \subseteq A\times B$ onto A and B, respectively. Then p_A is surjective and $\mathrm{Ker}\,(p_A) = \{0\}\times J$. Moreover, $p_B(A \bowtie^f J) = f(A)+J$ and $\mathrm{Ker}\,(p_B) = f^{-1}(J)\times\{0\}$. Hence the following canonical isomorphisms hold:
$$\frac{A \bowtie^f J}{(\{0\}\times J)} \cong A \quad \text{and} \quad \frac{A \bowtie^f J}{f^{-1}(J)\times\{0\}} \cong f(A)+J.$$

(4) Let $\gamma : A \bowtie^f J \to (f(A)+J)/J$ be the natural ring homomorphism defined by $(a, f(a)+j) \mapsto f(a)+J$. Then γ is surjective and $\mathrm{Ker}(\gamma) = f^{-1}(J)\times J$. Thus there exists a natural isomorphism

$$\frac{A \bowtie^f J}{f^{-1}(J)\times J} \cong \frac{f(A)+J}{J}.$$

In particular, when f is surjective, we have

$$\frac{A \bowtie^f J}{f^{-1}(J)\times J} \cong \frac{B}{J}.$$

The next result shows one more aspect of the essential role of the ring $f(A)+J$ for the construction $A \bowtie^f J$.

Proposition 2.10. [49, Proposition 5.2] With the notation of Proposition 2.9, assume J is a nonzero ideal of B. Then the following conditions are equivalent.

(1) $A \bowtie^f J$ is an integral domain.

(2) $f(A)+J$ is an integral domain and $f^{-1}(J) = 0$.

Remark 2.11. [49, Remark 5.3]

(1) Note that, if $A \bowtie^f J$ is an integral domain, then A is also an integral domain by Proposition 2.9(1).

(2) Let $B := A, f := \mathrm{id}_A$ and $J := I$ be an ideal of A. In this situation, the amalgamated duplication of A along I is never an integral domain, unless $I = \{0\}$ and A is an integral domain.

Now we characterize when the amalgamated algebra $A \bowtie^f J$ is a reduced ring.

Proposition 2.12. [49, Proposition 5.4] With the notation of Proposition 2.9, the following conditions are equivalent.

(1) $A \bowtie^f J$ is a reduced ring.

(2) A is a reduced ring and $\mathrm{Nil}(B)\cap J = \{0\}$.

In particular, if A and B are reduced, then $A \bowtie^f J$ is reduced. Conversely, if J is a radical ideal of B and $A \bowtie^f J$ is reduced, then A and B are reduced.

Recall that for an extension $A \subseteq B$ of rings, the conductor (ideal) of B into A is defined as $\{a \in A \mid aB \subseteq A\}$.

Proposition 2.13. With the notation of Proposition 2.9, $K := f^{-1}(J)\times J$ is the conductor of $A\times B$ into $A \bowtie^f J$.

The next result determines the integral closure of the ring $A \bowtie^f J$ in its total ring of quotients.

Proposition 2.14. [51, Proposition 3.1] Let $f : A \longrightarrow B$ be a ring homomorphism, J an ideal of B, and let $A \bowtie^f J$ be as in Proposition 2.9. Assume that $f^{-1}(J)$ and J are regular ideals of A and B respectively. Then $\mathrm{Tot}\big(A \bowtie^f J\big)$ is canonically isomorphic to $\mathrm{Tot}(A)\times\mathrm{Tot}(B)$.

Proposition 2.15. [51, Lemma 3.3] Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Then the ring $A\times(f(A)+J)$, a subring of $A\times B$, which contains $A \bowtie^f J$ is integral over $A \bowtie^f J$. More precisely, every element of $A\times(f(A)+J)$ has degree at most two over $A \bowtie^f J$.

Proposition 2.16. [51, Proposition 3.4] With the notation of Proposition 2.15, assume that J and $f^{-1}(J)$ are regular ideals of B and A respectively. Then $\overline{A \bowtie^f J}$ (i.e., the integral closure of $A \bowtie^f J$ in its total ring of quotients) coincides with $\overline{A} \times \overline{f(A)+J}$. In particular, if f is an integral homomorphism, then $\overline{A \bowtie^f J} = \overline{A} \times \overline{B}$.

The next proposition gives when the ring $A \bowtie^f J$ is integral over $\Gamma(f)(:= \{(a, f(a)) \mid a \in A\})$.

Proposition 2.17. [51, Lemma 3.6] Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Then the following conditions are equivalent:

(1) $f(A)+J$ is integral over $f(A)$.

(2) $A \bowtie^f J$ is integral over $\Gamma(f)$.

In particular, if f is an integral homomorphism, then $A \bowtie^f J$ is integral over $\Gamma(f)(\cong A)$.

The following result is due to Azimi, Sahandi and Shirmohammadi [16].

Proposition 2.18. [16, Lemma 2.1] We always have the inclusion $\mathrm{Z}(A \bowtie^f J) \subseteq \{(r, f(r)+j) \mid r \in \mathrm{Z}(A), j \in J\} \cup \{(r, f(r)+j) \mid r \in A, j \in J, j'(f(r)+j) = 0 \text{ for some } j' \in J \setminus \{0\}\}$, where equality holds if at least one of the following conditions hold:

(1) $f(Z(A)) \subseteq J$ and $f^{-1}(J) \neq 0$;

(2) $f(\mathrm{Z}(A))J = 0$ and $f^{-1}(J) \neq 0$;

(3) $J \subseteq f(A)$;

(4) J is a torsion A-module;

(5) $J \subseteq \mathrm{Nil}(B)$.

Proposition 2.19. [97, Proposition 2.2] Let R be a commutative ring and let I be an ideal of R. Then

$$\begin{aligned} \mathrm{Z}(R \bowtie I) = &\{(0,i) \mid i \in I\} \cup \{(i,-i) \mid i \in I\} \cup \{(x,i) \mid x \in \mathrm{Z}(R) \setminus \{0\}, i \in I\} \\ &\cup \{(x,i) \mid x \in R \setminus \mathrm{Z}(R), \text{ there exists } j \in I \setminus \{0\}, j(x+i) = 0\} \end{aligned}$$

Proposition 2.20. Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Then

$$\mathrm{Nil}(A \bowtie^f J) = \{(a, f(a)+j) \mid a \in \mathrm{Nil}(A), j \in \mathrm{Nil}(B) \cap J\}.$$

Proposition 2.21. [39, Lemma 2.5] Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B such that $J \cap \mathrm{Idem}(B) = 0$. Then $\mathrm{Idem}(A \bowtie^f J) = \{(e, f(e)) \mid e \in \mathrm{Idem}(A)\}$.

The next part covers some results from [50]. Let A be a ring and S be a subset of A. Then $V(S)$ denotes the closed subspace of $\mathrm{Spec}(A)$, consisting of all prime ideals of A containing S. The next result describes the structure of the prime spectrum of the ring $A \bowtie^f J$.

Proposition 2.22. [50, Corollary 2.5] With the notation of Proposition 2.9. Set $X := \mathrm{Spec}(A), Y := \mathrm{Spec}(B)$, and $W := \mathrm{Spec}(A \bowtie^f J), J_0 := \{0\} \times J$, and $J_1 := f^{-1}(J) \times \{0\}$. For all $P \in X$ and $Q \in Y$, set

$$\begin{aligned} &P'^f := P \bowtie^f J := \{(p, f(p)+j) \mid p \in P, j \in J\} \text{ and} \\ &\bar{Q}^f := \{(a, f(a)+j) \mid a \in A, j \in J, f(a)+j \in Q\}. \end{aligned}$$

Then the following statements hold:

(1) The map $P \mapsto P'^f$ establishes a closed embedding of X into W, and so its image, which coincides with $\mathrm{V}(J_0)$, is homeomorphic to X.

(2) The map $Q \mapsto \bar{Q}^f$ is a homeomorphism of $Y \setminus \mathrm{V}(J)$ onto $W \setminus \mathrm{V}(J_0)$.

(3) The prime ideals of $A \bowtie^f J$ are of the type P'^f or $\bar{Q}^f$ for P varying in X and Q in $Y \setminus \mathrm{V}(J)$.

(4) $W = \mathrm{V}(J_0) \cup \mathrm{V}(J_1)$ and the set $\mathrm{V}(J_0) \cap \mathrm{V}(J_1)$ is homeomorphic to $\mathrm{Spec}((f(A)+\mathrm{J})/J)$, via the continuous map associated to the natural ring homomorphism $\gamma : A \bowtie^f J \to (f(A)+J)/J, (a, f(a)+j) \mapsto f(a)+J$. In particular, we have that the closed subspace $\mathrm{V}(J_0) \cap \mathrm{V}(J_1)$ of W is homeomorphic to the closed subspace $\mathrm{V}(J)$ of Y when f is surjective.

Proposition 2.23. [50, Corollary 2.8] We preserve the notation of Proposition 2.22. Set

$$\mathcal{X} := \bigcup_{Q \in \mathrm{Spec}(B) \setminus \mathrm{V}(J)} f^{-1}(Q+J).$$

The following properties hold:

(1) The map defined by $Q \mapsto \bar{Q}^f$ establishes a homeomorphism of $\mathrm{Min}(B) \setminus \mathrm{V}(J)$ with $\mathrm{Min}(A \bowtie^f J) \setminus \mathrm{V}(\{0\} \times J)$.

(2) The map defined by $P \mapsto P'^f$ establishes a homeomorphism of $\mathrm{Min}(A) \setminus \mathcal{X}$ with $\mathrm{Min}(A \bowtie^f J) \cap \mathrm{V}(\{0\} \times J)$.

Therefore we have:

$$\mathrm{Min}(A \bowtie^f J) = \{P'^f \mid P \in \mathrm{Min}(A) \setminus \mathcal{X}\} \cup \{\bar{Q}^f \mid Q \in \mathrm{Min}(B) \setminus \mathrm{V}(J)\}.$$

Proposition 2.24. [50, Corollary 2.7] We preserve the notation of Proposition 2.22. Then:

(1) Let $P \in X = \mathrm{Spec}(A)$. Then P'^f is a maximal ideal of $A \bowtie^f J$ if and only if P is a maximal ideal of A.

(2) Let Q be a prime ideal of B not containing J. Then $\bar{Q}^f$ is a maximal ideal of $A \bowtie^f J$ if and only if Q is a maximal ideal of B. In particular, $\mathrm{Max}(A \bowtie^f J) = \left\{P'^f \mid P \in \mathrm{Max}(A)\right\} \cup \left\{\bar{Q}^f \mid Q \in \mathrm{Max}(B) \setminus V(J)\right\}$.

(3) $A \bowtie^f J$ is a local ring if and only if A is a local ring and $J \subseteq \mathrm{Jac}(B)$. In particular, if M is the unique maximal ideal of A, then $M'^f = M \bowtie^f J$ is the unique maximal ideal of $A \bowtie^f J$.

Proposition 2.25. [50, Corollary 2.5] With the notation of Proposition 2.9 and Proposition 2.22, the following statements hold:

(1) For any prime ideal $Q \in Y \setminus V(J)$, the ring $\left(A \bowtie^f J\right)_{\bar{Q}^f}$ is canonically isomorphic to B_Q.

(2) For any prime ideal $P \in X \setminus V\left(f^{-1}(J)\right)$, the localization $\left(A \bowtie^f J\right)_{P'^f}$ is canonically isomorphic to A_P.

(3) Let P be a prime ideal of A containing $f^{-1}(J)$. Consider the multiplicative subset $S := S_{(f,P,J)} := f(A \setminus P) + J$ of B, and set $B_S := S^{-1}B$ and $J_S := S^{-1}J$. If $f_P : A_P \longrightarrow B_S$ is the ring homomorphism induced by f, then the ring $(A \bowtie J)_{P'^f}$ is canonically isomorphic to $A_P \bowtie^{f_P} J_S$.

Proposition 2.26. [51, Proposition 2.7] Let $\alpha : A \to C, \beta : B \to C$ be ring homomorphisms and denote by p_A (resp., p_B) the restriction to the pullback $\alpha \times_C \beta$ of the projection of $A \times B$ onto A (resp., B). Assume β is surjective and let H' and H'' be prime ideals of D such that $H' \subsetneq H''$. Assume that $H' \in \mathrm{Spec}(D) \setminus V(\mathrm{Ker}(p_A)), H'' \in V(\mathrm{Ker}(p_A))$, and that H' and H'' are adjacent prime ideals. Then there exist two prime ideals Q' and Q'' of B, with $Q' \subsetneq Q''$, and moreover such that $Q' \notin V(\mathrm{Ker}(\beta)), p_B^{-1}(Q') = H'$, and $p_B^{-1}(Q'') = H''$.

Proposition 2.27. [50, Proposition 3.1] We preserve the notation of Proposition 2.9 and Proposition 2.22. The following properties hold:

(1) If I (respectively, H) is an ideal of A (respectively, $f(A)+J$) such that $f(I)J \subseteq H \subseteq J$, then $I \bowtie^f H := \{(i, f(i)+h) \mid i \in I, h \in H\}$ is an ideal of $A \bowtie^f J$.

(2) If I is an ideal of A, then the extension $I\left(A \bowtie^f J\right)$ of I to $A \bowtie^f J$ coincides with $I \bowtie^f (f(I)B)J := \{(i, f(i)+\beta) \mid i \in I, \beta \in (f(I)B)J\}$.

(3) If I is an ideal of A such that $f(I)B = B$, then $I\left(A \bowtie^f J\right) = I'^f = \{(i, f(i)+j) \mid i \in I, j \in J\} = I \bowtie^f J$.

We say that $A \bowtie^f J$ satisfies the property $(*)$ if every ideal has one of the following three forms:

(a) $I \times 0$, where $I \subseteq f^{-1}(J)$ is an ideal of A;

(b) $0 \times K$, where $K \subseteq J$ is an ideal of $f(A)+J$;

(c) $I \bowtie^f J$, where I is an ideal of A.

The following theorem provides necessary and sufficient conditions for the ring $A \bowtie^f J$ to satisfy the property $(*)$.

Theorem 2.28. [94, Theorem 2.1] Let A and B be two rings, J be a nonzero proper ideal of B and $f : A \to B$ be a ring homomorphism.

(1) If $A \bowtie^f J$ satisfies the property $(*)$, then the following conditions hold:

  (i) $f(A)$ is an integral domain;

  (ii) $f(A) \cap J = 0$;

  (iii) $0 \times J \subseteq ((a, f(a)+j))$ for all $a \in A \setminus \{0\}$ and all $j \in J$ such that $f(a)+j \neq 0$.

(2) If f is injective and $A \bowtie^f J$ satisfies the property $(*)$, then A is an integral domain;

(3) If f is not injective and A is a ring with zero-divisors with $A \bowtie^f J$ satisfying the property $(*)$, then $\mathrm{Ann}_{f(A)+J}(f(a)+j) \subseteq J$ for all $a \in A \setminus \{0\}$ and $j \in J$ with $f(a) \neq 0$. Moreover, if $f^{-1}(J) \not\subseteq Z(A)$, then $f(a)+j \in \mathrm{Reg}(f(A)+J)$ for all $a \in \mathrm{Reg}(A)$ and $j \in J$ with $f(a) \neq 0$, and $\mathrm{Ann}_{f(A)+J}(j) \subseteq f(\mathrm{Z}(A) \setminus f^{-1}(J)) + J$ for all $j \in J$;

(4) If f is not injective and A is an integral domain such that $A \bowtie^f J$ satisfies the property $(*)$, then the following conditions hold:

  (i) $f(A)+J$ is an integral domain;

  (ii) J is idempotent.

(5) If $0 \times J \subseteq ((a, f(a)+j))$ for all $a \in A \setminus \{0\}$ and all $j \in J$ such that $f(a)+j \neq 0$, then $A \bowtie^f J$ satisfies the property $(*)$.

3 Noetherianity and Krull dimension

This section covers results from [49] and [51, Section 4]. The following proposition provides an answer to the question of when $A \bowtie^f J$ is a Noetherian ring.

Proposition 3.1. [49, Proposition 5.6] With the notation of Proposition 2.9, the following statements are equivalent:

(1) $A \bowtie^f J$ is a Noetherian ring.

(2) A and $f(A)+J$ are Noetherians rings

Proposition 3.2. [49, Proposition 5.7] With the notation of Proposition 2.9, assume that at least one of the following conditions holds:

(1) J is a finitely generated A-module (with the structure naturally induced by f).

(2) J is a Noetherian A-module (with the structure naturally induced by f).

(3) $f(A)+J$ is Noetherian as an A-module (with the structure naturally induced by f).

(4) f is a finite homomorphism, i.e., B is a finitely generated A-module..

Then $A \bowtie^f J$ is Noetherian if and only if A is Noetherian. In particular, if A is a Noetherian ring and B is a Noetherian A-module (e.g., if f is a finite homomorphism [15, Proposition 6.5]), then $A \bowtie^f J$ is a Noetherian ring for any ideal J of B.

Proposition 3.3. [49, Proposition 5.8] We preserve the notation of Propositions 2.9 and 2.7. If B is a Noetherian ring and the ring homomorphism $\breve{f} : A \to B/J$ is finite, then $A \bowtie^f J$ is a Noetherian ring if and only if A is a Noetherian ring.

As a consequence of the previous proposition, we can characterize when rings of the form $A+XB[X]$ and $A+XB[[X]]$ are Noetherian. Note that S. Hizem and A. Benhissi [77] have already given a characterization of the Noetherianity of the power series rings of the form $A+XB[[X]]$. The next corollary provides a simple proof of Hizem and Benhissi's Theorem and shows that a similar characterization holds for the polynomial case (in several indeterminates). At the Fez Conference in June 2008, S. Hizem has announced to have proven a similar result in the polynomial ring case with a totally different approach.

Corollary 3.4. [49, Corollary 5.9] Let $A \subseteq B$ be a ring extension and $X := \{X_1, \ldots, X_n\}$ a finite set of indeterminates over B. Then the following conditions are equivalent.

(1) $A+XB[X]$ is a Noetherian ring.

(2) $A+XB[[X]]$ is a Noetherian ring.

(3) A is a Noetherian ring and $A \subseteq B$ is a finite ring extension.

The next result studies the Krull dimension of the ring $A \bowtie^f J$.

Proposition 3.5. [51, Proposition 4.1] Let $f : A \to B, J$, and $A \bowtie^f J$ be as in Proposition 2.9. Then $\dim\left(A \bowtie^f J\right) = \max\{\dim(A), \dim(f(A)+J)\}$. In particular, if f is surjective, then

$$\dim\left(A \bowtie^f J\right) = \max\{\dim(A), \dim(B)\} = \dim(A).$$

We already observed that the kind of results as in the previous proposition has a moderate interest, because the Krull dimension of $A \bowtie^f J$ is compared to the Krull dimension of $f(A)+J$, which is not easy to evaluate (moreover, if $f^{-1}(J) = \{0\}$, we have $A \bowtie^f J \cong f(A)+J$ (Proposition 2.9(3)). An easy case for evaluating $\dim\left(A \bowtie^f J\right)$ is the following:

Proposition 3.6. [51, Proposition 4.2] Let $f : A \to B, J$, and $A \bowtie^f J$ be as in Proposition 2.9. Let $f_\diamond : A \to B_\diamond := f(A)+J$ be the ring homomorphism induced from f. If we assume that $f_\diamond$ is integral (*e.g.*, f is integral), then $\dim\left(A \bowtie^f J\right) = \dim(A)$.

By Proposition 2.22, we know that $\operatorname{Spec}\left(A \bowtie^f J\right) = X \cup U$, where $X := \operatorname{Spec}(A)$ and $U := \operatorname{Spec}(B) \setminus V(J)$ (for the sake of simplicity, we identify X and U with their homeomorphic images in $\operatorname{Spec}\left(A \bowtie^f J\right)$). Furthermore, again from Proposition 2.22, we deduce that ideals of the form $\bar{Q}^f$ can be contained in ideals of the form P'^f, but not vice versa. Therefore, chains in $\operatorname{Spec}\left(A \bowtie^f J\right)$ are obtained by juxtaposition of two types of chains, one from U "on the bottom" and the other one from X "on the top" (where either one or the other may be empty or a single element). It follows immediately that both $\dim(X) = \dim(A)$ and $\dim(U)$ are lower bounds for $\dim\left(A \bowtie^f J\right)$ and $\dim(A)+\dim(U)+1$ is an upper bound for $\dim\left(A \bowtie^f J\right)$ (where, conventionally, we set $\dim(\emptyset) = -1$).

Remark 3.7. [51, Remark 4.3] Assume that $J \subseteq \operatorname{Jac}(B)$. By Proposition 2.22, we get that U does not contain maximal elements of $\operatorname{Spec}\left(A \bowtie^f J\right)$. Hence, in this case, $1+\dim(U) \leq \dim\left(A \times^f J\right)$.

Let us define the following subset of $U := \operatorname{Spec}(B) \setminus \mathrm{V}(J)$:

$$y_{(f,J)} := \left\{Q \in U \mid f^{-1}(Q+J) = \{0\}\right\}.$$

It is obvious that $y_{(f,J)}$ is stable under generizations, i.e., $Q \in y_{(f,J)}, Q' \in \operatorname{Spec}(B)$ and $Q' \subseteq Q$ imply $Q' \in y_{(f,J)}$. Hence $\dim\left(y_{(f,J)}\right) = \sup\left\{\operatorname{ht}_B(Q) \mid Q \in y_{(f,J)}\right\}$ and we will denote this integer by $\delta_{(f,J)}$.

Proposition 3.8. [51, Proposition 4.4] Let $f : A \to B, J$, and $A \bowtie^f J$ be as in Proposition 2.9; let $U = \operatorname{Spec}(B) \setminus V(J)$ and $\delta_{(f)} = \dim\left(y_{(f,J)}\right)$.

(1) Let $Q \in \operatorname{Spec}(B)$. Then $f^{-1}(Q+J) = \{0\}$ if and only if $\bar{Q}^f \left(= (A \times Q) \cap A \bowtie^f J\right)$ is contained in $J_0 (= \{0\} \times J)$.

(2) For every $Q \in y_{(f,J)}$, the corresponding prime $\bar{Q}^f$ of $A \times' J$ is contained in every prime of the form P'.

(3) $\max\left\{\dim(U), \dim(A)+1+\delta_{(f,J)}\right\} \leq \dim\left(A \bowtie^f J\right)$.

The next goal is to determine upper bounds for $\dim\left(A \bowtie^f J\right)$, possibly sharper than $\dim(A) + \dim(U)+1$.

Theorem 3.9. [51, Theorem 4.9] Let $f : A \to B, J$, and $A \bowtie^f J$ be as in Proposition 2.9. With the notation of Proposition 3.8, assume that $A \bowtie^f J$ has finite Krull dimension. Then :

$$\begin{aligned}\dim\left(A \bowtie^f J\right) &\leq \max\left\{\dim(A), \dim\left(A/f^{-1}(J)\right) + \min\{\dim(B), 1+\dim(U)\}\right\} \\ &\leq \min\left\{\dim(A)+\dim(U)+1, \max\left\{\dim(A), \dim\left(A/f^{-1}(J)\right)+\dim(B)\right\}\right\}.\end{aligned}$$

4 ϕ-prime ideals and ϕ-Krull dimension

This section is due to El Khalfaoui and Mahdou [59]. Anderson and Smith [10] defined a weakly prime ideal as a proper ideal P of R with the property that for $a,b \in R$, $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. Then the authors of [30] defined the notion of almost prime ideals, i.e., a proper ideal P with the property that if $a,b \in R$, $ab \in P \setminus P^2$, then either $a \in P$ or $b \in P$. Thus a weakly prime ideal is almost prime and any proper idempotent ideal is also almost prime. Moreover, an ideal P of R is almost prime if and only if P/P^2 is a weakly prime ideal of R/P^2. Anderson and Bataineh in [9] extended these concepts to ϕ-prime ideals. Let $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function. A proper ideal P of R is called ϕ-prime if for $x,y \in R$, $xy \in P \setminus \phi(P)$ implies $x \in P$ or $y \in P$. In fact, P is a ϕ-prime ideal of R if and only if $P/\phi(P)$ is a weakly prime ideal of $R/\phi(P)$. In 2017, J. Bagheri Harehdashti and H. Fazaeli Moghimi [23] defined the ϕ-radical of an ideal I as the intersection of all ϕ-prime ideals of R containing I and investigated when the set of all ϕ-prime ideals of R has a Zariski topology analogous to that of the prime spectrum. Since $P \setminus \phi(P) = P \setminus (P \cap \phi(P))$, there is no loss of generality in assuming that $\phi(P) \subseteq P$.

Now we study a generalization of prime ideals in the amalgamation of rings.

Proposition 4.1. [59, Proposition 2.7] Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let $\phi : \mathcal{I}(A) \to \mathcal{I}(A) \cup \{\emptyset\}$ and $\psi : \mathcal{I}(A \bowtie^f J) \to \mathcal{I}(A \bowtie^f J) \cup \{\emptyset\}$ be two functions such that

$$\psi(P \bowtie^f J) = \begin{cases} \phi(P) \bowtie^f K & \text{if } \phi(P) \neq \emptyset \\ \emptyset & \text{if } \phi(P) = \emptyset \end{cases}$$

where P is an ideal of A and K is a subideal of J. Then $P \bowtie^f J$ is ψ-prime if and only if P is ϕ-prime and for all $a,b \notin P$ such that $ab \in \phi(P)$ we have $(f(a)j + f(b)i + ij) \in K$ for all $i,j \in J$.

Remark 4.2. [59, Remark 2.8] Let $\psi : \mathcal{I}(A \bowtie^f J) \to \mathcal{I}(A \bowtie^f J) \cup \{\emptyset\}$ be a function defined by

$$\psi(P \bowtie^f J) = \begin{cases} \phi(P) \bowtie^f J & \text{if } \phi(P) \neq \emptyset \\ \emptyset & \text{if } \phi(P) = \emptyset \end{cases}$$

(1) Assume that $\phi(\{0\}) = \emptyset$. Then $0 \times J$ is ψ-prime if and only if $\{0\}$ is ϕ-prime [50, Corollary 2.5].
(2) Assume that $\phi(\{0\}) \neq \emptyset$. Then $0 \times J$ is always ψ-prime and $\{0\}$ is always ϕ-prime.

In the next proposition we will tackle the cases where ideals of the form $\overline{H}^f = \{(a, f(a)+j) \in A \bowtie^f J \mid (f(a)+j) \in H\}$, where H is an ideal of B, are ψ-prime.

Proposition 4.3. [59, Proposition 2.9] Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let $\phi : \mathcal{I}(A) \to \mathcal{I}(A) \cup \{\emptyset\}$, $\varphi : \mathcal{I}(f(A)+J) \to \mathcal{I}(f(A)+J) \cup \{\emptyset\}$ and $\psi : \mathcal{I}(A \bowtie^f J) \to \mathcal{I}(A \bowtie^f J) \cup \{\emptyset\}$ be three functions such that

$$\varphi(H) = \emptyset \Leftrightarrow \phi(I_H) = \emptyset$$

$$\psi(\overline{H}^f) = \begin{cases} \{(a, f(a)+j) \mid a \in \phi(I_H) \text{ and } f(a)+j \in \varphi(H)\} & \text{if } \varphi(H) \neq \emptyset \\ \emptyset & \text{if } \varphi(H) = \emptyset \end{cases}$$

for every ideal H of $f(A)+J$, where $I_H = \{a \in A \mid (a, f(a)+j) \in \overline{H}^f \text{ for some } j \in J\}$. Then $\overline{H}^f$ is a ψ-prime ideal if and only if H is a φ-prime ideal of $f(A)+J$ and for all $(f(a)+i), (f(b)+j) \notin H$ such that $(f(a)+i)(f(b)+j) \in \varphi(H)$ we have $ab \in \phi(I_H)$.

The next theorem gives a general form of ψ-prime ideals of the amalgamation of rings in some particular cases. Recall that I is a radical ideal of a ring R if $x^n \in I$ for any $x \in R$ and for any positive integer n implies that $x \in I$.

Theorem 4.4. [59, Theorem 2.10] Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let $\phi : \mathcal{I}(A) \to \mathcal{I}(A) \cup \{\emptyset\}$ and $\psi : \mathcal{I}(A \bowtie^f J) \to \mathcal{I}(A \bowtie^f J) \cup \{\emptyset\}$ be two functions such that

$$\psi(H) = \begin{cases} (\phi(I_H) \bowtie^f K) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A \mid (a, f(a) + i) \in H \text{ for some } i \in J\}$ and K is a subideal of J. Let P be a ψ-prime ideal of $A \bowtie^f J$ and $J \subseteq \text{Nil}(B)$.
(I) If $\phi(I_P) \neq \emptyset$ is a radical ideal, then

(1) either $P \subseteq \phi(I_P) \times J$ or $P = I_P \bowtie^f J$;

(2) if we suppose furthermore that $\psi(P)$ is a radical ideal of $A \bowtie^f J$, then either $P = \phi(I_P) \times J$ (necessarily in that case $f(\phi(I_P)) \subseteq J$) or $P = I_P \bowtie^f J$.

(II) Assume that $\phi(P) = \emptyset$. Then $P = H \bowtie^f J$, where H is a ϕ-prime ideal of A.

Corollary 4.5. [59, Corollary 2.12] Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let P be a weakly prime ideal of $A \bowtie^f J$ and $I_P = \{a \in A \mid (a, f(a) + i) \in P \text{ for some } i \in J\}$. If $\{0\}$ is a radical ideal of A and $J \subseteq \text{Nil}(B)$, then

(1) either $P \subseteq 0 \times J$ or $P = I_P \bowtie^f J$;

(2) if we suppose furthermore that $\{(0,0)\}$ is a radical ideal of $A \bowtie^f J$, then either $P = 0 \times J$ or $P = I_P \bowtie^f J$.

For homomorphic images, we give the following result.

Proposition 4.6. [59, Proposition 2.13] Let R be a ring, I an ideal of R and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$, $\psi : \mathcal{I}(R/I) \to \mathcal{I}(R/I) \cup \{\emptyset\}$ two functions such that

$$\psi(J/I) = \begin{cases} \phi(J)/I & \text{if} \quad I \subseteq \phi(J) \\ \{\overline{0}\} & \text{if} \quad \phi(J) \neq \emptyset \text{ and } I \not\subseteq \phi(J) \\ \emptyset & \text{if} \quad \phi(J) = \emptyset \end{cases}$$

Let P/I be an ideal of R/I.

(1) Assume that I is a weakly prime ideal of R. Then P is a ϕ-prime ideal of R provided P/I is a ψ-prime ideal of R/I.

(2) Assume that $I \subseteq \phi(P)$. Then P/I is a ψ-prime ideal of R/I if and only if P is a ϕ-prime ideal of R.

Now we will define and study a new notion considered as a generalization of Krull dimension in commutative rings.

Definition 4.7. [59, Definition 3.1] Let A be a ring and $\phi : \mathcal{I}(A) \to \mathcal{I}(A) \cup \{\emptyset\}$ a function. The ϕ-Krull dimension of A (denoted by ϕ-$\dim(A)$) is the supremum of the lengths of all chains of distinct ϕ-prime ideals of A.
Clearly we have

$$\phi\text{-}\dim(A) \geq \dim(A).$$

We start with the direct product of rings.

Theorem 4.8. [59, Theorem 3.2] Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings and let $\{\phi_i : \mathcal{I}(R_i) \to \mathcal{I}(R_i) \cup \{\emptyset\}\}_{i=1,\dots,n}$ be a family of functions such that for all ideals I_i of R_i, $\phi_i(I_i) \neq I_i$ $(i = 1,\dots,n)$. We consider $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function such that $\phi(\prod_{i=1}^{n} I_i) = \prod_{i=1}^{n} \phi_i(I_i)$ where I_i is an ideal of R_i for each $i = 1,\dots,n$. Then

$$\phi\text{-}\dim(R) = \sup_{i=1,\dots,n} \{\phi_i\text{-}\dim(R_i)\}.$$

For homomorphic images, we give the following result.

Proposition 4.9. [59, Proposition 3.3] Let R be a ring, I an ideal of R and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$, $\psi : \mathcal{I}(R/I) \to \mathcal{I}(R/I) \cup \{\emptyset\}$ two functions such that

$$\psi(J/I) = \begin{cases} \phi(J)/I & \text{if} \quad \phi(J) \neq \emptyset \text{ and } I \subseteq \phi(J) \\ \{\overline{0}\} & \text{if} \quad \phi(J) \neq \emptyset \text{ and } I \not\subseteq \phi(J) \\ \emptyset & \text{if} \quad \phi(J) = \emptyset \end{cases}$$

Assume that one of the two following conditions is satisfied :

(1) I is a weakly prime ideal of R.

(2) for all ψ-prime ideals P/I of R/I we have $I \subseteq \phi(P)$.

Then

$$\psi\text{-}\dim(R/I) \leq \phi\text{-}\dim(R).$$

For the amalgamation we have the following result.

Theorem 4.10. [59, Theorem 3.7] Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let $\phi : \mathcal{I}(A) \to \mathcal{I}(A) \cup \{\emptyset\}$ and $\psi : \mathcal{I}(A \bowtie^f J) \to \mathcal{I}(A \bowtie^f J) \cup \{\emptyset\}$ be two functions such that

$$\psi(H) = \begin{cases} (\phi(I_H) \bowtie^f J) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A \mid (a, f(a) + i) \in H \text{ for some } i \in J\}$. Then

(1) $\psi\text{-}\dim(A \bowtie^f J) \geq \phi\text{-}\dim(A)$.

(2) Suppose that for every ψ-prime ideal P of $A \bowtie^f J$, $\phi(I_p) \neq \emptyset$ and $\psi(P) \neq \emptyset$ are radical ideals and $J \subseteq \mathrm{Nil}(B)$. Then

$$\psi\text{-}\dim(A \bowtie^f J) = \phi\text{-}\dim(A).$$

With the condition that for each ψ-prime ideal P of $A \bowtie^f J$, $\phi(I_P) = \emptyset$ and $J \subseteq \mathrm{Nil}(B)$, we get the following result.

Corollary 4.11. [59, Corollary 3.8] Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B such that $J \subseteq \mathrm{Nil}(B)$. Then,

$$\dim(A \bowtie^f J) = \dim(A).$$

5 Local dimension

This section is due to El Khalfaoui, Mahdou and Yassemi [61]. Local dimension is introduced in [71] and it is an ordinal valued invariant that is in some sense a measure of how far a ring is from being local.

The purpose of this section is to study the local dimension of ring extensions. We give a link between the local dimension of a ring R and its extensions such as homomorphic image and the amalgamation of rings.

We define, by transfinite induction, classes ζ_α of rings for all ordinals α. Let ζ_1 be the class of local rings. Consider an ordinal $\alpha > 1$; if ζ_β has been defined for all ordinals $\beta < \alpha$, let ζ_α be the class of those rings R such that $R/I \in \bigcup_{\beta<\alpha} \zeta_\beta$ for every nonzero ideal I of R. If a ring R belongs to some ζ_α, then the least such α is called the local dimension of R and it is denoted by $\mathrm{ldim}(R)$. In this case we say R has local dimension. If a ring R does not belong to any ζ_α, then we say that R has no local dimension. It is known that R has local dimension if and only if R is Noetherian or local, see [71, Corollary 4.11]. In addition, R has finite local dimension if and only if R is Artinian or local [71, Theorem 4.12]. Using these two results, it is easy to see that R has local dimension equal to ω, where ω is the first infinite ordinal number, if and only if it is Noetherian non-local, non-Artinian and every proper homomorphic image of R is Artinian or local. We start with a result that makes a link between the local dimension of a ring and its proper homomorphic images.

Proposition 5.1. [61, Proposition 2.1] Let R be a ring. Then :

$$\mathrm{ldim}(R) = \sup_{I \neq \{0\}}\{\mathrm{ldim}(R/I)\} + 1,$$

where I is a nonzero ideal of R.

For localization, we have the following result.

Proposition 5.2. [61, Proposition 3.1] Let R be a ring and let S be a multiplicative subset of R.

(1) If $\mathrm{ldim}(R)$ exists, then so does $\mathrm{ldim}(S^{-1}R)$.

(2) If $\mathrm{ldim}(R)$ is finite, then so is $\mathrm{ldim}(S^{-1}R)$.

(3) If $\mathrm{ldim}(S^{-1}R)$ is infinite, then $\mathrm{ldim}(R) = \omega$ implies that $\mathrm{ldim}(S^{-1}R) = \omega$.

Remark 5.3. [61, Remark 3.2] The converses of statements (1) and (2) of Proposition 5.2 are not true.

(1) Let R be a non-Noetherian non-local ring and let P be a prime ideal of R. Then the localization R_P at P is a local ring. So $\mathrm{ldim}(R_P) = 1$ but R does not have local dimension.

(2) We consider a Noetherian non-Artinian and non-local ring R. Then R has infinite local dimension. Let P be a prime ideal of R. Then the localization R_P at P is a local ring. Therefore $\mathrm{ldim}(R_P) = 1$.

In the following we study the local dimension of the amalgamation of rings. First we bring a result which will be used in the main result:

Theorem 5.4. [61, Theorem 3.5] Let A and B be rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism. Then :

(1) $A \bowtie^f J$ has local dimension if and only if either A and $f(A)+J$ are Noetherian or A is local and $J \subseteq \mathrm{Jac}(B)$.

(2) If $\mathrm{ldim}(A \bowtie^f J)$ exists, then the following statements hold.

  (a) $\mathrm{ldim}(A)$ and $\mathrm{ldim}(f(A)+J)$ exist and we have:
  $\mathrm{ldim}(A \bowtie^f J) \geq \mathrm{ldim}(A)$ and $\mathrm{ldim}(A \bowtie^f J) \geq \mathrm{ldim}(f(A)+J)$.

  (b) If $\mathrm{ldim}(A \bowtie^f J) = \omega$, then A and $f(A)+J$ have finite local dimension.

  (c) Let $\mathrm{ldim}(A \bowtie^f J) = \omega$. Then $f(A)+J$ is local non-Artinian if A is Artinian. Also, if $f(A)+J$ is Artinian, then A is local non-Artinian. In addition, if A is local, then $J \not\subseteq \mathrm{Jac}(B)$.

In the following proposition, under some conditions, we will see when the amalgamation has local dimension equal to ω.

Recall that an ideal I of $A \bowtie^f J$ is called homogeneous if $I = K \bowtie^f J$ for some ideal K of A. If H is an ideal of $A \bowtie^f J$ such that $0 \times J \subseteq H$, then H is homogeneous [89].

Proposition 5.5. [61, Proposition 3.6] Suppose that all ideals of $A \bowtie^f J$ are homogeneous. Then:

(1) If $J \subseteq \mathrm{Jac}(B)$,

$$\mathrm{ldim}(A \bowtie^f J) = \omega \text{ if and only if } A \text{ is Artinian non-local and } f(A)+J \text{ is local non-Artinian.}$$

(2) If $\dim(A) > 0$,

$$\mathrm{ldim}(A \bowtie^f J) = \omega \text{ if and only if } A \text{ is local and } J \not\subseteq \mathrm{Jac}(B).$$

As a corollary, we give the result concerning the amalgamated duplication of a ring along an ideal.

Corollary 5.6. [61, Corollary 3.7] Let A be a ring and I an ideal of A. Then

(1) $A \bowtie I$ has local dimension if and only if A is Noetherian or local.

(2) $A \bowtie I$ has finite local dimension if and only if A has finite local dimension.

(3) If the local dimension of $A \bowtie I$ exists, then it is never equal to ω.

6 Valuation-like properties

This section covers results from [62, 87]. In [76], Hedstrom and Houston introduced a class of integral domains which is closely related to the class of valuation domains. An integral domain A with quotient field K is called a pseudo-valuation domain (PVD) when each prime ideal P of A is a strongly prime ideal, in the sense that for every $x, y \in K$, if $xy \in P$, then $x \in P$ or $y \in P$. An interesting survey article on pseudo-valuation domains is [19]. In [22], the study of pseudo-valuation domains was generalized to arbitrary rings (with zero-divisors). Recall from [22] that a prime ideal P of a ring A is said to be strongly prime if aP and bA are comparable for all $a, b \in A$. A ring A is called a pseudo-valuation ring (PV-ring) if each prime ideal of A is strongly prime. A ring A is a PV-ring if and only if it is local with its maximal ideal strongly prime [22, Lemma 3]. Also, an integral domain is a PV-ring if and only if it is a PVD by [12, Proposition 3.1], [13, Proposition 4.2] and [17, Proposition 3]. In [11], D. D. Anderson and M. Zafrullah introduced and studied the notion of almost valuation domains. An integral domain A is called an almost valuation domain (AVD) if for every nonzero $x \in K$, there exists an integer $n \geq 1$ such that either $x^n \in A$ or $x^{-n} \in A$. In [82] and [93], a generalization of the almost valuation domains to arbitrary commutative rings (with zero-divisors) is considered as follows: A is called an almost valuation ring (AV-ring) if for any two elements a and b in A, there exists a positive integer $n \geq 1$ such that a^n divides b^n or b^n divides a^n. An AV-ring is necessarily local [82, Proposition 2.2]. In [20], Badawi introduced a new class of integral domains as follows. A prime ideal P of an integral domain A is called a pseudo-strongly prime ideal if whenever $x, y \in K$ and $xyP \subseteq P$, there is an integer $n \geq 1$ such that either $x^n \in A$ or $y^n P \subseteq P$. If each prime ideal of A is a pseudo-strongly prime ideal, then A is called a pseudo-almost valuation domain (PAVD). Also, an integral domain A is a PAVD if and only if for every nonzero $x \in K$, there is a positive integer $n \geq 1$ such that either $x^n \in A$ or $ax^{-n} \in A$ for every nonunit $a \in A$. In [81], a generalization of the pseudo-almost valuation domains to arbitrary commutative rings (with zero-divisors) is considered as follows: A prime ideal P of a ring A is said to be a pseudo-strongly prime ideal if for every $a, b \in A$, there is an integer $n \geq 1$ such that either $a^n A \subseteq b^n A$ or $b^n P \subseteq a^n P$. A ring A is called a pseudo-almost valuation ring (PAV-ring) if each maximal ideal of A is pseudo-strongly prime. A PAV-ring is necessarily local. Also, an integral domain A is a PAV-ring if and only if A is a PAVD [81, Proposition 2.7].

The next two theorems study the transfer of the valuation property to amalgamation rings.

Theorem 6.1. [87, Theorem 2.1] Let A and B be a pair of rings, J an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. Then:

(1) If f is not injective, then $A \bowtie^f J$ is a valuation ring if and only if A is a valuation ring and $J = (0)$.

(2) If f is injective, then $A \bowtie^f J$ is a valuation ring if and only if $f(A) + J$ is a valuation ring and $f(A) \cap J = (0)$.

Remark 6.2. [87, Remark 2.2] Let $f : A \longrightarrow B$ be an injective ring homomorphism and let J be an ideal of B. If $A \bowtie^f J$ is a valuation ring and $J \neq (0)$, then A is a valuation domain. Indeed, suppose that the statement is false and choose an element $(a, b) \in A^2$ such that $a \neq 0, b \neq 0$ and $ab = 0$. For each $x \in J$ there is $(c, f(c) + y) \in A \bowtie^f J$ such that $(b, f(b))(c, f(c) + y) = (0, x)$. Then $bc = 0$ and $f(b)y = x$, therefore $f(a)x = 0$ and $f(a) \in (0 : J)$. For each $x \in J$, we can write $(a, f(a))(d, f(d) + z) = (0, x)$, where $(d, f(d) + z)$ is an element of $A \bowtie^f J$. Hence $x = f(a)z = 0$, which contradicts $J \neq (0)$.

Theorem 6.3. [62, Theorem 2.8] Let A and B be a pair of rings, $f : A \longrightarrow B$ be a ring homomorphism, and J be a nonzero proper ideal of B. Then $A \bowtie^f J$ is a valuation ring if and only if the following conditions holds:

(a) $f^{-1}(J) = 0$ and A is a valuation domain.

(b) $J = (f(a) + j)J$ for all $0 \neq a \in A$, $j \in J$.

(c) For each two element $i, j \in J$, $j(f(A) + J)$ and $i(f(A) + J)$ are comparable.

Remark 6.4. [62, Remark 2.9] Suppose that $A \bowtie^f J$ is a valuation ring when J is a nonzero finitely generated A-module (by the modulation $a.j := f(a)j$ for all $a \in A$ and $j \in J$). Then A must be a field. Indeed, since $A \bowtie^f J$ is a valuation ring, we get by Theorem 6.3 that A is a valuation domain, say with maximal ideal M, and for all $0 \neq a \in M$ we have $J = f(a)J$. If $M \neq 0$, then $J = MJ$. As J is a finitely generated A-module, by Nakayama lemma, we conclude that $J = 0$, which is a contradiction.

The following theorem studies the transfer of the PV-property to amalgamation of rings.

Theorem 6.5. [62, Theorem 2.11] Let A and B be a pair of rings, $f : A \longrightarrow B$ be a ring homomorphism, and J be a nonzero ideal of B such that $f^{-1}(J) \neq 0$. Then $A \bowtie^f J$ is a PV-ring if and only if A is a local ring with maximal ideal M such that $J^2 = 0$, $f(M)J = 0$, and $M^2 = 0$.

In light of Theorem 6.5, we have the following corollary.

Corollary 6.6. [62, Corollary 2.12] Let A be a ring and I be a nonzero proper ideal of A. Then $A \bowtie I$ is a PV-ring if and only if A is a local ring with maximal ideal M such that $M^2 = 0$.

If A is an integral domain and J is a nonzero proper ideal of B, then we have the following result:

Proposition 6.7. [62, Proposition 2.13] Let A and B be a pair of rings, $f : A \longrightarrow B$ be a ring homomorphism, and J be a nonzero proper ideal of B. Assume that A is an integral domain. Then $A \bowtie^f J$ is a PV-ring if and only if the following conditions holds:

(a) $f^{-1}(J) = 0$, $J \subseteq \mathrm{Jac}(B)$ and A is a PVD with maximal ideal M.

(b) $J = (f(a)+j)J$ for all $0 \neq a \in A$, $j \in J$.

(c) For each two element $i, j \in J$, $j(f(A)+J)$ and $i(f(M)+J)$ are comparable.

Recall that a commutative ring R is called a total quotient ring if $\mathrm{Tot}(R) = R$, equivalently every element of R is either a zero-divisor or a unit.

Remark 6.8. [62, Remark 2.14] If $J \neq 0$ and $A \bowtie^f J$ is a PV-ring, then either A is a total quotient ring or A is an integral domain. Indeed, assume that $A \bowtie^f J$ is a PV-ring. So A is a PV-ring with maximal ideal M. If f is not injective, then $f^{-1}(J) \neq 0$. Hence Theorem 6.5 implies that $M^2 = 0$, and so A is a total quotient ring. Now assume that f is injective such that A is not a total quotient ring. Let $a, b(\neq 0) \in A$ such that $ab = 0$ and $0 \neq j \in J$. Since $A \bowtie^f J$ is a PV-ring, we get that $A \bowtie^f J$ is a local ring with maximal ideal $M \bowtie^f J$. Moreover, we have $(0, j) \in (a, f(a))M \bowtie^f J$ or $(a, f(a))M \bowtie^f J \subseteq (0, j)A \bowtie^f J$. If $(a, f(a))M \bowtie^f J \subseteq (0, j)A \bowtie^f J$, then $(a, f(a))(d, f(d)) = (0, j)(t, f(t)+k)$ for some regular element $d \in M$ and $(t, f(t)+k) \in A \bowtie^f J$. Hence $ad = 0$ implies that $a = 0$, which gives a contradiction. Therefore $(0, j) = (a, f(a))(t, f(t)+k)$ for some $(t, f(t)+k) \in M \bowtie^f J$. Hence $at = 0$ and $j = f(a)(f(t)+k)$. Then $f(b)j = 0$ for each $j \in J$. By similar reasoning as above, we have $(0, j) \in (b, f(b))M \bowtie^f J$. Hence $j = f(b)k$ for some $k \in J$, and thus $j = 0$, which is a contradiction. Therefore A is an integral domain.

Proposition 6.9. [62, Theorem 2.17(1)] Let A and B be a pair of rings and $f : A \longrightarrow B$ be a ring homomorphism. Let J be a nonzero proper ideal of B having no nontrivial nilpotent element. Then $A \bowtie^f J$ is a PV-ring if and only if f is injective, $f(A) \cap J = 0$, and $f(A)+J$ is a PV-ring.

Now we will study the transfer of the AV-property and the PAV-property to the amalgamation of rings.

Theorem 6.10. [62, Theorem 2.15] Let A and B be a pair of rings and $f : A \longrightarrow B$ be a ring homomorphism. Suppose that A is a local ring with maximal ideal M, and J is a proper ideal of B such that $f(M)J = 0$ and $J \subseteq \mathrm{Nil}(B)$. Set $R := A \bowtie^f J$. Then:

(1) R is an AV-ring if and only if A is an AV-ring.

(2) R is a PAV-ring if and only if A is a PAV-ring.

The next corollary is an immediate application of Theorem 6.10.

Corollary 6.11. [62, Corollary 2.16] Let A be a local ring with maximal ideal M and I be a proper ideal of A such that $MI = 0$. Then:

(1) $A \bowtie I$ is an AV-ring if and only if A is an AV-ring.

(2) $A \bowtie I$ is a PAV-ring if and only if A is a PAV-ring.

Theorem 6.12. ([62, Theorem 2.17(2)] and [87, Theorem 2.4]) Let A and B be a pair of rings and $f : A \to B$ be a ring homomorphism. Assume that J is a nonzero proper ideal of B having no nontrivial nilpotent elements and A is reduced. Then $A \bowtie^f J$ is an AV-ring (resp., a PAV-ring) if and only if f is injective, $f(A) \cap J = 0$ and $f(A)+J$ is an AV-ring (resp., a PAV-ring).

The next corollary is an immediate application of Theorem 6.12.

Corollary 6.13. [62, Corollary 2.18] Let A be an integral domain and I a proper ideal of A. Then $A \bowtie I$ is an AV-ring (resp., a PAV-ring) if and only if A is an AV-ring (resp., a PAV-ring) and $I = 0$.

The next proposition gives a partial result about when an amalgamation is an AV-ring in case A is not reduced and J has nontrivial nilpotents. First we recall that a ring A is Gaussian if for every polynomials $f, g \in A[X]$, one has the content ideal equation $c(fg) = c(f)c(g)$ (see [108]).

Proposition 6.14. [87, Proposition 2.8] Let (A, B) be a pair of rings where A is a local ring with maximal ideal M, $f : A \to B$ be a ring homomorphism and J be a nonzero proper ideal of B contained in the Jacobson radical $\mathrm{Jac}(B)$ of B. If A is Gaussian, $J^2 = (0)$ and $f(a)J = f(a)^2 J$ for all $a \in M$, then $A \bowtie^f J$ is an AV-ring.

The results of the transfer enrich the literature with new examples as shown below.

Example 6.15. [62, Example 2.19] Let A be a valuation domain with quotient field K, and $B := A \ltimes K$ be the trivial ring extension of A by K. Consider:

$$f : A \hookrightarrow B$$

$$a \mapsto f(a) = (a, 0)$$

to be a ring homomorphism and $J := 0 \ltimes K$ a proper ideal of B. Therefore by Theorem 6.3, $A \bowtie^f J$ is a valuation ring.

Example 6.16. [62, Example 2.20] Let (A, M) be a valuation domain which is not a field, $B := \frac{A}{M^2}$, and $f : A \longrightarrow B$ be a homomorphism of ring such that $f(a) = \bar{a}$. Set $J := \frac{M}{M^2}$. Hence $f(M)J = 0$ and $J \subseteq \mathrm{Nil}(B)$. Therefore by Theorem 6.10, $A \bowtie^f J$ is an AV-ring (resp. a PAV-ring) which is not a PV-ring by Theorem 6.5.

Example 6.17. [62, Example 2.21] Let K be a field and E be a K-vector space. Set $A := K \ltimes E$ and $I := 0 \ltimes E$. Then:

(1) $A \bowtie I$ is an AV-ring (resp., a PAV-ring) by Theorem 6.10.

(2) $A \bowtie I$ is a PV-ring by Theorem 6.5 which is not a valuation ring by [87, Theorem 2.1].

The following is an example of a PAV-ring which is not an AV-ring.

Example 6.18. [62, Example 2.22] Let F be a finite field and X an indeterminate over F. Put $H := F(X)$, the quotient field of $F[X]$, and let Y be an analytic indeterminate over H. Set $A := F + HY^2 + Y^4H[[Y]]$. Then A is a local domain with maximal ideal $M = HY^2 + Y^4H[[Y]]$. Moreover, by [20, Example 3.5] A is a PAVD which is not an AVD. Now, set $E := \frac{A}{M}$, $B := A \ltimes E$ and $J := 0 \ltimes E$. Consider:

$$f : A \hookrightarrow B$$

$$a \mapsto f(a) = (a, 0)$$

to be a ring homomorphism. It is clear that $f(M)J = 0$ and $J \subseteq \mathrm{Nil}(B)$. Finally Theorem 6.10 gives that $A \bowtie^f J$ is a PAV-ring which is not an AV-ring.

The next example illustrates the failure of Theorem 6.12, in general, beyond the context where A is reduced and J has no nonzero nilpotent elements.

Example 6.19. [87, Example 2.7] Let $(A,M) = (\mathbb{Z}/4\mathbb{Z}, 2\mathbb{Z}/4\mathbb{Z})$. Then A is a local ring and $M^2 = (0)$. Set $R := \mathbb{Z}/4\mathbb{Z} \bowtie^f 2\mathbb{Z}/4\mathbb{Z}$ where $f = id_{\mathbb{Z}/4\mathbb{Z}}$. Then the following statements hold:

(1) $A = \mathbb{Z}/4\mathbb{Z}$ is not reduced.

(2) $J = 2\mathbb{Z}/4\mathbb{Z}$ has nonzero nilpotent elements ($\mathrm{Nil}(B) \cap J \neq (0)$).

(3) R is an AV-ring.

(4) $f(A) \cap J \neq 0$.

In view of Proposition 6.14, the following example shows that the assumptions A is reduced and J has no nonzero nilpotent element are not necessary conditions in the statement (2) of Theorem 6.12.

Example 6.20. [87, Example 2.9] Let (A,M) be a local Gaussian ring such that $M^2 \neq (0)$ and I a nonzero ideal of A such that $I^2 = (0)$ (for instance, $A := \mathbb{Q}[[X]] \ltimes \mathbb{Q}(\sqrt{2})$. Then A is a local Gaussian ring with maximal ideal $M = X\mathbb{Q}[[X]] \ltimes \mathbb{Q}(\sqrt{2})$ ([24, Theorem 3.1(2)]), and let $I = (0) \ltimes \mathbb{Q}(\sqrt{2})$). Let E be a nonzero $\frac{A}{M}$-vector space and $B := A \ltimes E$ be the trivial ring extension of A by E. Consider the injective ring homomorphism f given by:

$$\begin{array}{cccc} f: & A & \hookrightarrow & B \\ & a & \mapsto & f(a) = (a,0) \end{array}$$

and let $J := I \ltimes E$. Clearly $J \subseteq \mathrm{Jac}(B)$, $f(a)J = f(a)^2 J$ for all $a \in M$, and:

(1) A is not reduced.

(2) $J^2 = (0)$.

(3) $f(A) \cap J = (A \ltimes (0)) \cap (I \ltimes E) = I \ltimes (0) \neq (0)$.

(4) $A \bowtie^f J$ is an AV-ring.

Also Theorem 6.12 enriches the literature with new examples of AV-domains which are not Prüfer domains.

Example 6.21. [87, Example 2.10] Let T be the valuation domain $T = K[[X]] = K + M$, where K is a field and $M = XK[[X]]$ is the maximal ideal of T, and D be a proper subring of K which is a non-Prüfer AV-domain such that $qf(D) = K$. (For instance, let $k \subset F$ be a root extension of fields, i.e., for every $x \in F$, $x^n \in k$ for some positive integer n, and let t be an indeterminate over F. Set $K = F((t))$, $V = F[[t]] = F + tF[[t]]$ and $D = k + tF[[t]]$. Then $Qf(D) = F((t)) = K$, D is an AV-domain ([98, Lemma 2.2] and D is not a Prüfer domain). Set $R := D + M$ and let $f : D \hookrightarrow T$ be the natural embedding and $J := M$. Then :

(1) $D \bowtie^f J$ is an AV-domain.

(2) $D \bowtie^f J$ is not a Prüfer domain.

7 Coherent-like conditions

7.1 Coherent rings

This subsection is due to Alaoui and Mahdou [1]. It is to characterize the amalgamated algebra along an ideal $A \bowtie^f J$ to be a coherent ring. The main result examines the property of the coherence that the amalgamation $A \bowtie^f J$ might inherit from the ring A for some classes of ideals J and homomorphisms f, and hence generates new examples of non-Noetherian coherent rings.

Let $f : A \to B$ be a ring homomorphism, J be an ideal of B and let n be a positive integer. Consider the function $f^n : A^n \to B^n$ defined by $f^n((\alpha_i)_{i=1}^{i=n}) = (f(\alpha_i))_{i=1}^{i=n}$. Obviously f^n is a ring homomorphism and J^n is an ideal of B^n. This allows us to define $A^n \bowtie^{f^n} J^n$.

Moreover, let $\phi : (A \bowtie^f J)^n \to A^n \bowtie^{f^n} J^n$ defined by $\phi((a_i, f(a_i)+j_i)_{i=1}^{i=n}) = ((a_i)_{i=1}^{i=n}, f^n((a_i)_{i=1}^{i=n})+(j_i)_{i=1}^{i=n})$. It is easily checked that ϕ is a ring isomorphism. So $(A \bowtie^f J)^n$ and $A^n \bowtie^{f^n} J^n$ are isomorphic as rings.

Let U be a submodule of A^n. Then $U \bowtie^{f^n} J^n := \{(u, f^n(u) + j) \in A^n \bowtie^{f^n} J^n \mid u \in U, j \in J^n\}$ is a submodule of $A^n \bowtie^{f^n} J^n$.

Now the main result is the following.

Theorem 7.1. [1, Theorem 2.2] Let $f : A \to B$ be a ring homomorphism and let J be a proper ideal of B.

(1) If $A \bowtie^f J$ is a coherent ring, then so is A.

(2) Assume that J and $f^{-1}(J)$ are finitely generated ideals of $f(A)+J$ and A respectively. Then $A \bowtie^f J$ is a coherent ring if and only if A and $f(A)+J$ are coherent rings.

(3) Assume that J is a regular finitely generated ideal of $f(A)+J$. Then $A \bowtie^f J$ is a coherent ring if and only if A and $f(A)+J$ are coherent rings and $f^{-1}(J)$ is a finitely generated ideal of A.

Lemma 7.2. [1, Lemma 2.3] Let $f : A \to B$ be a ring homomorphism and let J be a proper ideal of B. Then:

(1) $\{0\} \times J$ (resp., $f^{-1}\{J\} \times \{0\}$) is a finitely generated ideal of $A \bowtie^f J$ if and only if J (resp., $f^{-1}\{J\}$) is a finitely generated ideal of $f(A)+J$ (resp., A).

(2) If $A \bowtie^f J$ is a coherent ring and $f^{-1}(J)$ is a finitely generated ideal of A, then $f(A)+J$ is a coherent ring.

Lemma 7.3. [1, Lemma 2.5] Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Assume that J and $f^{-1}(J)$ are finitely generated ideals of $f(A)+J$ and A respectively. Then $f^{-1}\{J\} \times \{0\}$ is a coherent $(A \bowtie^f J)$-module provided A is a coherent ring.

The following corollary is an immediate consequence of Theorem 7.1(3).

Corollary 7.4. [1, Corollary 2.7] Let $f : A \to B$ be a ring homomorphism, B be an integral domain, and let J be a proper and finitely generated ideal of $f(A)+J$. Then $A \bowtie^f J$ is a coherent ring if and only if A and $f(A)+J$ are coherent rings and $f^{-1}(J)$ is a finitely generated ideal of A.

The corollary below follows immediately from Theorem 7.1(2) which examines the case of the amalgamated duplication.

Corollary 7.5. [1, Corollary 2.8] Let A be a ring and I be a proper ideal of A. Then:

(1) If $A \bowtie I$ is a coherent ring, then so is A.

(2) Assume that I is a finitely generated ideal of A. Then $A \bowtie I$ is a coherent ring if and only if A is a coherent ring.

The next corollary is an immediate consequence of Theorem 7.1(2).

Corollary 7.6. [1, Corollary 2.9] Let A be a ring, I be an ideal of A, $B := \frac{A}{I}$, and let $f : A \to B$ be the canonical homomorphism ($f(x) = \overline{x}$).

(1) Assume that J and $f^{-1}(J)$ are finitely generated ideals of B and A respectively. Then $A \bowtie^f J$ is a coherent ring if and only if A and B are coherent rings.

(2) Assume that J is a regular finitely generated ideal of B. Then $A \bowtie^f J$ is a coherent ring if and only if A and B are coherent rings and $f^{-1}(J)$ is a finitely generated ideal of A.

The aforementioned result enriches the literature with new examples of coherent rings which are non-Noetherian rings.

Example 7.7. [1, Example 2.10] Let A be a non-Noetherian coherent ring, I be a finitely generated ideal of A, $f : A \to B(= \frac{A}{I})$ be the canonical homomorphism, and let J be a finitely generated ideal of A. Then $A \bowtie^f \bar{J}$ is a non-Noetherian coherent ring.

Example 7.8. [1, Example 2.11] Let $A := \mathbb{Z} + X\mathbb{Q}[X]$, where $\mathbb{Z}$ is the ring of integers and $\mathbb{Q}$ is the field of rational numbers. Let $I := X\mathbb{Q}[X]$, $B := \frac{A}{I}(\cong \mathbb{Z})$, $f : A \to B$ be the canonical homomorphism and let J be a nonzero ideal of B. Then $A \bowtie^f J$ is a non-Noetherian coherent ring.

7.2 n-Coherent rings

This subsection is due to Alaoui and Mahdou [4]. Let R be a ring. For a nonnegative integer n, an R-module E is called n-presented if there is an exact sequence of R-modules:

$$F_n \to F_{n-1} \to \dots F_1 \to F_0 \to E \to 0,$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are, respectively, finitely generated and finitely presented R-modules.

The ring R is said to be n-coherent if each $(n-1)$-presented ideal of R is n-presented, and R is said to be a strong n-coherent ring if each n-presented R-module is $(n+1)$-presented [53, 54] (This terminology is not the same as that of Costa (1994) [43], more precisely Costa's n-coherence is our strong n-coherence). In particular, 1-coherence coincides with coherence, and 0-coherence coincides with Noetherianity. Any strong n-coherent ring is n-coherent, and the converse holds for $n = 1$ or for coherent rings [54, Proposition 3.3]. The main theorem examines the transfer of the properties of strong n-coherence and n-coherence $(n \geq 2)$ to the amalgamated algebra along an ideal issued from local rings.

Before we announce the main result of this subsection (Theorem 7.10), we make the following useful proposition.

Proposition 7.9. [4, Proposition 2.1] Let (A, M) be a local ring, $f : A \to B$ be a ring homomorphism, and let J be a proper ideal of B such that $J \subseteq \mathrm{Jac}(B)$.

(1) $A \bowtie^f J$ is a local ring and $M \bowtie^f J$ is its maximal ideal.

(2) $f(A) + J$ is a local ring and $f(M) + J$ is its maximal ideal.

The following is the main result of this subsection.

Theorem 7.10. [4, Theorem 2.2] Let (A, M) be a local ring, $f : A \to B$ be a ring homomorphism, and let J be a proper ideal of B such that $f(M)J = 0$.

(1) Assume that $J \subseteq \mathrm{Jac}(B)$.

(a) (i) Assume that J is a finitely generated ideal of $f(A)+J$. If $A \bowtie^f J$ is a (strong) 2-coherent ring, then so is A.

(ii) Assume that $f(M) \subseteq J$ and M is a finitely generated ideal of A. If $A \bowtie^f J$ is a (strong) 2-coherent ring, then so is $f(A)+J$.

(iii) Assume that either $f(M) \subseteq J$ or $f(M) \cap J = \{0\}$, M and J are finitely generated ideals of A and $f(A)+J$ respectively, and $f(A)+J$ is a coherent ring. Then $A \bowtie^f J$ is a (strong) 2-coherent ring if and only if so is A.

(b) Assume that $n > 2$.

(i) Assume that either $f(M) \subseteq J$ or $f(M) \cap J = \{0\}$, M is an $(n-3)$-finitely presented ideal of A, and J is an $(n-2)$-finitely presented ideal of $f(A)+J$. If $A \bowtie^f J$ is a (strong) n-coherent ring, then so is A.

(ii) Assume that $f(M) \subseteq J$, M is an $(n-2)$-finitely presented ideal of A, and J is an $(n-3)$-finitely presented ideal of $f(A)+J$. If $A \bowtie^f J$ is a (strong) n-coherent ring, then so is $f(A)+J$.

(iii) Assume that $f(M) \subseteq J$, M and J are $(n-2)$-finitely presented ideals of A and $f(A)+J$ respectively, and $f(A)+J$ is a strong $(n-1)$-coherent ring. Then $A \bowtie^f J$ is a (strong) n-coherent ring if and only if so is A.

(2) Assume that $J^2 = 0$, $n \geq 2$, and J is a finitely generated ideal of $(f(A)+J)$. Then $A \bowtie^f J$ is a (strong) n-coherent ring if and only if so is A.

The following corollaries are immediate consequences of Theorem 7.10.

Corollary 7.11. [4, Corollary 2.5] Let (A, M) be a local ring, $f : A \to B$ be a ring homomorphism, and let J be a proper ideal of B such that $f(M)J = 0$.

(1) Assume that $J \subseteq \text{Jac}(B)$.

(a) Assume that either $f(M) \subseteq J$ or $f(M) \cap J = \{0\}$, M and J are finitely generated ideals of A and $f(A)+J$ respectively, and $f(A)+J$ is a coherent ring. Then $A \bowtie^f J$ is a (strong) 2-coherent ring which is not a coherent ring provided A is.

(b) Assume that $f(M) \subseteq J$, $n > 2$, M and J are $(n-2)$-finitely presented ideals of A and $f(A)+J$ respectively, and $f(A)+J$ is a strong $(n-1)$-coherent ring. Then $A \bowtie^f J$ is a (strong) n-coherent ring which is not a (strong) $(n-1)$-coherent ring provided A is.

(2) Assume that $J^2 = 0$, $n \geq 2$, and J is a finitely generated ideal of $f(A)+J$. Then $A \bowtie^f J$ is a (strong) n-coherent ring which is not a (strong) $(n-1)$-coherent ring provided A is.

Corollary 7.12. [4, Corollary 2.6] Let (A, M) be a local ring, I be a finitely generated ideal of A such that $MI = 0$, and $n \geq 2$. Then $A \bowtie I$ is a (strong) n-coherent ring which is not a (strong) $(n-1)$-coherent ring provided A is.

Theorem 7.10 enriches the literature with new examples of n-coherent rings that are not $(n-1)$-coherent rings $(n \geq 2)$.

Example 7.13. [4, Example 2.7] Let $T := K + M$ be a Bézout domain, where K is a field and M is a nonzero maximal ideal of T, D is a subring of K, the quotient field of D is $k = qf(D) \subseteq K$, $R = D + M$, and $T_0 = k + M$. Let $m(\neq 0) \in M$ and consider the canonical ring homomorphism $f : T_0 \to T_0/M^2$ $(f(x) = \overline{x})$. Assume that either $[K : k] = \infty$ or $1 \neq [K : k] < \infty$ and M is not a principal ideal of T. Then by Corollary 7.11(2), $T_0 \bowtie^f (\overline{m})$ is a 2-coherent ring which is not a coherent ring since T_0 is a 2-coherent ring which is not a coherent ring by [54, Theorem 2.1].

Example 7.14. [4, Example 2.8] Let R be a local 2-coherent domain which is not a field, $K = qf(R)$, $A := R \ltimes K$ be the trivial ring extension of R by K and M its maximal ideal, and let E be an $\frac{A}{M}$-vector space with finite dimension. Set $B := A \ltimes E$, $J := 0 \ltimes E$, and consider the ring homomorphism $f : A \to B$ ($f(x) = (x, 0)$). Then by Corollary 7.11(2), $A \bowtie^f J$ is a 2-coherent ring which is not a coherent ring since A is by [84, Theorem 3.1].

Example 7.15. [4, Example 2.9] Let $(V, \mathfrak{m})$ be a non discrete valuation domain, $A := V \ltimes \frac{V}{\mathfrak{m}}$ and $M := \mathfrak{m} \ltimes \frac{V}{\mathfrak{m}}$ its maximal ideal, $B := \frac{A}{M^2} \ltimes E$, where E is an $\frac{\frac{A}{M^2}}{\frac{M}{M^2}}$-vector space with finite dimension. Let $J := (\overline{m}) \ltimes E$, where $m(\neq 0) \in M$, and consider the ring homomorphism $f : A \to B$ ($f(x) = (\overline{x}, 0)$). Then by Corollary 7.11(2), $A \bowtie^f J$ is a 3-coherent ring which is not a 2-coherent ring since A is by [84, Example 3.8].

Example 7.16. [4, Example 2.10] Let V be a non-Noetherian valuation ring with rank$(V) > 1$. Let $A = V[[T]]$ be the power series ring in one variable T and M its maximal ideal. Set $B := (A \ltimes \frac{A}{M}) \times \frac{A}{M^2}$, $J := (0 \ltimes \frac{A}{M}) \times \frac{M}{M^2}$, and consider the ring homomorphism $f : A \to B$ ($f(x) = ((x, 0), 0)$). Then by Corollary 7.11(2), $A \bowtie^f J$ is a 2-coherent ring which is not a coherent ring since A is by [43, Example 4.4].

Example 7.17. [4, Example 2.11] Let K be a field, E be a K-vector space of infinite dimension, $A := K \ltimes E$ be the trivial extension ring of K by E, and $I := 0 \ltimes E'$, where E' is a finite dimensional K-subspace of E. Then by Corollary 7.12, $A \bowtie I$ is a 2-coherent ring wich is not a coherent ring since A is by [90, Theorem 3.4] and [84, Theorem 2.6].

7.3 On (n, d)-property

This subsection is due to Alaoui and Mahdou [3]. In 1994, Costa [43] introduced a doubly filtered set of classes of rings in order to categorize the structure of non-Noetherian rings: for nonnegative integers n and d, we say that a ring R is an (n, d)-ring if $\mathrm{pd}_R(E) \leq d$ for each n-presented R-module E (as usual, $\mathrm{pd}_R(E)$ denotes the projective dimension of E as an R-module). An integral domain with this property will be called an (n, d)-domain. For example, the $(n, 0)$-domains are the fields, the $(0, 1)$-domains are the Dedekind domains, and the $(1, 1)$-domains are the Prüfer domains [43].

For integers $n, d \geq 0$, Costa asks in [43] whether there is an (n, d)-ring which is neither an $(n, d-1)$-ring nor an $(n-1, d)$-ring? The answer is affirmative for $(0, d)$-rings, $(1, d)$-rings, $(2, d)$-rings, and $(3, d)$-rings. (See for instance [43, 44, 83, 84, 90, 91, 92, 111]). The goal of this subsection is to give examples of $(2, d)$-rings which are neither $(1, d)$-rings ($d = 0, 1, 2$) nor $(2, d-1)$-rings ($d = 1, 2$), and examples of $(3, d)$-rings which are neither $(2, d)$-rings ($d \geqslant 0$) nor $(3, d-1)$-rings ($d \geqslant 1$).

Now we have the following main result.

Theorem 7.18. [3, Theorem 2.2] Let (A, M) be a local ring, $f : A \to B$ be a ring homomorphism, and let J be a proper ideal of B such that $J \subseteq \mathrm{Jac}(B)$ and $f(M)J = 0$. Then:

(1) $A \bowtie^f J$ is a non-$(1, 2)$-ring. In particular, $A \bowtie^f J$ is a non-von Neumann regular ring.

(2) Assume that A is a $(2, 0)$-ring and M is not a finitely generated ideal of A. Then:

  (a) $A \bowtie^f J$ is a $(2, 0)$-ring.

  (b) Let $A_1 = A \bowtie^f J$, $d \leq 2$ be an integer, A_2 be a Noetherian ring of global dimension d, and let $C = A_1 \times A_2$ the direct product of A_1 and A_2. Then C is a $(2, d)$-ring which is neither a $(1, d)$-ring ($d = 0, 1, 2$) nor a $(2, d-1)$-ring ($d = 1, 2$).

The following corollary is a consequence of Theorem 7.18.

Corollary 7.19. [3, Corollary 2.4] Let (A, M) be a local $(2,0)$-ring such that M is not a finitely generated ideal of A, I be a proper ideal of A such that $IM = 0$. Let $A_1 := A \bowtie I$, $d \leq 2$ be an integer, A_2 be a Noetherian ring of global dimension d, and let $C := A_1 \times A_2$ the direct product of A_1 and A_2. Then $A \bowtie I$ is a $(2,0)$-ring that is a non-$(1,0)$-ring, and C is a $(2,d)$-ring which is neither a $(1,d)$-ring $(d = 0,1,2)$ nor a $(2,d-1)$-ring $(d = 1,2)$.

Theorem 7.18 enriches the literature with new classes of $(2,0)$, $(2,1)$, and $(2,2)$-rings.

Example 7.20. [3, Example 2.5] Let $(R, \mathfrak{m})$ be a local ring (for example $R = K[[X_1, \ldots, X_n, \ldots]]$ where K is a field), $E = (\frac{R}{\mathfrak{m}})^{\infty}$ is an infinite-dimensional $(\frac{R}{\mathfrak{m}})$-vector space, and let $A := R \ltimes E$ be the trivial extension ring of R by E and $M := \mathfrak{m} \ltimes E$ its maximal ideal. Let $B := \frac{A}{M^2} \times \frac{A}{M^3}$, $J := \frac{I}{M^2} \times \frac{L}{M^3}$, where I and L are two proper ideals of A, and consider the ring homomorphism $f : A \to B$ $(f(a) = (\overline{a}, 0))$. Let K be a field, $A_1 = A \bowtie^f J$, and let $A_2 := K[X_1]$ and $A_3 := K[X_1, X_2]$, where X_1, X_2 are indeterminate over K. Then by Theorem 7.18, we have

(1) A_1 is a $(2,0)$-ring that is not a $(1,0)$-ring since A is a $(2,0)$-ring by [91, Theorem 2.1].

(2) $A_1 \times A_2$ is a $(2,1)$-ring which is neither a $(1,1)$-ring nor a $(2,0)$-ring.

(3) $A_1 \times A_3$ is a $(2,2)$-ring which is neither a $(1,2)$-ring nor a $(2,1)$-ring.

Example 7.21. [3, Example 2.6] Let D be a local domain, $K := qf(D)$, and E be a K-vector space of infinite dimension, and let $A := K \ltimes E$ be the trivial extension ring of K by E and $M := 0 \ltimes E$ its maximal ideal. Let E' be an $\frac{A}{M}$-vector space, $B := (A \ltimes E') \times (D \ltimes K)$, $J := (M \ltimes E') \times (I \ltimes K)$, where I is a proper ideal of D, and consider the ring homomorphism $f : A \to B$ $(f(a) = ((a,0)), 0)$. Let $A_1 := A \bowtie^f J$, $A_2 := \mathbb{Z}$, and $A_3 := \mathbb{Z}[X]$, where X is an indeterminate over $\mathbb{Z}$. Then by Theorem 7.18:

(1) A_1 is a $(2,0)$ ring that is not a $(1,0)$-ring since A is a $(2,0)$ ring by [91, Corollary 2.3].

(2) $A_1 \times A_2$ is a $(2,1)$-ring which is neither a $(1,1)$-ring nor a $(2,0)$-ring.

(3) $A_1 \times A_3$ is a $(2,2)$-ring which is neither a $(1,2)$-ring nor a $(2,1)$-ring.

Example 7.22. [3, Example 2.7] Let K be a field and E be a K-vector space of infinite dimension, and let $A := K \ltimes E$ be the trivial extension ring of K by E and $M := 0 \ltimes E$ its maximal ideal. Let $A_1 = A \bowtie M$, and let $A_2 := K[X_1]$ and $A_3 := K[X_1, X_2]$, where X_1, X_2 are indeterminate over K. Then by Corollary 7.19:

(1) A_1 is a $(2,0)$-ring that is not a $(1,0)$-ring since A is a $(2,0)$-ring by [91, Corollary 2.3].

(2) $A_1 \times A_2$ is a $(2,1)$-ring which is neither a $(1,1)$-ring nor a $(2,0)$-ring.

(3) $A_1 \times A_3$ is a $(2,2)$-ring which is neither a $(1,2)$-ring nor a $(2,1)$-ring.

The aim of Theorem 7.23 is to construct a class of $(3,d)$-rings which are neither $(2,d)$-rings $(d \geqslant 0)$ nor $(3,d-1)$-rings $(d \geqslant 1)$.

Theorem 7.23. [3, Theorem 2.8] Let (A, M) be a local ring, $f : A \to B$ be a ring homomorphism, and let J be a proper ideal of B such that $J \subseteq \mathrm{Jac}(B)$ and $f(M)J = 0$. Then:

(1) Assume that M is not a finitely generated ideal of A. Then $A \bowtie^f J$ is a $(3,0)$-ring.

(2) Assume that M contains a regular element and J is a finitely generated ideal of $f(A) + J$. Then:

  (a) $A \bowtie^f J$ is a non-$(2,2)$-ring.

(b) Assume that M is not a finitely generated ideal of A. Let $A_1 = A \bowtie^f J$, A_2 be a Noetherian ring of global dimension d ($d \in \mathbb{N}$), and let $C = A_1 \times A_2$ the direct product of A_1 and A_2. Then:

(i) C is a $(3,d)$-ring which is neither a $(2,d)$-ring ($d = 0,1,2$) nor a $(3,d-1)$-ring ($d = 1,2$).

(ii) Assume that $J^2 = 0$. Then C is a $(3,d)$-ring which is neither a $(2,d)$-ring $d \geqslant 0$ nor a $(3,d-1)$-ring $d \geqslant 1$.

The following corollary follows immediately from Theorem 7.23.

Corollary 7.24. [3, Corollary 2.10] Let (A,M) be a local domain, $f : A \to B$ be a ring homomorphism, and let J be a proper ideal of B such that $f(M)J = 0$. Assume that M is not a finitely generated ideal of A and J is a finitely generated ideal of $f(A)+J$. Let $A_1 = A \bowtie^f J$, A_2 be a Noetherian ring of global dimension d ($d \in \mathbb{N}$), and let $C = A_1 \times A_2$ the direct product of A_1 and A_2. Then:

(1) Assume that $J \subseteq \mathrm{Jac}(B)$. Then $A \bowtie^f J$ is a $(3,0)$-ring which is a non-$(2,0)$-ring, and C is a $(3,d)$-ring which is neither a $(2,d)$-ring ($d = 0,1,2$) nor a $(3,d-1)$-ring ($d = 1,2$).

(2) Assume that $J^2 = 0$. Then C is a $(3,d)$-ring which is neither a $(2,d)$-ring $d \geqslant 0$ nor a $(3,d-1)$-ring $d \geqslant 1$.

Now we are able to give new examples of $(3,d)$-rings as shown below.

Example 7.25. [3, Example 2.11] Let $k \subseteq K$ be two fields such that $[K : k] = \infty$, $T = K[[X]] = K + M$, where X is an indeterminate over K and $M = XT$ be the maximal ideal of T, $A = k+M$. Set $B := A \ltimes E$, where E be an $\frac{A}{M}$-vector space with finite dimension, $J := 0 \ltimes E$, and consider the canonical ring homomorphism $f : A \to B$ ($f(a) = (a,0)$). Let $C = K[X_1, X_2, ..., X_d]$, where $d \geq 1$. Then by Corollary 7.24, $A \bowtie^f J$ is a $(3,0)$-ring which is a non-$(2,0)$-ring and $(A \bowtie^f J) \times C$ is a $(3,d)$-ring which is neither a $(2,d)$-ring nor a $(3,d-1)$-ring.

Example 7.26. [3, Example 2.12] Let K be any field and $X_1, X_2, \ldots, X_n, \ldots$ be indeterminate over K. Let $A = K[[X_1, X_2, \ldots, X_n, \ldots]]$ the power series ring in countably infinite variables over K, and let M be its maximal ideal. Set $B := \frac{A}{M^2}$, $J := \frac{I}{M^2}$, where I is a finitely generated proper ideal of A, and consider the canonical ring homomorphism $f : A \to B$ ($f(x) = \bar{x}$). Let $C = \mathbb{Z}[X_1, X_2, \ldots, X_{d-1}]$, where $d \geq 1$. Then by Corollary 7.24, $A \bowtie^f J$ is a $(3,0)$-ring which is a non-$(2,0)$-ring and $(A \bowtie^f J) \times C$ is a $(3,d)$-ring which is neither a $(2,d)$-ring nor a $(3,d-1)$-ring.

Example 7.27. [3, Example 2.13] Let K be a field and let $A = K[[X]] = K + M$, where $M = XA$. Set $B := A/M^2$, $J := (\overline{m})$ be an ideal of B, where $m \in M$ such that $\overline{m} \neq 0$, and consider the canonical ring homomorphism $f : A \to B$ ($f(x) = \overline{x}$). Then $A \bowtie^f J$ is not an (n,d)-ring for any integers $n, d \geq 0$.

7.4 Nil_*-coherence and special Nil_*-coherence

This subsection is due to Alaoui, Dobbs and Mahdou [5]. Let R be a ring and M be an R-module. Then M is called a Nil_*-coherent R-module if every finitely generated R-submodule of $\mathrm{Nil}(R)M$ is a finitely presented R-module; and that a ring R is said to be a Nil_*-coherent ring if it is Nil_*-coherent as an R-module, that is, if every finitely generated ideal of R that is contained in $\mathrm{Nil}(R)$ is finitely presented. Also an R-module M is said to be a *special* Nil_**-coherent R-module* if $\mathrm{Nil}(R)M$ is a coherent R-module, equivalently, if $\mathrm{Nil}(R)M$ is a finitely generated R-module and every finitely generated R-submodule of $\mathrm{Nil}(R)M$ is a finitely presented R-module. Then R is said to be a *special* Nil_**-coherent ring* if it is Nil_*-coherent as an R-module; equivalently, if $\mathrm{Nil}(R)$ is a coherent R-module; equivalently, if $\mathrm{Nil}(R)$ is a finitely generated ideal of R and each finitely generated ideal of R that is contained in $\mathrm{Nil}(R)$ is finitely presented.

The main result of this subsection (Theorem 7.28) characterizes the Nil_*-coherent and special Nil_*-coherent properties for certain constructions of an amalgamated algebra along an ideal.

Theorem 7.28. [5, Theorem 5.3] Let $f : R \to S$ be a ring homomorphism and let J be a proper ideal of B. Then:

(1) If $R \bowtie^f J$ is a Nil_*-coherent ring (resp., a special Nil_*-coherent ring), then so is R.

(2) If $f^{-1}(J)$ is a finitely generated ideal of R, $f^{-1}(J) \subseteq \mathrm{Nil}(R)$ and $R \bowtie^f J$ is a Nil_*-coherent ring, then $f(R)+J$ is also a Nil_*-coherent ring.

(3) Assume that $f^{-1}(J)$ and J are finitely generated ideals of R and $f(R)+J$ respectively, and that $f^{-1}(J) \subseteq \mathrm{Nil}(R)$. Then $R \bowtie^f J$ is a Nil_*-coherent ring if and only if R and $f(R)+J$ are Nil_*-coherent rings.

(4) Assume that $f^{-1}(J)$ and $\mathrm{Nil}(R)$ are finitely generated ideals of R, $f^{-1}(J) \subseteq \mathrm{Nil}(R)$, and both J and $J \cap \mathrm{Nil}(S)$ are finitely generated ideals of $f(R)+J$. Then $R \bowtie^f J$ is a special Nil_*-coherent ring if and only if R is a special Nil_*-coherent ring and $f(R)+J$ is a Nil_*-coherent ring.

The following two corollaries are direct consequences of Theorem 7.28.

Corollary 7.29. [5, Corollary 5.4] Let R be a ring and let I be a finitely generated ideal of R such that $I \subseteq \mathrm{Nil}(R)$. Then:

(1) $R \bowtie I$ is a Nil_*-coherent ring if and only if R is a Nil_*-coherent ring.

(2) Assume, in addition, that $\mathrm{Nil}(R)$ is a finitely generated ideal of R. Then $R \bowtie I$ is a special Nil_*-coherent ring if and only if R is a special Nil_*-coherent ring.

Corollary 7.30. [5, Corollary 5.5] Let $f : R \to S$ be an injective ring homomorphism and let J be a proper ideal of S. Assume that $f^{-1}(J)$ and $\mathrm{Nil}(R)$ are finitely generated ideals of R, J is a finitely generated ideal of $f(R)+J$, and $J \subseteq \mathrm{Nil}(S)$. Then $R \bowtie^f J$ is a special Nil_*-coherent ring if and only if R is a special Nil_*-coherent ring and $f(R)+J$ is a Nil_*-coherent ring.

Recall that it is characterized when certain amalgamated algebras are coherent rings [1, Theorem 2.2 (2)]. By using this result in conjunction with Theorem 7.28, one can prove the next two corollaries.

Corollary 7.31. [5, Corollary 5.6] Let $f : R \to S$ be a ring homomorphism and let J be a proper ideal of S. Assume that $f^{-1}(J)$ and J are finitely generated ideals of R and $f(R)+J$ respectively, and that $f^{-1}(J) \subseteq \mathrm{Nil}(R)$. Then:

(1) Assume, in addition, that R and $f(R)+J$ are Nil_*-coherent rings and that either R or $f(R)+J$ is not a coherent ring. Then $R \bowtie^f J$ is a Nil_*-coherent ring which is not a coherent ring.

(2) Assume, in addition, that $\mathrm{Nil}(R)$ is a finitely generated ideal of R, $J \subseteq \mathrm{Nil}(S)$, R is a special Nil_*-coherent ring, $f(R)+J$ is a Nil_*-coherent ring, and either R or $f(R)+J$ is not a coherent ring. Then $R \bowtie^f J$ is a special Nil_*-coherent ring which is not a coherent ring.

Corollary 7.32. [5, Corollary 5.7] Let R be a ring and let I be a finitely generated ideal of R such that $I \subseteq \mathrm{Nil}(R)$. Then:

(1) If R is a Nil_*-coherent ring that is not a coherent ring, then $R \bowtie I$ is also a Nil_*-coherent ring that is not a coherent ring.

(2) Assume, in addition, that $\mathrm{Nil}(R)$ is a finitely generated ideal of R. If R is a special Nil_*-coherent ring that is not a coherent ring, then $R \bowtie I$ is also a special Nil_*-coherent ring that is not a coherent ring.

We close this subsection by using the above results on amalgamated algebras to give four families of examples of special Nil_*-coherent rings which are not coherent rings.

Example 7.33. [5, Example 5.8] Let (R,M) be a local domain which is not a coherent ring, (for instance $R = k+M$, where $k \subset K$ is an infinite-dimensional field extension and $M = XK[[X]]$, where X is an analytic indeterminate over K). Then $(R \ltimes R/M) \bowtie (0 \ltimes R/M)$ is a special Nil_*-coherent ring which is not a coherent ring.

Example 7.34. [5, Example 5.9] Let $(R,\mathfrak{m})$ be a local integral domain which is not a coherent ring and let E' be a finite dimensional vector space over $R/\mathfrak{m}$. Set $A := R \ltimes E'$ and $M := \mathfrak{m} \ltimes E'$. Let E be a finite dimensional vector space over A/M. Let $f : A \to A \ltimes E$ be the usual embedding (given by $f(a) := (a,0)$) and set $J := \mathrm{Nil}(A) \ltimes E$. Then $A \bowtie^f J$ is a special Nil_*-coherent ring which is not a coherent ring.

Example 7.35. [5, Example 5.10] Let $f : R \to S$ be a ring homomorphism, where R is not a coherent ring and S is an integral domain. Let J be a proper ideal of S such that $f^{-1}(J) = \{0\}$. Then $R \bowtie^f J$ is a special Nil_*-coherent ring which is not a coherent ring.

Example 7.36. [5, Example 5.11] Let $f : R \to S$ be a ring homomorphism where R and S are integral domains and R is not a coherent ring. Let J be a proper ideal of S. Then $R \bowtie^f J$ is a special Nil_*-coherent ring which is not a coherent ring.

8 Duplication of module along an ideal

This section is due to Bouba, Mahdou, and Tamekkante [32]. Let R be a ring, I an ideal of R, and M an R-module, and set $\iota := \mathrm{id}_{M\otimes_R R/I}$. The particular pullback of M and M with respect to ι, denoted by $M \bowtie I$ and called the duplication of the R-module M along the ideal I, is

$$M \bowtie I := \{(m,m') \in M \times M \mid m \otimes \overline{1} = m' \otimes \overline{1}\},$$

and the $R \bowtie I$-module over $M \bowtie I$ is given by

$$(r,r+i).(m,m') = (rm,(r+i)m'), \quad \text{where} \quad r \in R,\, i \in I, \text{ and } (m,m') \in M \bowtie I.$$

This $R \bowtie I$-module may be expressed simply as

$$M \bowtie I = \{(m,m') \in M \times M \mid m - m' \in IM\}.$$

If $M = R$, then the duplication of the R-module R along the ideal I coincides with the amalgamated duplication of the ring R along the ideal I, which justified our notation of this kind of modules.

Notice that $0\times I$ is an ideal of $R \bowtie I$ and that $0\times IM = (0\times I)M \bowtie I$ and $IM\times IM$ are $R \bowtie I$-submodules of $M \bowtie I$. We begin this section with the following isomorphisms.

Proposition 8.1. [32, Proposition 2.1] We have the following isomorphisms (of R and $R \bowtie I$ modules):

(1) $\frac{M\bowtie I}{0\times IM} \cong M$.

(2) $\frac{M\bowtie I}{IM\times IM} \cong \frac{M}{IM}$.

Remark 8.2. [32, Remark 2.2] If R-modules are regarding as $R \bowtie I$-modules via the second projection $R \bowtie I \to R$; $(r,r+i) \mapsto r+i$, then with a similar proof as in the above proposition, $\frac{M\bowtie I}{IM\times 0} \cong M$ (isomorphism of R and $R \bowtie I$ modules).

Proposition 8.3. [32, Proposition 2.4] The $R \bowtie I$-module $M \bowtie I$ is Noetherian (resp., Artinian) if and only if the R-module M is Noetherian (resp., Artinian).

Proposition 8.4. [32, Proposition 2.6] If I is finitely generated, then $M \bowtie I$ is a coherent $R \bowtie I$-module if and only if M is a coherent R-module.

The above proposition recovers a known result for amalgamated duplication of a ring along an ideal.

Corollary 8.5. [1, Corollary 2.8] If I is finitely generated, then $R \bowtie I$ is a coherent ring if and only if R is a coherent ring.

Proposition 8.6. [32, Proposition 2.9] Suppose that I is a finitely generated ideal of R with $I \subseteq \text{Nil}(R)$ and that M is a finitely generated R-module. Then $M \bowtie I$ is a Nil_*-coherent $R \bowtie I$-module if and only if M is a Nil_*-coherent R-module.

Proposition 8.7. [32, Proposition 2.10] Suppose that I is a finitely generated ideal of R with $I \subseteq \text{Nil}(R)$ and M is a finitely generated R-module. Then $M \bowtie I$ is a special Nil_*-coherent $R \bowtie I$-module if and only if M is a special Nil_*-coherent R-module.

Propositions 8.6 and 8.7 recover a recent result for amalgamated duplication of a ring along an ideal.

Corollary 8.8. [5, Corollary 5.4] Suppose that I is a finitely generated ideal of R with $I \subseteq \text{Nil}(R)$. Then $R \bowtie I$ is a (resp., special) Nil_*-coherent ring if and only if R is a (resp., special) Nil_*-coherent ring.

A proper R-submodule N of an R-module M is said to be a prime submodule if for each $r \in R$ the trivial multiplication by r, $M/N \to M/N$ is either injective or zero. This implies that $\text{Ann}_R(M/N) = \mathfrak{p}$ is a prime ideal of R, and N is said to be $\mathfrak{p}$-prime submodule. We say M is a prime module if the zero submodule of M is a prime submodule of M. Clearly this is equivalent to the condition: for all $r \in R$ and $m \in M$ we have :

$$\Big(rm = 0\Big) \Longrightarrow \Big(m = 0 \text{ or } rM = 0\Big).$$

In particular, the ring R is a prime R-module if and only if R is an integral domain. Moreover, N is a prime submodule of M if and only if M/N is a prime module (for more details please see [106]).

Proposition 8.9. [32, Proposition 2.12]

(1) $M \bowtie I$ is a prime $R \bowtie I$-module if and only if $IM = 0$ and M is a prime R-module.

(2) $0 \times IM$ is a prime $R \bowtie I$-submodule of $M \bowtie I$ if and only if M is a prime R-module.

Now we give the necessary and sufficient conditions for a duplication of module along an ideal to be an injective module, a projective module, or a flat module.
Recall first that the annihilator of I in M is defined to be

$$\text{ann}_M(I) = \{m \in M \mid im = 0 \text{ for all } i \in I\}.$$

It is clear that $\text{ann}_M(I)$ is an R-submodule of M.

Theorem 8.10. [32, Theorem 3.3]

(1) The $R \bowtie I$-module $M \bowtie I$ is injective if and only if $\text{ann}_M(I)$ and IM are injective R-modules.

(2) The $R \bowtie I$-module $M \bowtie I$ is projective (resp., flat) if and only if M is a projective (resp., flat) R-module.

Example 8.11. [32, Example 3.4] For each positive integer k, the $\mathbb{Z} \bowtie k\mathbb{Z}$-module $\mathbb{Q} \bowtie k\mathbb{Z} = \mathbb{Q} \times \mathbb{Q}$ is flat and injective but not projective.

The characterization of $R \bowtie I$ to be self-injective was done in [38]. However we will recover it again as a consequence of Theorem 8.10.

Corollary 8.12. [38, Theorem 2.4] The ring $R \bowtie I$ is self-injective if and only if R is self-injective and I is generated by an idempotent.

Recall that over an arbitrary ring R the Krull dimension of an R-module M is defined by

$$\dim_R(M) := \dim\left(\frac{R}{\mathrm{Ann}_R(M)}\right).$$

Note that if N is an R-submodule of M then $\dim_R(N) \leq \dim_R(M)$.

Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring and M a nonzero finitely generated R-module. The depth of M is

$$\mathrm{depth}_R(M) := \min\{i \mid \mathrm{Ext}^i_R(k, M) \neq 0\}.$$

Clearly if M and M' are finitely generated R-modules, we have

$$\mathrm{depth}_R(M \oplus M') = \min\{\mathrm{depth}_R(M), \mathrm{depth}_R(M')\}.$$

In general, we have $\mathrm{depth}_R(M) \leq \dim_R(M)$. One calls M Cohen-Macaulay (CM) if $\mathrm{depth}_R(M) = \dim_R(M)$. If R itself is a CM R-module, then it is called a Cohen-Macaulay ring (CM-ring). A maximal Cohen-Macaulay R-module (MCM) is a CM R-module M such that $\dim_R(M) = \dim(R)$ (for more details please see [34]).

Next we investigate necessary and sufficient conditions for $M \bowtie I$ to be a CM (resp., an MCM) $R \bowtie I$-module. To do so, we first need to describe the Krull dimension of $M \bowtie I$.

Lemma 8.13. [32, Lemma 3.6]

$$\mathrm{Ann}_{R\bowtie I}(M \bowtie I) = \{(r, r+i) \in R \bowtie I \mid r \in \mathrm{Ann}_R(M) \text{ and } i \in \mathrm{Ann}_R(M) \cap I\}.$$

Recall that a module is said to be faithful if its annihilator is reduced to the zero-ideal. As a consequence of the preceding lemma, it is easy to check that the duplication $M \bowtie I$ is a faithful $R \bowtie I$-module if and only if M is a faithful R-module.

Lemma 8.14. [32, Lemma 3.7] $\dim_{R\bowtie I}(M \bowtie I) = \dim_R(M)$.

Theorem 8.15. [32, Theorem 3.8] Suppose that R is local Noetherian and that M is a nonzero finitely generated R-module.

(1) If $IM = 0$, then the $R \bowtie I$-module $M \bowtie I$ is CM if and only if M is a CM R-module.

(2) If $IM \neq 0$, then the $R \bowtie I$-module $M \bowtie I$ is CM if and only if M and IM are CM R-modules with $\dim_R(M) = \dim_R(IM)$.

Example 8.16. [32, Example 3.9] Let R be a ring with a CM module M and set $I := \mathrm{ann}_R(M)$. Then $M \bowtie I = \{(m, m) \mid m \in M\}$ is a CM $R \bowtie I$-module.

Corollary 8.17. [32, Corollary 3.10] Suppose that R is local Noetherian and that M is a nonzero finitely generated R-module.

(1) If $IM = 0$, then the $R \bowtie I$-module $M \bowtie I$ is MCM if and only if M is an MCM R-module.

(2) If $IM \neq 0$, then the following are equivalent:

(a) The $R \bowtie I$-module $M \bowtie I$ is MCM.

(b) M is a CM and IM is an MCM R-module.

(c) M and IM are MCM R-modules.

The above corollary recovers the first part of [7, Theorem 1.8].

Corollary 8.18. [32, Corollary 3.11] If $I \neq 0$, then the ring $R \bowtie I$ is a CM ring if and only if R is a CM-ring and I is an MCM R-module.

9 Amalgamated modules along an ideal

This section is due to El Khalfaoui, Mahdou, Sahandi, and Shirmohammadi [60]. Let $f : R \to S$ be a ring homomorphism, J be an ideal of S, M be an R-module, N be an S-module (which is an R-module induced naturally by f) and $\varphi : M \to N$ be an R-module homomorphism. We define the amalgamation of M and N along J with respect to φ by

$$M \bowtie^{\varphi} JN := \{(m, \varphi(m) + n) \mid m \in M \text{ and } n \in JN\}.$$

It can be seen that $M \bowtie^{\varphi} JN$ is an $R \bowtie^{f} J$-module by the following scaler product

$$(r, f(r) + j)(m, \varphi(m) + n) := (rm, \varphi(rm) + f(r)n + j\varphi(m) + jn).$$

Note that $\varphi(rm) = f(r)\varphi(m)$ since φ is an R-module homomorphism. If $M = R$, $N = S$ and $\varphi = f$, then the amalgamation of the R-module R and the S-module S along J with respect to φ coincides with the amalgamation of rings R and S along J with respect to f. Also, if $S = R$, $N = M$ and $\varphi = id_M$, then the amalgamation of M and N along J with respect to φ is exactly the duplication of the R-module M along the ideal J.

In this section, we present some basic properties of the amalgamation $M \bowtie^{\varphi} JN$ of M and N along J with respect to φ.

One can define $M \bowtie^{\varphi} JN$ by means of pullback of modules. Indeed, let $\pi : N \to N/JN$ be a natural homomorphism, $M \bowtie^{\varphi} JN \to N$ (resp., $M \bowtie^{\varphi} JN \to M$) be the restriction to $M \bowtie^{\varphi} JN$ of the projection of $M \times N$ onto N (resp., M). It can be seen that the following diagram is a pullback:

$$\begin{array}{ccc} M \bowtie^{\varphi} JN & \longrightarrow & N \\ \downarrow & & \downarrow \pi \\ M & \xrightarrow[\pi \circ \varphi]{} & N/JN. \end{array}$$

Remark 9.1. [60, Remark 2.1]

(1) $f(R) + J$ is a subring of S. So N is an $f(R) + J$-module. It is easy to see that $\varphi(M) + JN$ is an $f(R) + J$-submodule of N. Thus $\varphi(M) + JN$ is an $R \bowtie^{f} J$-module via $P_S : R \bowtie^{f} J \to f(R) + J$ defined by $P_S(r, f(r) + j) = f(r) + j$.

(2) $\pi_N : M \bowtie^{\varphi} NJ \to \varphi(M) + JN$ given by $\pi_N(m, \varphi(m) + n) = \varphi(m) + n$ is an $R \bowtie^{f} J$-module homomorphism.

(3) M is an $R \bowtie^{f} J$-module via the surjective homomorphism $p_R : R \bowtie^{f} J \to R$. It is easy to see that $\pi_M : M \bowtie^{\varphi} JN \to M$ given by $\pi_M(m, \varphi(m) + n) = m$ is an $R \bowtie^{f} J$-module homomorphism.

(4) It can be seen that JN is an $f(R)+J$-submodule of $\varphi(M)+JN$. Hence JN is an $R\bowtie^f J$-submodule of $\varphi(M)+JN$.

(5) We have the following exact sequence of $R\bowtie^f J$-modules and $R\bowtie^f J$-homomorphisms:

$$0 \to JN \xrightarrow{\iota} M\bowtie^{\varphi} JN \xrightarrow{\pi_M} M \to 0,$$

where $\iota : JN \to M\bowtie^{\varphi} JN$ defined by $\iota(n) = (0,n)$.

Proposition 9.2. [60, Proposition 2.2] Let $f : R \to S$ be a ring homomorphism, J be an ideal of S, M be an R-module, N be an S-module and $\varphi : M \to N$ an R-module homomorphism. Then the following hold:

(1) $\frac{M\bowtie^{\varphi} JN}{\{0\}\times JN} = M$.

(2) $\frac{M\bowtie^{\varphi} JN}{F\bowtie^{\varphi} JN} = \frac{M}{F}$, where F is a submodule of M.

(3) $\frac{M\bowtie^{\varphi} JN}{\varphi^{-1}(JN)\times\{0\}} = \varphi(M)+JN$.

Remark 9.3. [60, Remark 2.3] If we consider R-modules in Proposition 9.2 as $R\bowtie^f J$-modules, then the isomorphisms are also $R\bowtie^f J$-isomorphisms.

We have the following results about localization.

Proposition 9.4. [60, Proposition 2.4] With the notation of Proposition 9.2, the following statements hold:

(1) For $\mathfrak{p}\in \mathrm{Spec}(R)$ and $\mathfrak{q}\in\mathrm{Spec}(S)\setminus \mathrm{V}(J)$, set

$$\mathfrak{p}'^f := \mathfrak{p}\bowtie^f J := \{(p, f(p)+j) \mid p\in\mathfrak{p}, j\in J\},$$
$$\overline{\mathfrak{q}}^f := \{(r, f(r)+j) \mid r\in R, j\in J, f(r)+j\in\mathfrak{q}\}.$$

Then one has the following:

(a) The prime ideals of $R\bowtie^f J$ are of the type $\overline{\mathfrak{q}}^f$ or $\mathfrak{p}'^f$ for $\mathfrak{q}$ varying in $\mathrm{Spec}(S)\setminus \mathrm{V}(J)$ and $\mathfrak{p}$ in $\mathrm{Spec}(R)$.

(b) $\mathrm{Max}(R\bowtie^f J) = \{\mathfrak{p}'^f \mid \mathfrak{p}\in\mathrm{Max}(R)\}\cup\{\overline{\mathfrak{q}}^f \mid \mathfrak{q}\in\mathrm{Max}(S)\setminus\mathrm{V}(J)\}$.

(2) The following formulas for localizations hold:

(a) For any $\mathfrak{q}\in\mathrm{Spec}(S)\setminus\mathrm{V}(J)$, the localization $(M\bowtie^{\varphi} JN)_{\overline{\mathfrak{q}}^f}$ is canonically isomorphic to $N_{\mathfrak{q}}$. This isomorphism maps the element $(x,\varphi(x)+y)/(r,f(r)+j)$ to $(\varphi(x)+y)/(f(r)+j)$.

(b) For any $\mathfrak{p}\in\mathrm{Spec}(R)\setminus\mathrm{V}(f^{-1}(J))$, the localization $(M\bowtie^{\varphi} JN)_{\mathfrak{p}'^f}$ is canonically isomorphic to $M_{\mathfrak{p}}$. This isomorphism maps the element $(x,\varphi(x)+y)/(r,f(r)+j)$ to x/r.

(c) For any $\mathfrak{p}\in\mathrm{Spec}(R)$ containing $f^{-1}(J)$, consider the multiplicative subset $T_{\mathfrak{p}} := f(R\setminus\mathfrak{p})+J$ of S and set $N_{T_{\mathfrak{p}}} := T_{\mathfrak{p}}^{-1}N$ and $J_{T_{\mathfrak{p}}} := T_{\mathfrak{p}}^{-1}J$. If $f_{\mathfrak{p}} : R_{\mathfrak{p}} \to S_{T_{\mathfrak{p}}}$ is the ring homomorphism induced by f and $\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{T_{\mathfrak{p}}}$ is the $R_{\mathfrak{p}}$-homomorphism induced by φ, then the $R_{\mathfrak{p}}$-module $(M\bowtie^{\varphi} JN)_{\mathfrak{p}'^f}$ is canonically isomorphic to $M_{\mathfrak{p}}\bowtie^{\varphi_{\mathfrak{p}}} J_{T_{\mathfrak{p}}}N_{T_{\mathfrak{p}}}$. This isomorphism maps the element $(x,f(x)+y)/(r,f(r)+j)$ to $(x/r,(f(x)+y)/(f(r)+j))$.

In [49, Propositions 5.6 and 5.7], the authors determined the Noetherian property of the amalgamated algebra $R\bowtie^f J$. We will now see when the amalgamation of a module along an ideal is Noetherian.

Proposition 9.5. [60, Proposition 3.1] With the notation of Proposition 9.2, the amalgamation $M \bowtie^{\varphi} JN$ is a Noetherian $R \bowtie^f J$-module if and only if $\varphi(M) + JN$ is a Noetherian $f(R) + J$-module and M is a Noetherian R-module.

Proposition 9.6. [60, Proposition 3.2] With above notation, assume that at least one of the following conditions holds:

(1) JN is a Noetherian R-module (with the structure naturally induced by f).

(2) $\varphi(M) + JN$ is a Noetherian R-module (with the structure naturally induced by f).

Then $M \bowtie^{\varphi} JN$ is a Noetherian $R \bowtie^f J$-module if and only if M is a Noetherian R-module. In particular, if M is a Noetherian R-module and N is a Noetherian R-module (with the structure naturally induced by f), then $M \bowtie^{\varphi} JN$ is a Noetherian $R \bowtie^f J$-module for all ideals J of S.

Let $f : R \to S$ be a ring homomorphism, J be an ideal of S, M be an R-module, N be an S-module and $\varphi : M \to N$ be an R-module homomorphism and let n be a positive integer. Consider the function $\varphi^n : M^n \to N^n$ defined by $\varphi^n((m_i)_{i=1}^{i=n}) = (\varphi(m_i))_{i=1}^{i=n}$. Obviously φ^n is an R-module homomorphism and $JN^n = (JN)^n$ is a submodule of N^n. This allows us to define $M^n \bowtie^{\varphi^n} (JN)^n$.

We recall that the S-module N is an R-module induced by f, and so it is the same for N^n. Hence $rx = f(r)x$ for all $r \in R$ and $x \in N^n$, and so

$$\varphi^n(rm) = r\varphi^n(m) = f(r)\varphi^n(m) \text{ for all } r \in R \text{ and } x \in N^n.$$

Proposition 9.7. [60, Proposition 4.1] Let F be a submodule of M^n. Then the following hold:

(1) Assume that F is a finitely generated R-module and JN is a finitely generated $f(R) + J$-module. Then $F \bowtie^{\varphi^n} (JN)^n$ is a finitely generated $R \bowtie^f J$-module.

(2) Suppose that $\varphi^n(F) \subseteq (JN)^n$. Then $F \bowtie^{\varphi^n} (JN)^n$ is a finitely generated $R \bowtie^f J$-module if and only if F is a finitely generated R-module and JN is a finitely generated $f(R) + J$-module.

Theorem 9.8. [60, Theorem 4.2]

(1) Assume that J and JN are finitely generated $f(R) + J$-modules. If $M \bowtie^{\varphi} JN$ is a coherent $R \bowtie^f J$-module, then M is a coherent R-module.

(2) Assume that J is a finitely generated ideal of $f(R) + J$ and φ is surjective. If $M \bowtie^{\varphi} JN$ is a coherent $R \bowtie^f J$-module, then M is a coherent R-module.

(3) Assume that J and JN are finitely generated $f(R)+J$-modules and $\varphi^{-1}(JN)$ is a finitely generated R-module. Then $M \bowtie^{\varphi} JN$ is a coherent $R \bowtie^f J$-module if and only if M is a coherent R-module and $\varphi(M) + JN$ is a coherent $f(R) + J$-module.

(4) Assume that J is a finitely generated $f(R) + J$-module, φ is surjective and $\varphi^{-1}(JN)$ is a finitely generated R-module. Then $M \bowtie^{\varphi} JN$ is a coherent $R \bowtie^f J$-module if and only if M is a coherent R-module and $\varphi(M) + JN$ is a coherent $f(R) + J$-module.

Lemma 9.9. [60, Lemma 4.3]

(1) $\{0\} \times JN$ (resp., $\varphi^{-1}(JN) \times \{0\}$) is a finitely generated $R \bowtie^f J$-module if and only if JN (resp., $\varphi^{-1}(JN)$) is a finitely generated $f(R) + J$-module (resp., R-module).

(2) If $M \bowtie^{\varphi} JN$ is a coherent $R \bowtie^f J$-module and $\varphi^{-1}(JN)$ is a finitely generated R-module, then $\varphi(M) + JN$ is a coherent $f(R) + J$-module.

Lemma 9.10. [60, Lemma 4.4] Assume that J and $\varphi^{-1}(JN)$ are finitely generated $f(R)+J$-module and R-module respectively. If M is a coherent R-module, then $\varphi^{-1}(JN) \times \{0\}$ is a coherent $R \bowtie^f J$-module.

10 Bi-Amalgamated algebras along ideals

This section is due to Kabbaj, Louartiti, and Tamekkante [85]. Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C respectively such that $f^{-1}(J) = g^{-1}(J')$. The bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) is the subring of $B \times C$ given by

$$A \bowtie^{f,g} (J, J') := \left\{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\right\}.$$

Notice first that every amalgamated duplication is an amalgamated algebra and every amalgamated algebra is a bi-amalgamated algebra as seen below.

Example 10.1. [85, Example 2.1] (The amalgamated algebra) Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Set $I := f^{-1}(J)$ and $\iota := \mathrm{id}_A$. Thus

$$\begin{aligned} A \bowtie^{\iota,f} (I, J) &= \left\{(a + i, f(a) + j) \mid a \in A, (i, j) \in I \times J\right\} \\ &= \left\{(a + i, f(a + i) + j - f(i)) \mid a \in A, (i, j) \in I \times J\right\} \\ &= \left\{(a, f(a) + j) \mid a \in A, j \in J\right\} \\ &= A \bowtie^f J. \end{aligned}$$

Further the subring $f(A) + J$ of B can be regarded as a bi-amalgamation, precisely:

Remark 10.2. [85, Remark 2.2] Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Set $I := f^{-1}(J)$ and consider the canonical projection $\pi : A \to A/I$. Then one can easily check that

$$\begin{aligned} f(A) + J &\cong \left\{(\bar{a}, f(a) + j) \mid a \in A, j \in J\right\} \\ &= A \bowtie^{\pi,f} (0, J). \end{aligned}$$

In particular, Boisen-Sheldon's CPI-extensions [31] can also be viewed as bi-amalgamations.

Example 10.3. [85, Example 2.3] (The CPI-extension) Let A be a ring and let I be an ideal of A. Then $\overline{S} := (A/I) \setminus \mathrm{Z}(A/I)$ and $S := \{s \in A \mid \bar{s} \in \overline{S}\}$ are multiplicative subsets of A/I and A respectively. Let $\varphi : S^{-1}A \to Q(A/I) = (\overline{S})^{-1}(A/I)$ and $f : A \to S^{-1}A$ be the canonical ring homomorphisms. Then the subring

$$C(A, I) := \varphi^{-1}(A/I) = f(A) + S^{-1}I$$

of $S^{-1}A$ is called the CPI-extension of A with respect to I (in the sense of Boisen-Sheldon). Now, let $\pi : A \to A/I$ be the canonical projection. From Remark 10.2, we have

$$A \bowtie^{\pi,f} (0, S^{-1}I) \cong f(A) + S^{-1}I = C(A, I).$$

Other known families of rings stem from Remark 10.2; namely, those issued from extensions of rings $A \subset B$ (including classical pullbacks).

Example 10.4. [85, Example 2.4] (The ring $A + J$) Let $i : A \hookrightarrow B$ be an embedding of rings, J an ideal of B, $I := A \cap J$, and $\pi : A \to A/I$ the canonical projection. From Remark 10.2, the subring $A + J$ of B can arise as a bi-amalgamation via

$$A + J \cong A \bowtie^{\pi,i} (0, J)$$

and consequently, so do most classical pullback constructions such as $A + XB[X]$ (via $A \subset B[X]$ and $XB[X]$), $A + XB[[X]]$ (via $A \subset B[[X]]$ and $XB[[X]]$), and $D + M$ (via $D \subset T$ and M ideal of T with $D \cap M = 0$).

In the next, as an application of Proposition 10.8, we will see that some glueings of prime ideals [102, 105, 107, 110] can be viewed as bi-amalgamations. Now we give an explicit (non-classical pullback) example; namely, the ring $R := \mathbb{Z}[X] + (X^2+1)\mathbb{Q}[X]$ which lies between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.

Example 10.5. [85, Example 2.5] Let $i : \mathbb{Z}[X] \hookrightarrow \mathbb{Q}[X]$ be the natural embedding and consider the ring homomorphism $\pi : \mathbb{Z}[X] \to \mathbb{Z}[i]$, $p(X) \mapsto p(i)$. Clearly $(X^2+1)\mathbb{Q}[X] \cap \mathbb{Z}[X] = (X^2+1)$ and $\dfrac{\mathbb{Z}[X]}{(X^2+1)} \cong \mathbb{Z}[i]$ so that

$$R := \mathbb{Z}[X] + (X^2+1)\mathbb{Q}[X] \cong \mathbb{Z}[X] \bowtie^{\pi,i} \left(0, (X^2+1)\mathbb{Q}[X]\right).$$

Throughout, let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and J, J' two ideals of B and C, respectively, such that $I := f^{-1}(J) = g^{-1}(J')$. Let $A \bowtie^{f,g} (J,J')$ denote the bi-amalgamation of A with (B,C) along (J,J') with respect to (f,g).

Next, we shed light on the correlation between pullback constructions and bi-amalgamations. We first show how every bi-amalgamation can arise as a natural pullback.

Proposition 10.6. [85, Proposition 3.1] Consider the ring homomorphisms $\alpha : f(A)+J \to A/I$, $f(a)+j \mapsto \bar{a}$ and $\beta : g(A)+J' \to A/I$, $g(a)+j' \mapsto \bar{a}$. Then the bi-amalgamation is determined by the following pullback

$$\begin{array}{ccc} A \bowtie^{f,g} (J,J') & \twoheadrightarrow & f(A)+J \\ \downarrow & & \downarrow \alpha \\ g(A)+J' & \xrightarrow{\beta} & A/I \end{array}$$

that is

$$A \bowtie^{f,g} (J,J') = \alpha \times_{\frac{A}{I}} \beta.$$

Next we see how bi-amalgamations can be represented as conductor squares.

Proposition 10.7. [85, Proposition 3.2] Consider the following ring homomorphisms

$$\begin{array}{cccc} \iota_1 : & \dfrac{A}{I} & \longrightarrow & \dfrac{f(A)+J}{J} \times \dfrac{g(A)+J'}{J'} \\ & \bar{a} & \longmapsto & \left(\overline{f(a)}, \overline{g(a)}\right) \end{array}$$

$$\begin{array}{cccc} \mu_2 : & A \bowtie^{f,g} (J,J') & \longrightarrow & \dfrac{A}{I} \\ & (f(a)+j, g(a)+j') & \longmapsto & \bar{a} \end{array}$$

Then the following diagram

$$\begin{array}{ccc} A \bowtie^{f,g} (J,J') & \xrightarrow{\iota_2} & (f(A)+J) \times (g(A)+J') \\ \downarrow \mu_2 & & \downarrow \mu_1 \\ \dfrac{A}{I} & \xrightarrow{\iota_1} & \dfrac{f(A)+J}{J} \times \dfrac{g(A)+J'}{J'} \end{array}$$

is a conductor square with conductor $\mathrm{Ker}(\mu_1) = J \times J'$, where ι_2 is the natural embedding and μ_1 is the canonical surjection.

The next result characterizes pullbacks that can arise as bi-amalgamations.

Proposition 10.8. [85, Proposition 3.3] Consider the following diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow \alpha \\ C & \xrightarrow{\beta} & D \end{array}$$

of ring homomorphisms and let $\pi : B \times C \to B$ be the canonical projection. Then the following conditions are equivalent:

(1) $\alpha \times_D \beta = A \bowtie^{f,g} (J, J')$ for some ideals J of B and J' of C with $f^{-1}(J) = g^{-1}(J')$;

(2) The above diagram is commutative with $\alpha \circ \pi(\alpha \times_D \beta) = \alpha \circ f(A)$.

In view of Example 10.1, Proposition 10.8 recovers the special case of amalgamated algebras, as recorded in the next corollary.

Now we give a brief discussion on Traverso's glueings of prime ideals [102, 105, 107] which are special pullbacks [110, Lemma 2]. So they can also be viewed as special bi-amalgamations if they satisfy condition (2) of Proposition 10.8. Precisely, from [110, Lemma 1], let A be a Noetherian ring and B an overring of A such that B is a finitely generated A-module. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the prime ideals of B lying over $\mathfrak{p}$. For each i, $\frac{A_{\mathfrak{p}}}{\mathfrak{p}A_{\mathfrak{p}}}$ is a subfield of $\frac{B_{\mathfrak{p}_i}}{\mathfrak{p}_i B_{\mathfrak{p}_i}}$, and let $\overline{\frac{b}{t}}^i$ denote the class of the element $\frac{b}{t}$ of $B_{\mathfrak{p}_i}$ modulo $\mathfrak{p}_i B_{\mathfrak{p}_i}$. The ring A' obtained from B by glueing over $\mathfrak{p}$ is the subring of B (containing A) given by

$$A' := \left\{ b \in B \mid \exists \frac{a_o}{s_o} \in A_{\mathfrak{p}} \text{ with } \overline{\frac{b}{1}}^i = \overline{\frac{a_o}{s_o}}^i \ \forall i \text{ and, for } \frac{a}{s} \in A_{\mathfrak{p}}, \ \overline{\frac{b}{1}}^i = \overline{\frac{a}{s}}^i \Leftrightarrow \overline{\frac{b}{1}}^j = \overline{\frac{a}{s}}^j \ \forall i, j \right\}.$$

Now consider the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\iota} & B \\ \downarrow \mu & & \downarrow \Phi \\ \frac{A_{\mathfrak{p}}}{\mathfrak{p}A_{\mathfrak{p}}} & \xrightarrow{\Psi} & D := \frac{B_{\mathfrak{p}_1}}{\mathfrak{p}_1 B_{\mathfrak{p}_1}} \times \cdots \times \frac{B_{\mathfrak{p}_n}}{\mathfrak{p}_n B_{\mathfrak{p}_n}} \end{array}$$

where ι is the natural embedding, $\mu(a) = \overline{\frac{a}{1}}\ \forall a \in A$, $\Phi(b) = (\overline{\frac{b}{1}}^1, \ldots, \overline{\frac{b}{1}}^n)\ \forall b \in B$, and $\Psi(\overline{\frac{a}{s}}) = (\overline{\frac{a}{s}}^1, \ldots, \overline{\frac{a}{s}}^n)\ \forall \frac{a}{s} \in A_{\mathfrak{p}}$. Set $J := \operatorname{Ker}(\Phi)$ and $J' := \operatorname{Ker}(\Psi)$. Then note that

$$\mathfrak{p} = \iota^{-1}(J) = \mu^{-1}(J').$$

Corollary 10.9. [85, Corollary 3.5] Under the above notation, the following assertions are equivalent:

(1) $A' = A \bowtie^{\iota,\mu} (J, J')$;

(2) For any $(\frac{a}{s}, b) \in A_{\mathfrak{p}} \times B :\ a - sb \in \bigcap_{1 \leq i \leq n} \mathfrak{p}_i \Rightarrow a - sa_o \in \mathfrak{p}$ for some $a_o \in A$.

For example, if $A := \mathbb{Z}$ and $p := 2\mathbb{Z}$, then for any finitely generated $\mathbb{Z}$-module B (e.g., $\mathbb{Z}[i]$) Condition (2) of Corollary 10.9 always holds since, for any $n \in \mathbb{Z}$ and $s \in \mathbb{Z} \setminus 2\mathbb{Z}$, $n - sn \in 2\mathbb{Z}$.

Throughout, let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and J, J' two ideals of B and C respectively such that $I_o := f^{-1}(J) = g^{-1}(J')$. Let

$$A \bowtie^{f,g} (J,J') := \left\{(f(a)+j, g(a)+j') \mid a \in A, (j,j') \in J \times J'\right\}$$

be the bi-amalgamation of A with (B,C) along (J,J') with respect to (f,g).

Next we study some basic algebraic properties of bi-amalgamations. Precisely we investigate necessary and sufficient conditions for a bi-amalgamation to be a Noetherian ring, a domain, or a reduced ring. We will show that the transfer of these notions is made via the special rings $f(A)+J$ and $g(A)+J'$ (which correspond to B and C respectively in the case when f and g are surjective).

We start with some basic ideal-theoretic properties of bi-amalgamations. For this purpose, notice first that $0 \times J'$, $J \times 0$, and $J \times J'$ are particular ideals of $A \bowtie^{f,g} (J,J')$; and if I is an ideal of A, then the set

$$I \bowtie^{f,g} (J,J') := \left\{(f(i)+j, g(i)+j') \mid i \in I, (j,j') \in J \times J'\right\}$$

is an ideal of $A \bowtie^{f,g} (J,J')$ containing $J \times J'$.

Proposition 10.10. [85, Proposition 4.1] Let I be an ideal of A. We have the following canonical isomorphisms:

(1) $\dfrac{A \bowtie^{f,g} (J,J')}{I \bowtie^{f,g} (J,J')} \cong \dfrac{A}{I+I_o}$.

(2) $\dfrac{A \bowtie^{f,g} (J,J')}{0 \times J'} \cong f(A)+J$ and $\dfrac{A \bowtie^{f,g} (J,J')}{J \times 0} \cong g(A)+J'$.

(3) $\dfrac{A}{I_o} \cong \dfrac{A \bowtie^{f,g} (J,J')}{J \times J'} \cong \dfrac{f(A)+J}{J} \cong \dfrac{g(A)+J'}{J'}$.

Proposition 10.11. [85, Proposition 4.2] Under the above notation, we have:

$A \bowtie^{f,g} (J,J')$ is Noetherian if and only if $f(A)+J$ and $g(A)+J'$ are Noetherian.

As an illustrative example for Proposition 10.11 (of an original Noetherian ring which arises as a bi-amalgamation) is provided in Example 10.16.

Recall that the prime spectrum of a ring R is said to be Noetherian if R satisfies the ascending chain condition on radical ideals (or, equivalently, every prime ideal of R is the radical of a finitely generated ideal) [101].

Proposition 10.12. [85, Proposition 4.4] Under the above notation, we have:

$\mathrm{Spec}\big(A \bowtie^{f,g} (J,J')\big)$ is Noetherian if and only if $\mathrm{Spec}\big(f(A)+J\big)$ and $\mathrm{Spec}\big(g(A)+J'\big)$ are Noetherian.

The next result characterizes bi-amalgamations without zero divisors.

Proposition 10.13. [85, Proposition 4.5] Under the above notation, the following assertions are equivalent:

(1) $A \bowtie^{f,g} (J,J')$ is a domain;

(2) either $J = 0$ and $g(A)+J'$ is a domain or $J' = 0$ and $f(A)+J$ is a domain.

The next result characterizes bi-amalgamations without nilpotent elements.

Proposition 10.14. [85, Proposition 4.7] Under the above notation, consider the following conditions:

(a) $f(A)+J$ is reduced and $J'\cap \mathrm{Nil}(C)=0$,

(b) $g(A)+J'$ is reduced and $J\cap \mathrm{Nil}(B)=0$,

(c) $A\bowtie^{f,g}(J,J')$ is reduced,

(d) $J\cap \mathrm{Nil}(B)=0$ and $J'\cap \mathrm{Nil}(C)=0$.

Then:

(1) $(a)\Rightarrow(c)\Rightarrow(d)$ and $(b)\Rightarrow(c)$.

(2) If I_o is radical, then the above four conditions are equivalent.

(3) Assume that f is surjective and $\mathrm{Ker}(f)\subseteq \mathrm{Ker}(g)$. Then:

$$A\bowtie^{f,g}(J,J') \text{ is reduced if and only if } B \text{ is reduced and } J'\cap \mathrm{Nil}(C)=0.$$

Remark 10.15. [85, Remark 4.8] If $f(A)+J$ and $g(A)+J'$ are both reduced, then $A\bowtie^{f,g}(J,J')$ is reduced by Proposition 10.14. The converse is not true in general. A counter-example (for the special case of amalgamated algebras) is given in [49, Remark 5.5 (3)].

As an illustrative example for Propositions 10.11, 10.13, and 10.14, we provide an original reduced Noetherian ring with zero divisors which arises as a bi-amalgamation.

Example 10.16. [85, Example 4.10] Consider the surjective ring homomorphism $f:\mathbb{Z}[X]\twoheadrightarrow\mathbb{Z}[\sqrt{2}]$, $p(X)\mapsto p(\sqrt{2})$ and the principal ideal $J:=(\sqrt{2})$ of $\mathbb{Z}[\sqrt{2}]$. Let $p\in\mathbb{Z}[X]$ and write it as $p=(X^2-2)q(X)+aX+b$ for some $a,b\in\mathbb{Z}$ and $q\in\mathbb{Z}[X]$. Then one can verify that $p(\sqrt{2})\in J$ if and only if $b\in 2\mathbb{Z}$. That is,

$$I_o:=f^{-1}(J)=\left\{p\in\mathbb{Z}[X]\mid p(0)\in 2\mathbb{Z}\right\}.$$

Now consider the ring homomorphism $\alpha:\mathbb{Z}[\sqrt{2}]\twoheadrightarrow\dfrac{\mathbb{Z}[X]}{I_o}$, $a+b\sqrt{2}\mapsto\bar{a}$. It follows, by Proposition 10.6 and Propositions 10.11 & 10.13 & 10.14, that

$$\mathbb{Z}[X]\bowtie^{f,f}(J,J)=\alpha\times_{\frac{\mathbb{Z}[X]}{I_o}}\alpha=\left\{(a+b\sqrt{2},c+d\sqrt{2})\mid a,b,c,d\in\mathbb{Z},\ a-c\in 2\mathbb{Z}\right\}$$

is a reduced Noetherian ring that is not a domain (since $\mathbb{Z}[\sqrt{2}]$ is a Noetherian domain and $J\neq 0$).

Throughout, let $f:A\to B$ and $g:A\to C$ be two ring homomorphisms and J,J' two ideals of B and C respectively such that $I_o:=f^{-1}(J)=g^{-1}(J')$. Let

$$A\bowtie^{f,g}(J,J'):=\left\{(f(a)+j,g(a)+j')\mid a\in A,(j,j')\in J\times J'\right\}$$

be the bi-amalgamation of A with (B,C) along (J,J') with respect to (f,g).

Now we describe the prime ideal structure of bi-amalgamations and their localizations at prime ideals. We also establish necessary and sufficient conditions for a bi-amalgamation to be local.

Next we describe the prime (and maximal) ideals of bi-amalgamations. For this purpose, let's adopt the following notation:

$$\begin{aligned} Y &:= \mathrm{Spec}(f(A)+J)\\ Y' &:= \mathrm{Spec}(g(A)+J') \end{aligned}$$

and, for $L \in Y$ and $L' \in Y'$, consider the prime ideals of $A \bowtie^{f,g} (J,J')$ given by:

$$\begin{aligned} \bar{L} &:= \Big(L \times (g(A)+J')\Big) \cap \Big(A \bowtie^{f,g} (J,J')\Big) \\ &= \Big\{(f(a)+j, g(a)+j') \mid a \in A, (j,j') \in J \times J', f(a)+j \in L\Big\}, \\ \bar{L'} &:= \Big((f(A)+J) \times L'\Big) \cap \Big(A \bowtie^{f,g} (J,J')\Big) \\ &= \Big\{(f(a)+j, g(a)+j') \mid a \in A, (j,j') \in J \times J', g(a)+j' \in L'\Big\}. \end{aligned}$$

Proposition 10.17. [85, Proposition 5.3] Under the above notation, let P be a prime ideal of $A \bowtie^{f,g} (J,J')$. Then

(1) $J \times J' \subseteq P$ if and only if there exists a unique $\mathfrak{p} \supseteq I_o$ in Spec(A) such that $P = \mathfrak{p} \bowtie^{f,g} (J,J')$. In this case, there exist $L \supseteq J$ in Y and $L' \supseteq J'$ in Y' such that $P = \bar{L} = \bar{L'}$.

(2) $J \times J' \nsubseteq P$ if and only if there exists a unique $L \in Y$ (or Y') such that $J \nsubseteq L$ (or $J' \nsubseteq L$) and $P = \bar{L}$. In this case, $(A \bowtie^{f,g} (J,J'))_P \cong (f(A)+J)_L$ $\Big($ or $(A \bowtie^{f,g} (J,J'))_P \cong (g(A)+J')_L\Big)$.

Consequently, we have

$$\mathrm{Spec}\Big(A \bowtie^{f,g} (J,J')\Big) = \Big\{\bar{L} \mid L \in \mathrm{Spec}\Big(f(A)+J\Big) \cup \mathrm{Spec}\Big(g(A)+J'\Big)\Big\}.$$

Next, as an application of Proposition 10.17, we establish necessary and sufficient conditions for a bi-amalgamation to be local. Notice at this point that, in the presence of the equality $f^{-1}(J) = g^{-1}(J')$, $J \neq B$ if and only if $J' \neq C$.

Proposition 10.18. [85, Proposition 5.4] Under the above notation, we have

(1) $A \bowtie^{f,g} (J,J')$ is local if and only if $J \neq B$ and $f(A)+J$ and $g(A)+J'$ are local. Moreover, the maximal ideal of $A \bowtie^{f,g} (J,J')$ has the form $\mathfrak{m} \bowtie^{f,g} (J,J')$, where $\mathfrak{m}$ is the unique maximal ideal of A containing I_o.

(2) Suppose that A is local. Then $A \bowtie^{f,g} (J,J')$ is local *if and only if* $J \times J' \subseteq \mathrm{Jac}(B \times C)$.

In view of Example 10.1, Proposition 10.18 recovers the special case of amalgamated algebras and amalgamated duplications, as recorded in the next corollaries.

Corollary 10.19. [85, Corollary 5.5] Under the above notation, the following assertions are equivalent:

(1) $A \bowtie^{f} J$ is local;

(2) $J \neq B$, and A and $f(A)+J$ are local;

(3) A is local and $J \subseteq \mathrm{Jac}(B)$.

Next we describe the localizations of $A \bowtie^{f,g} (J,J')$ at its prime ideals which contain $J \times J'$. Recall that, given a ring R, an ideal I of R, and S a multiplicative subset of R with $S \cap I = \emptyset$, it follows that $S+I$ is a multiplicative subset of R.

Proposition 10.20. [85, Proposition 5.7] Let $\mathfrak{p}$ be a prime ideal of A containing I_o and set $P := \mathfrak{p} \bowtie^{f,g} (J,J')$. Consider the multiplicative subsets $S := f(A \setminus \mathfrak{p}) + J$ of B and $S' := g(A \setminus \mathfrak{p}) + J'$ of C. Let $f_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_S$ and $g_{\mathfrak{p}} : A_{\mathfrak{p}} \to C_{S'}$ be the ring homomorphisms induced by f and g respectively. Then

$$f_{\mathfrak{p}}^{-1}(J_S) = g_{\mathfrak{p}}^{-1}(J'_{S'}) = (I_o)_{\mathfrak{p}}$$

and

$$\Big(A \bowtie^{f,g} (J,J')\Big)_P \cong A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}},g_{\mathfrak{p}}} (J_S, J'_{S'}).$$

Remark 10.21. [85, Remark 5.8] If P is a prime ideal of $A \bowtie^{f,g} (J,J')$ which contains $J \times J'$, then by Proposition 10.17, there exists a (unique) prime ideal $\mathfrak{p}$ (which contains I_o) such that $P = \mathfrak{p} \bowtie^{f,g} (J,J')$. Thus by Proposition 10.7 and Proposition 10.20, one can obtain a conductor square of the form:

$$
\begin{array}{ccc}
(A \bowtie^{f,g} (J,J'))_P & \xrightarrow{\iota_2} & (f_{\mathfrak{p}}(A_{\mathfrak{p}}) + J_S) \times (g_{\mathfrak{p}}(A_{\mathfrak{p}}) + J'_{S'}) \\
\Big\downarrow{\scriptstyle \mu_2} & & \Big\downarrow{\scriptstyle \mu_1} \\
\dfrac{A_{\mathfrak{p}}}{I_o A_{\mathfrak{p}}} & \xrightarrow{\iota_1} & \dfrac{A_{\mathfrak{p}}}{I_o A_{\mathfrak{p}}} \times \dfrac{A_{\mathfrak{p}}}{I_o A_{\mathfrak{p}}}
\end{array}
$$

References

[1] K. I. Alaoui and N. Mahdou, Coherence in amalgamated algebra along an ideal, Bull. Iranian. Math. Soc. 41(3) (2015), 1–9.

[2] K. I. Alaoui and N. Mahdou, Finite conductor property in amalgamated algebra, J. Taibah Univ. Sci. 9 (2015), 332–339.

[3] K. I. Alaoui and N. Mahdou, On (n,d)-property in amalgamated algebra, Asian Eur. J. Math. 9(1) (2016), 1650014.

[4] K. I. Alaoui and N. Mahdou, n-coherence property in amalgamated algebra along an ideal, Acta Math. Univ. Comenianae 86(1) (2017), 59–72.

[5] K. I. Alaoui, D.E. Dobbs and N. Mahdou, Commutative rings and modules that are Nil_*-coherent or special Nil_*-coherent, J. Algebra Appl, 16(10) (2017), 1750187.

[6] F. A. A. Almahdi and M. Tamekkante, Short note on Cohen-Macaulay and Gorenstein amalgamated duplication, Palest. J. Math. 7(1) (2018), 6–10.

[7] H. Ananthnarayan, L. Avramov and W. Frank Moore, Connected sums of Gorenstein local rings, J. Reine Angew. Math. 2012(667) (2011), 149–176.

[8] D. D. Anderson and M. Winders, Idealization of a module, J. Commut. Algebra 1(1) (2009), 3–56.

[9] D. D. Anderson and M. Bataineh, Generalization of prime ideals, Comm. Algebra 36(2) (2008), 686–696.

[10] D. D. Anderson and E. Smith, Weakly prime ideals, Houston J. Math. 29(4) (2003), 831–840.

[11] D. D. Anderson and M. Zafrullah, Almost Bézout domains, J. Algebra 142(2) (1991), 285Ű-309.

[12] D. F. Anderson, Comparability of ideals and valuation overrings, Houston J. Math. 5(4) (1979), 451–463.

[13] D. F. Anderson, When the dual of an ideal is a ring, Houston J. Math. 9(4) (1989), 325–332.

[14] D. F. Anderson, A. Badawi and D. E. Dobbs, Pseudo-valuation rings, in Commutative Ring Theory (Fès, 1995), Lecture Notes Pure Appl. Math. 185(1997), Dekker, New York, 57–67.

[15] M. F. Atiyah and I. G. MacDonald, Introduction to Commutative Algebra, Addison-Wesley, Reading, 1969.

[16] Y. Azimi, P. Sahandi and N. Shirmohammadi, Prüfer conditions under the amalgamated construction, Commun. Algebra 47(5) (2019), 2251–2261.

[17] A. Badawi, On domains which have prime ideals that are linearly ordered, Comm. Algebra 23(12) (1995), 4365–4373.

[18] A. Badawi, On divided commutative rings, Comm. Algebra 27(3), (1999), 1465–1474.

[19] A. Badawi, Pseudo-valuation domains: A survey, Proceedings of the Third Palestinian International Conference on Mathematics, 38-59, Word Scientific Publishing Co., New York-London, 2002.

[20] A. Badawi, On pseudo-almost valuation domains, Comm. Algebra 35(4) (2007), 1167–1181.

[21] A. Badawi, On weakly semiprime ideals of commutative rings, Beir Algebra Geom 57 (2016), 589–597.

[22] A. Badawi, D. F. Anderson and D. E. Dobbs, Pseudo-valuation rings, 57–67, in Commutative Ring Theory (Fès, 1995), Lecture Notes Pure Appl. Math. 185, Dekker, New York, 1997.

[23] J. Bagheri Harehdashti and H. Fazaeli Moghimi, A generalization of the prime radical of ideals in commutative rings, Commun. Korean Math. Soc. (32)(3) (2017), 543–552.

[24] C. Bakkari, S. Kabbaj and N. Mahdou, Trivial extensions defined by Prüfer conditions, J. Pure App. Algebra 214(1) (2010), 53–60.

[25] H. Bass, Finitistic dimension and a homological characterisation of semiprimary rings, Trans. Amer. Math. Soc. 95 (1960), 466–488.

[26] H. Bass, Lectures on Topics in Algebraic K-Theory, Notes by A. Roy, Tata Institute of Fundamental Research, Bombay (Mumbai), 1967.

[27] E. Bastida and R. Gilmer, Overing and divisorial ideals in rings of the form $D+M$, Michigan Math. J. 20 (1973), 79–95.

[28] S. Bazzoni and S. Glaz, Gaussian properties of total rings of quotients, J. Algebra 310 (2007), 180–193.

[29] A. J. Berrick and M. E. Keating, An introduction to Rings and Modules with K-theoryin view (1st ed.), Cambridge Univ. Press, 2000.

[30] S. M. Bhatwadekar and P. K. Sharma, Unique factorization and birth of almost primes, Comm. Algebra 33(1) (2005), 43–49.

[31] M. Boisen Jr. and P. Sheldon, CPI-extension: overrings of integral domains with special prime spectrum, Canad. J. Math. 29 (1977) 722–737.

[32] E. M. Bouba, N. Mahdou and M. Tamekkante, Duplication of a module along an ideal, Acta Math. Hungar. 154 (2018), 29–42.

[33] N. Bourbaki, Algèbre Commutative, Chaps. 1-2, Eléments de Mathématique XXVII, Hermann, Paris, 1961.

[34] W. Bruns and J. Herzog, Cohen-Macaulay Rings (revised ed.), Cambridge Univ. Press, 1998.

[35] P. J. Cahen, M. Fontana, S. Frisch and S. Glaz, Open problems in commutative ring theory, Commutative Algebra. Springer, (2014), 353–375.

[36] H. Cartan and S. Eilenberg, Homological Algebra, Princeton Univ. Press, Princeton, 1956.

[37] S. U. Chase, Direct products of modules, Trans. Amer. Math. Soc. 97 (1960), 457–473.

[38] M. Chhiti, N. Mahdou and M. Tamekkante, Self injective amalgamated duplication of a ring along an ideal, J. Algebra Appl. 12(7) (2013), 1350033.

[39] M. Chhiti, N. Mahdou and M. Tamekkante, Clean property in amalgamated algebras along an ideal, Hacet. J. Math. Stat. 44(1) (2015), 41–49.

[40] M. Chhiti and N. Mahdou, Some homological properties of amalgamated duplication of a ring along an ideal, Bull. Iranian Math. Soc. 38(2) (2012), 507–515.

[41] H. S. Choi and A. Walker, The radical of an n-absorbing ideal, Comm. Algebra, 12(2) (2020), 171–177.

[42] B. S. Chwe and J. Neggers, Local rings with left vanishing radical, J. London Math. Soc. 4 (1971), 374–378.

[43] D. Costa, Parameterizing families of non-Noetherian rings, Comm. Algebra 22 (1994), 3997–4011.

[44] D. Costa and S. Kabbaj, Classes of $D+M$ rings defined by homological conditions, Comm. Algebra 24 (1996), 891–906.

[45] S. H. Cox and R. L. Pendleton, Rings for which certain flat modules are projective, Trans. Amer. Math. Soc. 150 (1970), 139–156.

[46] M. D'Anna, A construction of Gorenstein rings, J. Algebra 306(2) (2006), 507–519.

[47] M. D'Anna and M. Fontana, Amalgamated duplication of a ring along a multiplicative-canonical ideal, Ark. Mat. 45(2) (2007), 241–252.

[48] M. D'Anna and M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, J. Algebra Appl. 6(3) (2007), 443–459.

[49] M. D'Anna, C.A. Finocchiaro and M. Fontana, Amalgamated algebras along an ideal, in: Commutative Algebra and Applications, Proceedings of the Fifth International Fez Conference on Commutative Algebra and Applications, Fez, Morocco, 2008, W. de Gruyter Publisher, Berlin, 2009, pp. 155–172.

[50] M. D'Anna, C. A. Finocchiaro and M. Fontana, New algebraic properties of an amalgamated algebra along an ideal, Comm. Algebra (44) (2016), 1836–1851.

[51] M. D'Anna, C. A. Finocchiaro and M. Fontana, Properties of chains of prime ideals in amalgamated algebras along an ideal, J. Pure Appl. Algebra (214) (2010), 1633–1641.

[52] D. E. Dobbs, Remarks on faithful flatness, Portugal. Math. 32 (1973), 53–57.

[53] D. E. Dobbs, S. Kabbaj and N. Mahdou, n-coherent rings and modules, Lecture Notes in Pure and Appl. Math., Dekker, 185 (1997), 269–281.

[54] D. E. Dobbs, S. Kabbaj, N. Mahdou and M. Sobrani, When is $D+M$ n-coherent and an (n,d)-domain?, Lecture Notes in Pure and Appl. Math., Dekker, 205 (1999), 257–270.

[55] D. E. Dobbs and I. Papick, When is $D+M$ coherent?, Proc. Amer. Math. Soc. 56 (1976), 51–54.

[56] J. L. Dorroh, Concerning adjunctions to algebras, Bull. Amer. Math. Soc. 38 (1932), 85–88.

[57] J. L. Dorroh, Concerning the direct product of algebras, Ann. Math. 36 (1935), 882–885.

[58] S. Ebrahimi Atani and F. Farzalipour, On weakly primary ideals, Georgian Math. J. 12(3) (2005), 423–429.

[59] R. El Khalfaoui and N. Mahdou, The ϕ-Krull dimension of some commutative extensions, Comm. Algebra, 48(9) (2020), 3800–3810.

[60] R. El Khalfaoui, N. Mahdou, P. Sahandi and N. Shirmohammadi, Amalgamated modules along an ideal, Commun. Korean Math. Soc. 36(1) (2021), 1–10.

[61] R. El Khalfaoui, N. Mahdou and S. Yassemi, Local dimension of trivial extension and amalgamation of rings, J. Algebra Appl. Accepted for publication.

[62] A. El Khalfi and N. Mahdou, Amalgamated algebras along an ideal defined by valuation conditions, submitted.

[63] A. El Khalfi and N. Mahdou, Almost Bézout rings and almost GCD-rings, Asian-Eur. J. Math. 13(6) (2020), 2050107 (10 pages).

[64] A. El Khalfi, A. R. Hassani and N. Mahdou, Bi-amalgamation of rings defined by certain annihilator conditions, Rocky Mountain J. Math. 50(3) (2020), 947–956.

[65] A. El Khalfi, H. Kim and N. Mahdou, Amalgamated algebras issued from ϕ-chained rings and ϕ-PVRs, Bull. Iranian Math. Soc. 47 (2021), 1599–1609.

[66] M. El Maalmi and H. Mouanis, On Steinitz-like properties in amalgamated algebras along an ideal, J. Algebra Appl. 19(12) (2020), 2050237.

[67] M. Fontana, Topologically defined classes of commutative rings, Ann. Mat. Pura Appl. 123 (1980), 331-Ű355.

[68] M. Fontana, J. A. Huckaba and I. J. Papick, Prüfer Domains, Marcel Dekker, New York, 1997.

[69] R. Fossum, Commutative extensions by canonical modules are Gorenstein rings, Proc. Amer. Math. Soc. 40 (1973), 395–400.

[70] R. Fossum, P. Griffith and I. Reiten, Trivial extensions of Abelian categories. Homological algebra of trivial extensions of Abelian categories with applications to ring theory, Lecture Notes in Mathematics 456, Springer-Verlag, Berlin, 1975.

[71] A. Ghorbani and M. Naji Esfahani, Local dimension of coatomic modules, Comm. Algebra 43(12) (2015), 5205–5216.

[72] R. Gilmer, Multiplicative Ideal Theory, M. Dekker, New York, 1972.

[73] S. Glaz, Commutative Coherent Rings, Lecture Notes in Math. 1371, Springer-Verlag, Berlin, 1989.

[74] M. E. Harris, Some results on coherent rings, Proc. Amer. Math. Soc. 17 (1966), 474–479.

[75] M. E. Harris, Some results on coherent rings II, Glasgow Math. J. 8 (1967), 123–126.

[76] J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, Pacific J. Math. 75(1) (1978), 137–147.

[77] S. Hizem and A. Benhissi, When is $A + XB[[X]]$ Noetherian?, C. R. Acad. Sci. Paris, Ser. I 340 (2005), 5–7.

[78] J. A. Huckaba, Commutative Rings with Zero Divisors, Dekker, New York, 1988.

[79] M. Issoual, N. Mahdou and M. Moutui, On (m,n)-closed ideals in amalgamated algebra, Int. Electron. J. Algebra 29 (2021), 134–147.

[80] M. Issoual, N. Mahdou and M. A. Moutui, On n-absorbing and strongly n-absorbing ideals of amalgamation, J. Algebra Appl. 19(10) (2020), 2050199.

[81] R. Jahani-Nezhad and F. Khoshayand, Pseudo-almost valuation rings, Bull. Iranian Math. Soc. 41(4) (2015), 815–824.

[82] R. Jahani-Nezhad and F. Khoshayand, Almost valuation rings, Bull. Iranian Math. 43(3) (2017), 807–816.

[83] S. Kabbaj and N. Mahdou, Trivial extensions of local rings and a conjecture of Costa, Lecture Notes in Pure and Appl. Math., Dekker, 231 (2003), 301–311.

[84] S. Kabbaj and N. Mahdou, Trivial extensions defined by coherent-like conditions, Comm. Algebra 32(10) (2004), 3937–3953.

[85] S. Kabbaj, K. Louartiti and M. Tamekkante, Bi-amalgamated algebras along ideals, J. Commu. Algebra 9(1) (2017), 65–87.

[86] S. Kabbaj and A. Mimouni, Zero-divisor graphs of amalgamations, Math. Scand. 123(2) (2018), 174–190.

[87] M. Kabbour and N. Mahdou, Arithmetical property in amalgamated algebras along an ideal, Palest. J. Math. 3(1) (2014), 395–399.

[88] I. Kaplansky, Commutative Rings, The University of Chicago Press, Chicago, 1974.

[89] K. Lourtiti and N. Mahdou, Transfer of multiplication-like conditions in amalgamated algebra along an ideal, Afr. Diaspora J. Math. 14(1) (2012), 119–125.

[90] N. Mahdou, On Costa's conjecture, Comm. Algebra 29 (2001), 2775–2785.

[91] N. Mahdou, On 2-von Neumann regular rings, Comm. Algebra 33(10) (2005), 3489–3496.

[92] N. Mahdou, Sufficient condition to resolve Costa's first conjecture, Comm. Algebra 38(3) (2010), 1066–1074.

[93] N. Mahdou, A. Mimouni and M. A. S. Moutui, On almost valuation and almost Bézout rings, Comm. Algebra 43(1) (2015), 297–308.

[94] N. Mahdou, S. Moussaoui and S. Yassemi, The divided, going-down, and Gaussian properties of amalgamation of rings, Comm. Algebra 49(5) (2021), 1938–1949.

[95] N. Mahdou and M. A. Moutui, Amalgamated algebras along an ideal defined by Gaussian condition, J. Taibah Univ. Sci. 9 (2015), 373–379.

[96] N. Mahdou and M. Tamekkante, On an amalgamated duplication of a ring along an ideal which are quasi-Frobenius, Beitr Algebra Geom 52 (2011), 343–347.

[97] H. R. Maimani and S. Yassemi, Zero divisor graph of amalgamated duplication of a ring along an ideal, J. Pure Appl. Algebra 212 (2008), 168–174.

[98] A. Mimouni, Prüfer-like conditions and pullbacks, J. Algebra 279(2) (2004), 685–693.

[99] A. Molkhasi and K. P. Shum, On Nagata rings with amalgamation properties, Asian-Eur. J. Math. 14(8) (2021), 2150136.

[100] M. Nagata, Local Rings, Interscience, New York, 1962.

[101] J. Ohm and R. L. Pendleton, Rings with Noetherian spectrum, Duke Math. J. 35 (1968), 631–639.

[102] C. Pedrini, Incollamenti di ideali primi e gruppi di Picard, Rend. Sem. Math. Univ. Padova 48 (1973), 39–66.

[103] J. J. Rotman, An Introduction to Homological Algebra, Academic Press, New York, 1979.

[104] J. Shapiro, On a construction of Gorenstein rings proposed by M. D'Anna, J. Algebra 323 (2010), 1155–1158.

[105] G. Tamone, Sugli incollamenti di ideali primi, Bollettino U.M.I. 14 (1977), 810–825.

[106] Y. Tiras, A. Tercan and A. Harmanci, Prime modules, Honam Math. J. 18 (1996), 5–15.

[107] C. Traverso, Seminormality and Picard group, Ann. Sc. Norm. Sup. Pisa 24 (1970), 585–595.

[108] H. Tsang, Gauss's Lemma, Ph.D. Thesis, University of Chicago, Chicago, 1965.

[109] Y. Xiang and L. Ouyang, Nil_*-coherent rings, Bull. Korean Math. Soc. 51 (2014), 579–594.

[110] H. Yanagihara, On glueings of prime ideals, Hiroshima Math. J. 10(2) (1980), 351–363.

[111] D. Zhou, On n-coherent rings and (n,d)-rings, Comm. Algebra 32 (2004), 2425–2441.