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Author(s):

Sana Hizem

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Formal power series rings with one absorbing factorization

Sana Hizem
Department of Mathematics, Faculty of Sciences, University of Monastir, Tunisia
e-mail: *hizems@yahoo.fr*

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Abstract. Let R be a commutative ring with identity. A proper ideal I of R is said to be 1- absorbing prime ideal if whenever $xyz \in I$ for some nonunit elements $x,y,z \in R$, then either $xy \in I$ or $z \in I$. The ring R is called a 1- absorbing prime factorization ring (OAF - ring) if every proper ideal has an OA - factorization. In this note, we characterize commutative rings R (respectively, ring extensions $A \subset B$) for which the ring of formal power series $R[[X]]$ (respectively, the ring $A + XB[X]$ or $A + XB[[X]]$) is an OAF - ring.

Key Words: OA - ideals - OAF - rings - formal power series rings.
2010 MSC: 13A15; 13B25; 3F25; 13F15; 13B99.

1 Introduction

All rings considered in this paper are commutative with identity. In [10], Yassine et al. introduced the concept of 1- absorbing prime ideal in the following way: a proper ideal I of a commutative ring R is a 1- absorbing prime ideal (OA - ideal) if whenever x,y,z are nonunit elements of R such that $xyz \in I$, then either $xy \in I$ or $z \in I$. Note that a prime ideal is 1- absorbing prime. They prove that if R admits a 1- absorbing prime ideal that is not a prime ideal, then R is a quasilocal ring. They also prove that the radical of a 1- absorbing prime ideal is a prime ideal and characterize 1- absorbing prime ideals of PID's, valuation domains and Prüfer domains. Recall also from [1], that an ideal I of R is a 2- absorbing ideal if whenever $x,y,z \in R$ and $xyz \in I$, then $xy \in I$ or $xz \in I$ or $yz \in I$, so any 1- absorbing prime ideal is a 2- absorbing ideal. TA- factorization rings that is rings in which every proper ideal can be written as a finite product of 2- absorbing ideal were investigated in [7].

In [3], El Khalfi et al. studied commutative rings whose proper ideals have an OA - factorization. If I is a proper ideal of R, then an OA - factorization of I is an expression of I as a finite product $\prod_{i=1}^{n} I_i$ of OA - ideals. The ring R is called a 1- absorbing prime factorization ring (OAF - ring) if every proper ideal has an OA - factorization. They prove that if R is a non local ring then R is a general Z.P.I ring if and only if R is an OAF - ring. Recall that a general Z.P.I ring R is a ring whose proper ideals can be written as a product of prime ideals which is by [9] equivalent to the fact that R satisfies the following two conditions:

1. R is Noetherian.
2. Each maximal ideal M of R is simple that is there exist no ideals properly between M^2 and M.

In case of a local domain (R,M), they show that R is OAF if and only if R is atomic such that M^2 is universal.

Moreover, they investigate rings whose proper principal ideals have an OA - factorization and rings whose proper (principal) ideals are OA - ideals.

In the first section of this paper, we characterize commutative rings R such that the ring of formal power series $R[[X]]$ is an OAF - ring. More precisely, we prove that $R[[X]]$ is OAF if and only if R is a finite product of fields.

In the second (respectively, third) part of this paper, we consider composite rings of the form $A + XB[X]$ (respectively, $A + XB[[X]]$) where $A \subset B$ is an extension of commutative rings. Recall that $A + XB[X] = \{f \in B[X] \mid f(0) \in A\}$ (respectively, $A + XB[[X]] = \{f = \sum_{i=0}^{+\infty} a_i X^i \in B[[X]] \mid a_0 \in A\}$). These rings which are special cases of pullbacks are very useful in order to construct examples and counterexamples in commutative algebra.
We prove that if $A \subset B$ is an extension of commutative rings, then the ring $A + XB[X]$ is OAF if and only if $A = B$ and A is a finite product of fields. For the formal power series case, we show that if A is a local ring, then $A + XB[[X]]$ is OAF if and only if $A \subset B$ is an extension of fields and if A is not local then $A + XB[[X]]$ is OAF if and only if $A = B$ and A is a finite direct product of fields.

Let R be a commutative ring, $\dim(R)$ denotes the Krull dimension of R, $Max(R)$ denotes the set of maximal ideals of R, $Spec(R)$ denotes the set of prime ideals of R, $Nil(R)$ denotes the nilradical of R and $R[[X]]$ denotes the ring of formal power series over R.

2 OAF formal power series rings.

Let R be a commutative ring and $R[[X]]$ be the ring of formal power series in one indeterminate. It is clear by [[3], Proposition 2.2] that if $R[[X]]$ is OAF then so is the ring R as $R \sim R[[X]]/XR[[X]]$.

In the next theorem, we characterize those rings for which $R[[X]]$ is OAF.

Theorem 2.1. Let R be a commutative ring. The ring $R[[X]]$ is OAF if and only if R is a finite direct product of fields.

Proof. 1. First case: Suppose that R is a non local ring, so $R[[X]]$ is also a non local ring and $Max(R[[X]]) = \{M + XR[[X]] \mid M \in Max(R)\}$. By [[3], Remark 2.4], $R[[X]]$ is an OAF - ring if and only if R is a general Z.P.I ring, which is also equivalent by [[9], Theorem 3] to the fact that $R[[X]]$ is Noetherian and for every $M \in Max(R)$, the ideal $M + XR[[X]]$ is a simple ideal that is there exists no ideal properly between $(M + XR[[X]])^2$ and $M + XR[[X]]$. Note that the ring $R[[X]]$ is Noetherian if and only if R is. On the other hand, if $M \in Max(R)$ then $(M+XR[[X]])^2 = M^2+XM+X^2R[[X]]$. Let $I = M+XM+X^2R[[X]]$ then I is an ideal of $R[[X]]$ and $(M + XR[[X]])^2 \subset I \subsetneq M + XR[[X]]$, so if $M + XR[[X]]$ is simple then $M^2 = M$. Consequently, if $R[[X]]$ is an OAF - ring then R is Noetherian and for each $M \in Max(R)$, $M^2 = M$ which implies that R is isomorphic to a finite direct product of fields by [[8],Theorem 3.2]. Conversely, we prove that if for each $M \in Max(R)$, $M^2 = M$ then the ideals $M + XR[[X]]$ are simple. In fact, let I be an ideal of $R[[X]]$ such that $(M + XR[[X]])^2 \subset I \subset M + XR[[X]]$ and let $I_1 = \{a \in R \mid$ there exists $f = \sum_{i=0}^{+\infty} a_i X^i \in I$ with $a_1 = a\}$. Then I_1 is an ideal of R and $M \subset I_1 \subset R$. As M is a maximal ideal of R then $I_1 = M$ or $I_1 = R$. If $I_1 = M$ then we show that $I = M + XM + X^2R[[X]]$. In fact $M + XM + X^2R[[X]] \subset I$. Conversely, let $f = \sum_{i=0}^{+\infty} a_i X^i \in I \subset M + XR[[X]]$, then $a_1 \in I_1 = M$ so $f \in M + XM + X^2R[[X]]$. If $I_1 = R$, then we show that $I = M + XR[[X]]$. In fact, $I \subset M + XR[[X]]$. Conversely, $M + XM + X^2R[[X]] \subset I$, so $M \subset I$ and $X^2R[[X]] \subset I$. We prove that $X \in I$. As $I_1 = R$, then $1 \in I_1$, so there exists $g = \sum_{i=0}^{+\infty} b_i X^i \in I$ with $b_1 = 1$, which implies that $X = g - b_0 - \sum_{i=2}^{+\infty} b_i X^i \in I$. So $XR \subset I$ and $M + XR[[X]] \subset I$.
In conclusion, if R is not local then $R[[X]]$ is an OAF - ring if and only if R is a finite direct product of fields.

2. Second case: Suppose that R is a local ring, so $R[[X]]$ is also local. We prove that $R[[X]]$ is an OAF - ring if and only if R is a field. Suppose that $R[[X]]$ is an OAF - ring, as $\dim R[[X]] \geq 1$ then

by [[3], Proposition 3.3], $R[[X]]$ is an integral domain, so R is an integral domain. Moreover, by [[3], Theorem 3.5], we have $\dim R[[X]] \leq 1$. So $\dim R[[X]] = 1$. As $\dim R[[X]] \geq 1 + \dim R$ then $\dim R = 0$. So R is an integral domain with Krull dimension equal to zero and then it is a field. Conversely, if R is a field then the set of ideals of $R[[X]]$ is $\{0\} \cup \{X^n R[[X]] \mid n \in \mathbb{N}\}$ and $Spec(R[[X]]) = \{(0), XR[[X]]\}$. So every proper ideal of $R[[X]]$ is a finite product of prime ideals. As each prime ideal is an OA - ideal then every proper ideal of $R[[X]]$ is a finite product of OA - ideals and $R[[X]]$ is an OAF - ring.

□

From the preceding proof, we can conclude that for a non local commutative ring R, the ring $R[[X]]$ is general Z.P.I if and only if R is a finite direct product of fields. Note that the proof is also available in the case of a local ring R. So we get the following proposition.

Proposition 2.2. *Let R be a commutative ring with identity. The ring $R[[X]]$ is general Z.P.I if and only if R is a finite direct product of fields.*

Remark 2.3. We finish this section with some remarks on 1- absorbing prime ideals of the form $I[[X]]$ and $I + XR[[X]]$ of $R[[X]]$, where I is a proper ideal of the ring R.

1. Note that if R is not local then $I[[X]]$ (respectively, $I + XR[[X]]$) is 1- absorbing prime if and only if it is a prime ideal which is equivalent to the fact that I is a prime ideal of R.

2. If (R, M) is a local ring then $I[[X]]$ is 1- absorbing prime if and only if $I[[X]]$ is prime or $(M + XR[[X]])^2 \subset I[[X]] \subset M + XR[[X]]$. But $I[[X]]$ is prime if and only if I is prime. On the other hand $(M + XR[[X]])^2 \subset I[[X]]$ implies that $X^2 R[[X]] \subset I[[X]]$ which is impossible as I is a proper ideal of R. So in the case of a local ring, the ideal $I[[X]]$ is 1- absorbing prime if and only if I is prime.

3. If (R, M) is a local ring then $I + XR[[X]]$ is 1- absorbing prime if and only $I + XR[[X]]$ is prime or $(M + XR[[X]])^2 \subset I + XR[[X]] \subset M + XR[[X]]$ which is equivalent to the fact that I is a prime ideal of R or $M^2 \subset I \subset M$ that is I is a 1- absorbing ideal of R.

3 OAF - rings of the form $A + XB[X]$.

In [3], the authors prove that for a commutative ring R, the ring $R[X]$ is an OAF - ring if and only if R is a finite direct product of fields. In this section we investigate OAF - rings of the form $A + XB[X]$, where $A \subset B$ is an extension of commutative rings. Note that the ring $A + XB[X]$ is never local since X and $1 + X$ are nonunit elements of $A + XB[X]$. So $A + XB[X]$ is OAF if and only if it is general Z.P.I which is equivalent to the fact that $A + XB[X]$ is Noetherian and each maximal ideal of $A + XB[X]$ is simple. By [[5], Proposition 2.1] the ring $A + XB[X]$ is Noetherian if and only if A is Noetherian and B is a finitely generated A- module.

Proposition 3.1. *Let $A \subset B$ be an extension of commutative rings. If $A + XB[X]$ is OAF then A is a finite direct product of fields.*

Proof. By the preceding remarks, if $A + XB[X]$ is OAF then A is Noetherian. We prove first that $\dim A = 0$. Note that if $A + XB[X]$ is OAF then it is general Z.P.I, so by [[6], Exercice 10, Page 225], it is also a multiplication ring so $\dim(A + XB[X]) \leq 1$, by [[6], Page 210]. If $p_1 \subset p_2 \subset ... \subset p_n$ is a chain of prime ideals of A then $p_1 + XB[X] \subset p_2 + XB[X] \subset ... \subset p_n + XB[X]$ is a chain of prime ideals of $A + XB[X]$ which implies that $\dim A \leq \dim(A + XB[X])$ then $\dim A \leq 1$. Let $m \in Max(A)$ and suppose that there exists $p \in Spec(A)$ such that $p \subsetneq m$, then $M = m + XB[X] \in Max(A + XB[X])$, $P = p + XB[X] \in$

$Spec(A+XB[X])$ and $P \subsetneq M$. By [[6], Proposition 9.15], $P = \bigcap_{n=1}^{\infty} M^n \subset M^2 = m^2 + mBX + X^2B[X]$, so $B \subset mB$. But $A \subset B$ is an integral extension of rings as B is a finitely generated A- module. So there exists $q \in Spec(B)$ such that $m \subset q$, then $mB \subset q \subsetneq B$ which is a contradiction. So $\dim A = 0$. As A is Noetherian we deduce that A is artinian. Moreover, for each $m \in Max(A)$ the maximal ideal $M = m + XB[X]$ of $A + XB[X]$ is simple. Take $I = m + mBX + X^2B[X]$, then I is an ideal of $A + XB[X]$ and $M^2 \subset I \subsetneq M$, so $M^2 = I$ and then $m^2 = m$. By [[8],Theorem 3.2], A is isomorphic to a finite direct product of fields. □

Remark 3.2. Let $A \subset B$ be an integral extension of commutative rings such that A is isomorphic to a finite direct product of fields and A is not local.

1. For each $m \in Max(A)$, $mB \in Max(B)$. In fact as $A \subset B$ is an integral extension then there exists $M \in Spec(B)$ such that $M \cap A = m$ so $mB \subset M$ and $mB \cap A \subset M \cap A = m$. As $m \subset mB \cap A$, then $mB \cap A = m$. So $A/m \subset B/mB$ is an extension of integral domains. As A/m is a field then B/mB is a field and then $mB \in Max(B)$.

2. We show that $Max(B) = \{mB \mid m \in Max(A)\}$. In fact, if $M \in Max(B)$ then $M \cap A \in Spec(A) = Max(A)$ (as A is artinian). So there exists $m \in Max(A)$ such that $M \cap A = m$, then $m \subset M$ and $mB \subset M$. As $mB \in Max(B)$, by **1**, then $mB = M$.
   Note that if m and m' are two distincts maximal ideals of A then $mB \neq m'B$ (as $m = mB \cap A$ and $m' = m'B \cap A$).

3. For each $m \in Max(A)$, there exists an idempotent $a \in A \backslash (0)$ such that $m = eA$.
   Moreover, if m_1 and m_2 are two distincts maximal ideals of A and e_1, e_2 are two idempotents of A such that $m_1 = e_1A$ and $m_2 = e_2A$ then $e_1e_2 = 0$ (in fact A is isomorphic to a finite direct product of fields).

4. As A is artinian and $A \subset B$ is an integral extension then B is also artinian.

5. Note also that B is a reduced ring. In fact, let $Max(A) = \{m_1, ..., m_n\}$ with $n \geq 2$ and for each $i \in \{1, .., n\}$ there exists an idempotent $e_i \in A$ such that $m_i = e_iA$, then $Max(B) = \{e_1B, ..., e_nB\}$. So $Nil(B) = e_1B \cap ... \cap e_nB$. Let $x \in Nil(B)$, then for each $i \in \{1, ..., n\}$ there exists $b_i \in B$ such that $x = e_ib_i$. As $n \geq 2$, then $x = e_1b_1 = e_2b_2$ so $e_1x = e_1e_2b_2 = 0$ and $e_2x = e_1e_2b_1 = 0$. As $e_1A + e_2A = A$ there exists $(a_1, a_2) \in A^2$ such that $1 = e_1a_1 + e_2a_2$ so $x = xe_1a_1 + xe_2a_2 = 0$. So B is a reduced ring.

6. Consequently, B is an artinian reduced ring so B is isomorphic to a finite direct product of fields.

7. Let $A = K_1 \times ... \times K_n$ $(n \geq 2)$ and $B = L_1 \times ... \times L_m$ as $card(Max(A)) = card(Max(B))$ then $m = n$.

Theorem 3.3. Let $A \subset B$ be an extension of commutative rings. The ring $A+XB[X]$ is OAF if and only if $A = B$ and A is a finite direct product of fields.

Proof. Suppose that $A = B$ and A is a finite direct product of fields. By [[3], Corollary 2.6], $A+XB[X] = A[X]$ is an OAF - ring. Conversely, Suppose that $A + XB[X]$ is an OAF - ring. By the preceding proposition and remarks, A and B are finite direct products of fields so we can suppose that $A = K_1 \times ... \times K_n$ and $B = L_1 \times ... \times L_n$ where K_i and L_i are commutative fields and $K_i \subset L_i$, for each $i \in \{1, ..., n\}$. We prove that $A = B$. For each $m \in Max(A)$, $m + XB[X]$ is a simple ideal of $A + XB[X]$. Note that if N is an A- submodule of B containing mB then $I = m + NX + X^2B[X]$ is an ideal of $A + XB[X]$ such that $m + mBX + X^2B[X] \subset I \subset m + XB[X]$. So as $m + XB[X]$ is simple then there exists no A- submodule N of B such that $mB \subsetneq N \subsetneq B$. Let $i \in \{1, ..., n\}$ and $m_i = K_1 \times ... \times K_{i-1} \times \{0\} \times K_{i+1} \times ... \times K_n \in Max(A)$ then $m_iB = L_1 \times ... \times L_{i-1} \times \{0\} \times L_{i+1} \times ... \times L_n \in Max(B)$. Let $x \in L_i \backslash \{0\}$ then $N = L_1 \times ... \times L_{i-1} \times xK_i \times L_{i+1} \times ... \times L_n$

is an A- submodule of B containing strictly m_iB so $N = B$ which implies that $xK_i = L_i$, so there exists $y \in K_i \backslash \{0\}$ such that $xy = 1$ that is $x = \frac{1}{y} \in K_i$ so $L_i = K_i$ and $A = B$. □

We can now recover Corollary 2.6 of [10].

Corollary 3.4. *Let R be a commutative ring. The following statements are equivalent.*

1. *$R[X]$ is an OAF - ring.*
2. *R is a Noetherian von Neumann regular ring.*
3. *R is a finite direct product of fields.*

4 OAF - rings of the form $A + XB[[X]]$.

Let $A \subset B$ be an extension of commutative rings. In this section we characterize OAF - rings of the form $A+XB[[X]]$. More precisely we prove that if A is a local ring then $A+XB[[X]]$ is OAF if and only if $A \subset B$ is an extension of fields and if A is not local then $A + XB[[X]]$ is OAF if and only if $A = B$ and A is a finite direct product of fields.

In this section some proofs are inspired from the case of composite polynomial rings $A + XB[X]$ but for the sake of completeness these proofs are included here.

Theorem 4.1. Let $A \subset B$ be an extension of commutative rings and suppose that A is local. Then $A + XB[[X]]$ is an OAF - ring if and only if $A \subset B$ is an extension of fields.

Proof. Note that if A is local then $A + XB[[X]]$ is local. Moreover as $\dim(A + XB[[X]]) \geq 1$, then by [[3], Proposition 3.3], if $A + XB[[X]]$ is OAF then it is an integral domain so A and B are also integral domains. By [[3], Theorem 3.5], $\dim(A + XB[[X]]) \leq 1$, which implies that $\dim(A + XB[[X]]) = 1$. As $1 + \max(\dim B[[X]][X^{-1}], \dim A + \lambda_{A,B}) \leq \dim(A + XB[[X]])$, by [[2], Theorem 1.1], where $\lambda_{A,B} = \sup\{\dim B[[X]]_{q[[X]]} \mid q \in spec(B) \text{ and } q \cap A = (0)\}$, we get $\max(\dim B[[X]][X^{-1}], \dim A + \lambda_{A,B}) = 0$. Then $\dim A = \dim B[[X]][X^{-1}] = 0$. As A is an integral domain we deduce that A is a field. The equality $\dim B[[X]][X^{-1}] = 0$ implies also that $\dim B = 0$ and then B is a field. Conversely, from [[3], Page 2698], if $A \subset B$ is an extension of fields then $A + XB[[X]]$ is an OAF - ring as it is a local one dimensional domain with maximal ideal $M = XB[[X]]$ and $(M : M) = B[[X]]$ which is a DVR with maximal ideal M. □

Proposition 4.2. *Let $A \subset B$ be an extension of commutative rings such that A is not local. If the ring $A + XB[[X]]$ is OAF then A is a finite direct product of fields and B is a finitely generated A- module.*

Proof. Note that if A is not local then $A + XB[[X]]$ is not local and $Max(A + XB[[X]]) = \{m + XB[[X]] \mid m \in Max(A)\}$. So $A+XB[[X]]$ is an OAF - ring if and only if it is general Z.P.I which is equivalent to the fact that $A+XB[[X]]$ is Noetherian and for each $m \in Max(A)$, $m+XB[[X]]$ is simple. Suppose then that $A + XB[[X]]$ is OAF. By [[4], Theorem 4], $A + XB[[X]]$ is Noetherian if and only if A is Noetherian and B is a finitely generated A- module. On the other hand, let $m \in Max(A)$ then $m + XB[[X]]$ is simple that is there exist no ideals of $A+XB[[X]]$ properly between $(m+XB[[X]])^2$ and $m+XB[[X]]$. Note that $(m+XB[[X]])^2 = m^2 + mBX + X^2B[[X]]$ and that $I = m + mBX + X^2B[[X]]$ is an ideal of $A + XB[[X]]$ such that $(m + XB[[X]])^2 \subset I \subset m + XB[[X]]$. So $m^2 = m$ or $mB = B$. But B is a finitely generated A- module so the extension $A \subset B$ is an integral extension of rings which implies that there exists $P \in Spec(B)$ such that $P \cap A = m$. So $m \subset P$ and $mB \subset P \subsetneq B$ then $m^2 = m$ for each maximal ideal m of A. As A is Noetherian then A is isomorphic to a finite direct product of fields by [[8],Theorem 3.2]. □

In the next lemma, we characterize all ideals I of $A + XB[[X]]$ such that $m + mBX + X^2B[[X]] \subset I \subset m + XB[[X]]$, where m is a maximal ideal of A.

Lemma 4.3. *Let $A \subset B$ be an extension of commutative rings. Let m be a maximal ideal of A and I be an ideal of $A + XB[[X]]$ such that $m + mBX + X^2B[[X]] \subset I \subset m + XB[[X]]$. Set $I_1 = \{b \in B \mid$ there exists $f = \sum_{i=0}^{\infty} b_i X^i \in I$ with $b_1 = b\}$. Then I_1 is an A- submodule of B and $I = m + I_1X + X^2B[[X]]$.*

Proof. It is clear that I_1 is an A- submodule of B and $mB \subset I_1$. We prove that $I = m + I_1X + X^2B[[X]]$. As $I \subset m + XB[[X]]$ and using the definition of I_1, it is obvious that $I \subset m + I_1X + X^2B[[X]]$. Conversely, as $m + mBX + X^2B[[X]] \subset I$ then $m \subset I$ and $X^2B[[X]] \subset I$. We show that $I_1X \subset I$. Let $b \in I_1$, there exists $f = \sum_{i=0}^{\infty} b_i X^i \in I$ such that $b = b_1$, so $bX = f - b_0 - \sum_{i=2}^{\infty} b_i X^i \in I$.
Note that $I_1 = \{b \in B \mid bX \in I\}$. □

Proposition 4.4. *Let $A \subset B$ be an extension of commutative rings such that A is not local and $A + XB[[X]]$ is an OAF - ring, then $A = B$.*

Proof. By the preceding proposition and remark 3.2, if $A + XB[[X]]$ is OAF then A and B are finite direct products of fields and we can suppose that $A = K_1 \times \dots \times K_n$ and $B = L_1 \times \dots \times L_n$ with $n \geq 2$. Moreover for each $m \in Max(A)$, $m + XB[[X]]$ is a simple ideal of $A + XB[[X]]$. Note that if N is an A- submodule of B containing mB then $I = m + NX + X^2B[[X]]$ is an ideal of $A + XB[[X]]$ such that $m + mBX + X^2B[[X]] \subset I \subset m + XB[[X]]$. So as $m + XB[[X]]$ is simple then there exists no A- submodule N of B such that $mB \subsetneq N \subsetneq B$. Let $i \in \{1, \dots, n\}$ and $m_i = K_1 \times \dots \times K_{i-1} \times \{0\} \times K_{i+1} \times \dots \times K_n \in Max(A)$ then $m_iB = L_1 \times \dots \times L_{i-1} \times \{0\} \times L_{i+1} \times \dots \times L_n \in Max(B)$. Let $x \in L_i \backslash \{0\}$ then $N = L_1 \times \dots \times L_{i-1} \times xK_i \times L_{i+1} \times \dots \times L_n$ is an A- submodule of B containing strictly m_iB so $N = B$ which implies that $xK_i = L_i$, so there exists $y \in K_i \backslash \{0\}$ such that $xy = 1$ that is $x = \frac{1}{y} \in K_i$ so $L_i = K_i$ and $A = B$. □

Now we can state our second theorem of this section.

Theorem 4.5. Let $A \subset B$ be an extension of commutative rings such that A is not local then $A + XB[[X]]$ is OAF if and only if $A = B$ and A is a finite direct product of fields.

Proof. By the preceding propositions if $A + XB[[X]]$ is OAF then $A = B$ and A is a finite direct product of fields. The converse is true by Theorem 1. □

We can now recover Theorem 2.1.

Corollary 4.6. *Let R be a commutative ring. $R[[X]]$ is an OAF - ring if and only if R is a finite direct product of fields.*

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