Moroccan Journal of Algebra and Geometry with Applications
Supported by Sidi Mohamed Ben Abdellah University, Fez,
Volume 1, Issue 1 (2022), pp 108-121
ISSN: 2820-7114

Title :

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Moroccan Journal of Algebra and Geometry with Applications
Vol. 1(1) (2022), 108–121
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco
ISSN: 2820-7114

On two classes of modules related to CS trivial extensions

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Communicated by Najib Mahdou
(Received 15 October 2021, Revised 20 January 2022, Accepted 28 January 2022)

Abstract. All rings considered are commutative. In this article we introduce and study two notions of modules which are stronger than CS modules, namely weakly IN modules and strongly CS modules. Our main aim is to characterize when a trivial extension $A = R \propto M$ (of a ring R by an R-module M) is a CS ring.

Key Words: CS-ring, strongly CS module (ring), trivial extension, weakly IN module (ring).
2010 MSC: Primary 16D10, 16D70; Secondary 16D80.

1 Introduction

Throughout this article, all rings considered are assumed to be commutative rings with an identity and R denotes such a ring. All modules are unital. We denote respectively by $Spec(R)$, $Max(R)$ and $Min(R)$ the set of all prime ideals of R, the set of all maximal ideals of R and the set of all minimal prime ideals of R. The nilradical of R and the Jacobson radical of R are denoted by $Nil(R)$ and $J(R)$, respectively. Let M be an R-module and let $x \in M$. By $Ann_R(x)$ and $Ann_R(M)$ we denote the *annihilator* of x and M, respectively; i.e. $Ann_R(x) = \{r \in R \mid rx = 0\}$ and $Ann_R(M) = \{r \in R \mid rM = 0\}$. The notation $N \subseteq M$ means that N is a subset of M; $N \leq M$ means that N is a submodule of M; and $N \subseteq^{ess} M$ means that N is an essential submodule of M. By $\mathbb{Q}$ and $\mathbb{Z}$ we denote the ring of rational and integer numbers, respectively.

An R-module M is called CS (or extending) if every submodule of M is essential in a direct summand of M and a ring R is called CS if the R-module R is CS. By [11, Theorem 6], R is CS if and only if for any ideals I and J of R with $I \cap J = 0$, $Ann_R(I) + Ann_R(J) = R$. In our attempt to characterize CS trivial extensions, we were lead to introduce two types of CS modules. The first one comes form the above characterization of CS rings. For consider $A = R \propto M$, the trivial extension of R by an R-module M, and let $\phi : M \to A$ be the natural monomorphism. Assume that A is a CS ring. Then given two submodules N and L of M such that $N \cap L = 0$, it is clear that $\phi(N)$ and $\phi(L)$ are two ideals of A such that $\phi(N) \cap \phi(L) = 0$. Thus $Ann_A\phi(N) + Ann_A\phi(L) = A$ which implies that $Ann_R(N) + Ann_R(L) = R$. Recall that a ring R is called an Ikeda-Nakayama ring (or IN ring) if for any two ideals T and T' of R, we have $Ann_R(T \cap T') = Ann_R(T) + Ann_R(T')$ (see [13, p. 148]). Call an R-module M weakly IN if $Ann_R(N) + Ann_R(L) = R$ whenever N and L are submodules of M such that $N \cap L = 0$. The second type is due to the definition itself of CS rings. Suppose that A is a CS ring and let N be a submodule of M. Then $\phi(N)$ is an ideal of A and hence $\phi(N)$ is essential in fA for some idempotent f of A. But f is of the form $(e, 0)$ for some idempotent e of R, so N is essential in eM. Call an R-module M strongly CS if for every submodule N of M, there exists an idempotent e of R such that N is an essential submodule of eM. The above discussion justifies our choice of the title and motivates our study in this paper.

In Section 2, we study some properties of weakly IN modules and strongly CS modules. We investigate the connection between these two classes of modules. It is shown that an R-module M is strongly CS if and only if M is weakly IN and idempotents lift modulo $Ann_R(M)$ (Theorem 2.8). Then we characterize the class of rings R over which every cyclic R-module is weakly IN (strongly CS) (Theorems 2.11 and 2.12).

In section 3, we focus on when a finite direct sum of weakly IN (strongly CS) modules is also weakly IN (strongly CS). Let an R-module $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be a direct sum of submodules $M_1, M_2, \ldots, M_n$. We show that M is weakly IN if and only if each M_i is weakly IN and $Ann_R(M_i) + Ann_R(M_j) = R$ for all distinct i, j in $\{1, \ldots, n\}$ (Theorem 3.1). It is also shown that M is a strongly CS R-module if and only if $R = R_1 \times R_2 \times \cdots \times R_n$ such that M_i is a strongly CS R_i-module for all $i \in \{1, \ldots, n\}$ (Theorem 3.3). As an application, we prove that R is a clean ring (i.e. every element of R is a sum of a unit and an idempotent) if and only if every weakly IN R-module is strongly CS if and only if $R/\mathfrak{m} \oplus R/\mathfrak{m}'$ is a strongly CS R-module for every distinct maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ of R (Theorem 3.5). Also, we characterize the class of rings R for which $R/\mathfrak{p} \oplus R/\mathfrak{p}'$ is a strongly CS R-module for every distinct minimal prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ of R as that of the purified rings (Theorem 3.9). Recall that a ring R is said to be a purified (or coclean) ring if for every distinct minimal prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R, there exists an idempotent e of R such that $e \in \mathfrak{p}$ and $1 - e \in \mathfrak{q}$.

In section 4, we characterize weakly IN and strongly CS modules over Dedekind domains. In particular, we show that if R is a Dedekind domain and M is an R-module, then M is weakly IN if and only if M is cyclic, or $M \cong E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of R, or M is isomorphic to an R-submodule of the quotient field K of R (Theorem 4.3).

The last section is devoted to the study of CS trivial extensions. Let $A = R \propto M$ denotes the trivial extension of a ring R by an R-module M. In the main result of this section, we show that A is a CS ring if and only if M is weakly IN (or strongly CS) and there exists an idempotent e of R such that $Ann_R(M) = eR$ and eR is a CS ring if and only if $M \oplus Ann_R(M)$ is a weakly IN (or strongly CS) R-module (Theorem 5.4). When M is a flat R-module, we prove that A is a CS ring if and only if R and M are weakly IN (or strongly CS) R-modules and $Ann_R(M)$ is a direct summand of R (Corollary 5.6).

2 Weakly IN and Strongly CS Modules

Recall that an R-module M is called CS if every submodule of M is essential in a direct summand of M. A ring R is called CS if R is a CS R-module. Using [11, Theorem 6], we obtain the following proposition.

Proposition 2.1. *The following are equivalent for a ring R:*
(1) *R is a CS ring;*
(2) *For any ideal I of R, there exists* $e = e^2 \in R$ *such that* $I \subseteq^{ess} eR$*;*
(3) *For any ideals I and J of R with* $I \cap J = 0$, $Ann_R(I) + Ann_R(J) = R$.

A ring R is called an Ikeda-Nakayama ring (or an IN-ring) if $Ann_R(I \cap J) = Ann_R(I) + Ann_R(J)$ for all ideals I and J of R (see [8]). Using the endomorphism ring, Wisbauer, Yousif and Zhou generalized this notion to a module theoretic version in 2002 [17]. Here, we will consider another generalization. We will call an R-module M an *s.IN-module* (*scalar IN-module*) if $Ann_R(N \cap L) = Ann_R(N) + Ann_R(L)$ for all submodules N an L of M. Next, we introduce two notions. The first one is weaker than that of s.IN-modules and the latter one is stronger than that of CS modules.

Definition 2.2. Let M be an R-module.

(1) M is called *weakly IN* if for any submodules N and L of M with $N \cap L = 0$, $Ann_R(N) + Ann_R(L) = R$.
(2) M is called *strongly CS* if for any submodule N of M, there exists $e = e^2 \in R$ such that $N \subseteq^{ess} eM$.
(3) A ring R is called weakly IN (strongly CS) if R, as an R-module, has the corresponding property.

Remark 2.3. (1) From Proposition 2.1, it follows easily that a ring R is CS if and only if R is weakly IN if and only if R is strongly CS.

(2) Consider the ring $\mathbb{Z}_2[x_1, x_2, \ldots]$ where $x_i^3 = 0$ for all i, $x_i x_j = 0$ for all $i \neq j$ and $x_i^2 = x_j^2 \neq 0$ for all i and j. It was shown in [8, Example 6] that R is a CS ring and hence R is weakly IN, but R is not an IN ring. It follows that the R-module R is weakly IN, but it is not an s.IN-module.

(3) If M is a uniform R-module, then clearly M is strongly CS. If R is indecomposable or M is indecomposable, then the converse also holds.

Proposition 2.4. *Any submodule of a strongly CS (weakly IN) module is strongly CS (weakly IN).*

Proof. Let M be a strongly CS module and let N be a submodule of M. Let L be a submodule of N. Then $L \subseteq^{ess} eM$ for some idempotent e of R. Hence $L \subseteq^{ess} eM \cap N = eN$. The other assertion is evident. □

Proposition 2.5. *A free R-module F is weakly IN if and only if $rank(F) = 1$ and R is a CS ring.*

Proof. Let F be a weakly IN free R-module. Suppose that $rank(F) \geq 2$. Then F contains a submodule isomorphic to $R \oplus R$. Therefore $Ann_R(R) + Ann_R(R) = R$, a contradiction. So $rank(F) = 1$ and hence R is a CS ring (Remark 2.3(1)). The converse is clear. □

From the preceding proposition, one can directly infer that over a field K, a module M is weakly IN if and only if $M \cong K$.

Recall that a module M is called *quasi-continuous* if M is CS and, for any two direct summands M_1, M_2 with $M_1 \cap M_2 = 0$, $M_1 \oplus M_2$ is also a direct summand (see [12]). From [17, Corollary 4], it follows that a faithful R-module M is weakly IN if and only if M is a quasi-continuous R-module and for any $f^2 = f \in End_R(M)$, there exists $r \in R$ such that $f(x) = rx$ for any $x \in M$. This fact can be extended to the general case as shown in the following characterization of weakly IN modules.

Proposition 2.6. *Let R be a ring and let M be a nonzero R-module. Then the following are equivalent:*
(1) M is a weakly IN R-module;
(2) M is a weakly IN $R/Ann_R(M)$-module;
(3) M is a quasi-continuous R-module and for any $f^2 = f \in End_R(M)$, there exists $r \in R$ such that $f(x) = rx$ for any $x \in M$;
(4) M is a CS R-module and for any direct summand N of M, there exists $r \in R$ such that $N = rM$ and $r - r^2 \in Ann_R(M)$;
(5) For any submodule N of M, there exists $r \in R$ such that $N \subseteq^{ess} rM$ and $r - r^2 \in Ann_R(M)$;
(6) M is strongly CS as an $R/Ann_R(M)$-module.

Proof. (1) $\Leftrightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3) Let $\overline{R} = R/Ann_R M$. It is easy to see that M has a natural structure of $(\overline{R}, R)$-bimodule and M is a faithful left $\overline{R}$-module. Moreover, the R-submodules and the $\overline{R}$-submodules of M are the same. Using [17, Corollary 4], we see that M is a quasi-continuous R-module and for any $f^2 = f \in End_R(M)$, there exists $\overline{r} = r + Ann_R(M) \in \overline{R}$ such that $f(x) = \overline{r}x = rx$ for all $x \in M$.

(3) $\Rightarrow$ (4) It is clear that M is a CS R-module. Let N and K be two submodules of M such that $M = N \oplus K$. Let f be the projection on N along K. Then $f^2 = f$ and so there exists $r \in R$ such that $f(x) = rx$ for any $x \in M$. Hence, $N = f(M) = rM$. But $f^2 = f$, so $r - r^2 \in Ann_R(M)$.

(4) $\Rightarrow$ (5) Let N be a submodule of M. Since M is CS, there exists a direct summand K of M such that $N \subseteq^{ess} K$. Moreover, by assumption, $K = rM$ for some $r \in R$ with $r - r^2 \in Ann_R(M)$.

(5) ⇒ (1) Let N and L be two submodules of M with $N \cap L = 0$. Then there exist r and s in R such that $N \subseteq^{ess} rM$, $L \subseteq^{ess} sM$, $r - r^2 \in Ann_R(M)$ and $s - s^2 \in Ann_R(M)$. Thus $N \cap L \subseteq^{ess} rM \cap sM$. Since $N \cap L = 0$, we have $rM \cap sM = 0$. This implies that $rsM \subseteq rM \cap sM = 0$ and hence $rs \in Ann_R(M)$. Moreover, since $r - r^2 \in Ann_R(M)$ and $s - s^2 \in Ann_R(M)$, we have $(1-r) \in Ann_R(rM) \subseteq Ann_R(N)$ and $(1-s) \in Ann_R(sM) \subseteq Ann_R(L)$. It follows that

$$1 = (1-r) + (1-s)r + sr \in Ann_R(N) + Ann_R(L).$$

Therefore, $R = Ann_R(N) + Ann_R(L)$. Thus, M is weakly IN.

(5) ⇔ (6) This is immediate. □

In the next example, we present a CS module which is not weakly IN, and some CS modules which are not strongly CS. More examples are provided in the next two sections.

Example 2.7. (1) Let R be a self injective ring. Then clearly the R-module $M = R \oplus R$ is CS. However, M is not weakly IN by Proposition 2.5.

(2) Let R be an indecomposable ring (e.g., R is a local ring or a domain).

(a) Consider the R-module $M = S_1 \oplus S_2$ where S_1 and S_2 are simple modules. It is clear that M is a CS module. On the other hand, M is not strongly CS (see Remark 2.3(3)).

(b) Let M be an injective R-module which is not indecomposable (e.g., we can take the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Q}$ or $M = \mathbb{Z}(p_1^\infty) \oplus \mathbb{Z}(p_2^\infty)$ for some prime numbers p_1 and p_2). Then M is CS, but M is not strongly CS by Remark 2.3(3).

Let R be a ring and let I be a proper ideal of R. We say that idempotents lift modulo I if whenever r is an element of R such that $r - r^2 \in I$, then there exists $e = e^2 \in R$ such that $r - e \in I$.

Theorem 2.8. Let R be a ring and let M be a nonzero R-module. Then the following are equivalent:
(1) M is a strongly CS R-module;
(2) For any submodules N and L of M with $N \cap L = 0$, there exists an idempotent $e \in R$ such that $e \in Ann_R(N)$ and $1 - e \in Ann_R(L)$;
(3) M is a weakly IN R-module and idempotents lift modulo $Ann_R(M)$;
(4) M is a CS module and for every direct summand K of M, there exists an idempotent e of R such that $K = eM$.

Proof. (1) ⇒ (2) Let N and L be two submodules of M with $N \cap L = 0$. Then there exist $e = e^2$ and $f = f^2$ in R such that $N \subseteq^{ess} fM$ and $L \subseteq^{ess} eM$. Using the fact that $N \cap L = 0$, we get $fM \cap eM = 0$. Hence, $efM \subseteq fM \cap eM = 0$. So $e \in Ann_R(fM) \subseteq Ann_R(N)$ and $1 - e \in Ann_R(eM) \subseteq Ann_R(L)$.

(2) ⇒ (3) Let N and L be two submodules of M with $N \cap L = 0$. By hypothesis, there exists an idempotent $e \in R$ such that $e \in Ann_R(N)$ and $1-e \in Ann_R(L)$. But $e+1-e = 1$, so $Ann_R(N)+Ann_R(L) = R$. It follows that M is weakly IN. Now let $r \in R$ such that $r - r^2 \in Ann_R(M)$ and let $rx = (1-r)y \in rM \cap (1-r)M$ where $x, y \in M$. Then $r^2x = (r - r^2)y = 0$ since $r - r^2 \in Ann_R(M)$. Using once again the fact that $r - r^2 \in Ann_R(M)$, we get $rx = 0$. Therefore $rM \cap (1-r)M = 0$. Hence, by (2), there exists $f = f^2 \in R$ such that $1 - f \in Ann_R(rM)$ and $f \in Ann_R((1-r)M)$. This means that $r - fr \in Ann_R(M)$ and $f - fr \in Ann_R(M)$. Consequently,

$$f - r = (f - fr) - (r - fr) \in Ann_R(M).$$

This shows that idempotents lift modulo $Ann_R(M)$.

(3) ⇒ (1) Let N be a submodule of M. Since M is a weakly IN R-module, there exists $r \in R$ such that $N \subseteq^{ess} rM$ and $r - r^2 \in Ann_R(M)$ (see Proposition 2.6). But idempotents lift modulo $Ann_R(M)$, so there exits an idempotent e of R such that $r - e \in Ann_R(M)$. Therefore $rM = eM$ and consequently $N \subseteq^{ess} eM$.

(1) ⇔ (4) This is clear. □

The preceding theorem shows that the class of weakly IN modules contains that of strongly CS modules. Next, we present an example illustrating that this inclusion is proper, in general.

Example 2.9. Let p and q be two different prime numbers and consider the ring $R = \{m/n \in \mathbb{Q} \mid p \nmid n$ and $q \nmid n$ (m/n in lowest terms)$\}$. It is well known that pR and qR are the only maximal ideals in R. Moreover, idempotents do not lift modulo $J(R) = pR \cap qR$. Let $M = R/pR \oplus R/qR$. Clearly, $Ann_R(M) = J(R)$. From Corollary 3.4, we infer that M is a weakly IN R-module which is not strongly CS.

Let R be a ring. Recall that the socle of R, denoted by $Soc(R)$, is the sum of all its minimal ideals. In the following corollary, we provide sufficient conditions for a weakly IN module to be strongly CS.

Corollary 2.10. *Let M be a nonzero R-module such that $Ann_R(M)$ satisfies any one of the following conditions:*
(1) *$Ann_R(M)$ is a nil ideal of R (i.e., $Ann_R(M) \subseteq Nil(R)$);*
(2) *$Ann_R(M)$ is a direct summand of R (for instance, M is faithful);*
(3) *$Ann_R(M) = Soc(R)$.*
Then M is a weakly IN R-module if and only if M is strongly CS.

Proof. The sufficiency follows from Theorem 2.8. Conversely, suppose that M is a weakly IN R-module. Applying again Theorem 2.8, we only need to show that idempotents lift modulo $Ann_R(M)$.

(1) This follows from [2, Proposition 27.1].

(2) We will show that idempotent lift modulo every direct summand of R. Let $e = e^2 \in R$ and let $r \in R$ such that $r - r^2 \in eR$. Then $(1-e)(r-r^2) = 0$ and hence $((1-e)r)^2 = (1-e)r$. Thus $(1-e)r$ is an idempotent of R. Moreover, we have $r - ((1-e)r) = er \in eR$.

(3) This follows from [18, Lemma 1.2]. □

Let $n \geq 2$. By Proposition 2.5, there exists no ring R for which every n-generated R-module is weakly IN. This fact should be contrasted with [9, Corollary 13.8]. On the other hand, take a valuation ring R. It is easily seen that every cyclic R-module is uniform. Therefore every cyclic R-module is strongly CS and weakly IN (see Remark 2.3(3) and Theorem 2.8). Next, we will characterize the class of rings R for which every cyclic R-module is weakly IN (strongly CS).

A ring R is called a *CF-ring* if every R–module M which is a direct sum of finitely many cyclic R–modules has a *canonical form decomposition*, i.e. $M \cong R/I_1 \oplus \cdots \oplus R/I_n$ where the ideals I_j $(1 \leq j \leq n)$ of R satisfy $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subsetneq R$ (see [15]). For the definitions of the other types of rings used in the following two results, we refer the reader to [7].

Theorem 2.11. The following are equivalent for a ring R:
(1) Any cyclic R-module is weakly IN;
(2) R/I is a CS ring for every proper ideal I of R;
(3) R is a CF-ring;
(4) R is a finite direct product of valuation rings, h-local Prüfer domains and torch rings.

Proof. Let I be a proper ideal of R and consider the R-module $M = R/I$. By Proposition 2.6, M is a weakly IN R-module if and only if M is a strongly CS $R/Ann_R(M)$-module. This is equivalent to the condition that R/I is a CS ring (Remark 2.3(1)). Now use [11, Theorem 9]. □

Recall that a ring R is called *clean* if every element of R is a sum of a unit and an idempotent.

Theorem 2.12. The following are equivalent for a ring R:
(1) Any cyclic R-module is strongly CS;
(2) R is a clean CF-ring;
(3) R is a finite direct product of valuation rings.

Proof. (1) $\Leftrightarrow$ (2) From Theorem 2.8, it follows that the assertion (1) is equivalent to the condition that for any ideal I of R, R/I is a weakly IN R-module and idempotents lift modulo $Ann_R(R/I) = I$. Now using Theorem 2.11 and the fact that R is a clean ring if and only if idempotents lift modulo every ideal of R (see [1, Theorem 5.1]), we obtain the desired equivalence.

(2) $\Rightarrow$ (3) Since R is a CF-ring, we have $R = R_1 \times R_2 \times \cdots \times R_n$ where each R_i is an indecomposable ring which is either a valuation ring or an h-local Prüfer domain or a torch ring ([15, Theorems 3.10 and 3.12]). Let $i \in \{1,\ldots,n\}$. Since R is clean, it follows that R_i is also clean by [3, Proposition 2]. Moreover, R_i is a local ring by [3, Theorem 3]. Hence R_i could not be a torch ring since every torch ring has at least two maximal ideals ([7, page 38]). In addition, note that any local Prüfer domain is a valuation ring.

(3) $\Rightarrow$ (2) By Theorem 2.11, R is a CF-ring. Moreover, note that any valuation ring is local. Thus, using [3, Proposition 2(1)-(3)], we conclude that R is a clean ring. □

3 Finite Direct Sums of Weakly IN (Strongly CS) Modules

We begin by providing necessary and sufficient conditions for a finite direct sum of modules to be weakly IN.

Theorem 3.1. Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be a direct sum of submodules $M_i (1 \leq i \leq n)$. Then the following are equivalent:
(1) M is weakly IN;
(2) (a) M_i is a weakly IN R-module for all $i \in \{1,\ldots,n\}$, and
(b) $Ann_R(M_j) + Ann_R(M_k) = R$ for all $j \neq k \in \{1,\ldots,n\}$.

Proof. (1) $\Rightarrow$ (2) (a) follows from Proposition 2.4 and (b) follows from the definition of a weakly IN module.

(2) $\Rightarrow$ (1) Let N and L be two submodules of M such that $N \cap L = 0$. Using (b) and [6, Lemma 2.6], we get $N = \oplus_{i=1}^n (N \cap M_i)$ and $L = \oplus_{i=1}^n (L \cap M_i)$. Fix $i \in \{1,\ldots,n\}$. As $N \cap L = 0$, we have $(N \cap M_i) \cap (L \cap M_i) = 0$. Since M_i is weakly IN, we have $Ann_R(N \cap M_i) + Ann_R(L \cap M_i) = R$. Moreover, using (b), it follows that for any $i \neq j \in \{1,\ldots,n\}$, we have $Ann_R(N \cap M_i) + Ann_R(L \cap M_j) = R$ as $Ann_R(M_i) \subseteq Ann_R(N \cap M_i)$ and $Ann_R(M_j) \subseteq Ann_R(L \cap M_j)$. So $(\cap_{i=1}^n Ann_R(N \cap M_i)) + (\cap_{i=1}^n Ann_R(L \cap M_i)) = R$ by [19, Theorem 31]. Consequently, $Ann_R(N) + Ann_R(L) = R$. This completes the proof. □

In the next theorem, we provide a characterization of when a finite direct sum of modules is strongly CS. We need the following lemma.

Lemma 3.2. *Let R_1 and R_2 be two rings and let M_i be an R_i-module $(i = 1,2)$. Then the following hold true:*
(1) *$M_1 \times M_2$ is a strongly CS $R_1 \times R_2$-module if and only if M_i is a strongly CS R_i-module for each $i = 1,2$.*
(2) *$R = R_1 \times R_2$ is a CS ring if and only if so are R_1 and R_2.*

Proof. (1) We will use the following elementary property: $(*)$ given an R_1-submodule N_1 of M_1 and an R_2-submodule N_2 of M_2, $N_1 \times N_2 \subseteq^{ess} M_1 \times M_2$ if and only if $N_1 \subseteq^{ess} M_1$ and $N_2 \subseteq^{ess} M_2$.

Now suppose that $M_1 \times M_2$ is a strongly CS $R_1 \times R_2$-module and let N_1 be an R_1-submodule of M_1 and N_2 an R_2-submodule of M_2. Then there exists $(e_1,e_2) = (e_1,e_2)^2 = (e_1^2,e_2^2) \in R_1 \times R_2$ such that $N_1 \times N_2 \subseteq^{ess} (e_1,e_2)(M_1 \times M_2) = e_1M_1 \times e_2M_2$. By $(*)$, it follows that $N_1 \subseteq^{ess} e_1M_1$ and $N_2 \subseteq^{ess} e_2M_2$. This clearly implies that each M_i $(i = 1,2)$ is a strongly CS R_i-module. Conversely, let N be an $R_1 \times R_2$-submodule of $M_1 \times M_2$. Then $N = N_1 \times N_2$, where N_1 is an R_1-submodule of M_1 and N_2 is an R_2-submodule of M_2. Let $e_1^2 = e_1 \in R_1$ and $e_2^2 = e_2 \in R_2$ such that $N_1 \subseteq^{ess} e_1M_1$ and $N_2 \subseteq^{ess} e_2M_2$. Again by $(*)$, we have $N_1 \times N_2 \subseteq^{ess} e_1M_1 \times e_2M_2 = (e_1,e_2)(M_1 \times M_2)$. Since $(e_1,e_2)^2 = (e_1,e_2)$, we conclude that $M_1 \times M_2$ is a strongly CS $R_1 \times R_2$-module.

(2) Apply (1) for $M_1 = R_1$ and $M_2 = R_2$ (see Remark 2.3(1)). □

Theorem 3.3. Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be a direct sum of submodules $M_i (1 \le i \le n)$. Then the following are equivalent:
(1) M is strongly CS;
(2) M satisfies the following two conditions:
(a) M_i is a strongly CS R-module for every $i \in \{1, \ldots, n\}$, and
(b) There exists a complete set of orthogonal idempotents $\{e_1, \ldots, e_n\}$ of R such that $e_i M_i = M_i$ for all $i \in \{1, \ldots, n\}$;
(3) $R = R_1 \times R_2 \times \cdots \times R_n$ such that M_i is a strongly CS R_i-module for all $i \in \{1, \ldots, n\}$.

Proof. (1) ⇒ (2) (a) follows by using Proposition 2.4. Let us show (b) by induction on n. Suppose that $M = M_1 \oplus M_2$ is strongly CS. By Theorem 2.8, there exists an idempotent e of R such that $1 - e \in Ann_R(M_1)$ and $e \in \mathrm{Ann}_R(M_2)$. This implies that $M_1 = eM$ and $M_2 = (1 - e)M$. Therefore (b) is true for $n = 2$. Now assume (b) holds for n; we will prove it for $n + 1$. Let $M = M_1 \oplus \cdots \oplus M_n \oplus M_{n+1}$ be strongly CS. From the case $n = 2$, we infer that there exists an idempotent e of R such that $e(M_1 \oplus \cdots \oplus M_n) = M_1 \oplus \cdots \oplus M_n$ and $(1 - e)M_{n+1} = M_{n+1}$. But $M_1 \oplus \cdots \oplus M_n$ is a strongly CS R-module as it is a submodule of M, so, by induction hypothesis there exists a complete set of orthogonal idempotents $\{f_1, \ldots, f_n\}$ of R such that $M_i = f_i M_i$ for all $i \in \{1, \ldots, n\}$. It is easy to see that $\{ef_1, \ldots, ef_n, 1 - e\}$ is a complete set of orthogonal idempotents of R such that $ef_i M_i = eM_i = M_i$ for all $i \in \{1, \ldots, n\}$ and $(1 - e)M_{n+1} = M_{n+1}$.

(2) ⇒ (3) By (b), $R = R_1 \times R_2 \times \cdots \times R_n$ where $R_i = e_i R$ for all $i \in \{1, \ldots, n\}$. Fix $j \in \{1, \ldots, n\}$. Since $e_j M_j = M_j$, M_j has a natural structure of an R_j-module. Let N be an R_j-submodule of M_j. By (a), M_j is a strongly CS R-module. So there exists $e^2 = e \in R$ such that $N \subseteq^{ess} eM_j = ee_j M_j$ as R-modules and also as $e_j R$-modules. Note that ee_j is an idempotent of R_j. Thus M_j is a strongly CS R_j-module.

(3) ⇒ (1) This follows from Lemma 3.2(1). □

As an application of the preceding two theorems, we have the following corollary.

Corollary 3.4. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be two prime ideals of a ring R. Then the following hold true:*
(1) *The R-module $R/\mathfrak{p} \oplus R/\mathfrak{q}$ is weakly IN if and only if $\mathfrak{p} + \mathfrak{q} = R$.*
(2) *The following are equivalent:*
(a) *The R-module $R/\mathfrak{p} \oplus R/\mathfrak{q}$ is strongly CS;*
(b) *There exists an idempotent e of R such that $e \in \mathfrak{p}$ and $1 - e \in \mathfrak{q}$;*
(c) *$\mathfrak{p} + \mathfrak{q} = R$ and idempotents lift modulo $\mathfrak{p} \cap \mathfrak{q}$.*

Proof. Since the R-modules $R/\mathfrak{p}$ and $R/\mathfrak{q}$ are uniform, they are strongly CS (and hence also weakly IN).

(1) Use Theorem 3.1 and the fact that $R/\mathfrak{p}$ and $R/\mathfrak{q}$ are weakly IN.

(2) (a) ⇔ (c) Use (1), Theorem 2.8 and the fact that $Ann_R(R/\mathfrak{p} \oplus R/\mathfrak{q}) = \mathfrak{p} \cap \mathfrak{q}$.

(a) ⇒ (b) By Theorem 2.8, there exists an idempotent e of R such that $e \in Ann_R(R/\mathfrak{p}) = \mathfrak{p}$ and $1 - e \in Ann_R(R/\mathfrak{q}) = \mathfrak{q}$.

(b) ⇒ (a) Let e be an idempotent of R such that $e \in \mathfrak{p}$ and $1 - e \in \mathfrak{q}$. Then $(1 - e)(R/\mathfrak{p}) = R/\mathfrak{p}$ and $e(R/\mathfrak{q}) = R/\mathfrak{q}$. Now using Theorem 3.3((2) ⇒ (1)) and the fact that $R/\mathfrak{p}$ and $R/\mathfrak{q}$ are strongly CS R-modules, we deduce that $R/\mathfrak{p} \oplus R/\mathfrak{q}$ is a strongly CS R-module. □

In contrast to Example 2.9, we characterize in the next theorem the class of rings R for which every weakly IN R-module is strongly CS.

Theorem 3.5. The following are equivalent for a ring R:
(1) Any weakly IN R-module is strongly CS;
(2) $R/\mathfrak{m} \oplus R/\mathfrak{m}'$ is a strongly CS R-module for every distinct maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ of R;

(3) Idempotents of R lift modulo $\mathfrak{m} \cap \mathfrak{m}'$ for every distinct maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ of R;
(4) Idempotents lift modulo every ideal of R;
(5) For every distinct maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ of R, there exists an idempotent e of R such that $e \in \mathfrak{m}$ and $1 - e \in \mathfrak{m}'$;
(6) R is a clean ring.

Proof. The equivalences (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) follow from [1, Theorem 5.1].

The equivalences (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) follow from Corollary 3.4(2).

(4) $\Rightarrow$ (1) Use the equivalence (1) $\Leftrightarrow$ (3) in Theorem 2.8.

(1) $\Rightarrow$ (2) Let $M = R/\mathfrak{m} \oplus R/\mathfrak{m}'$ where $\mathfrak{m}$ and $\mathfrak{m}'$ are two distinct maximal ideals of R. By Corollary 3.4(1), M is a weakly IN R-module and so it is strongly CS by (1). □

In the same vein of Example 2.9, we exhibit the following examples.

Example 3.6. (1) Let R be a ring which is not clean. By Theorem 3.5, there exists an R-module M such that M is weakly IN but not strongly CS. To construct an explicit example of such a ring R and such a module M, consider the ring $R = \mathbb{Z}$ and the $\mathbb{Z}$-module $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Since $2\mathbb{Z} + 3\mathbb{Z} = \mathbb{Z}$, M is a weakly IN $\mathbb{Z}$-module (Corollary 3.4(1)). However, note that M is not uniform. Then M can not be a strongly CS R-module since R is indecomposable (Remark 2.3(3)).

(2) Let $R = \mathbb{Z}$ and consider the R-module $N = \mathbb{Q}/\mathbb{Z}_{(2\mathbb{Z})}$. By [15, Example 3.13], $A = R \propto N$ is a Torch ring. Thus, every cyclic A-module is weakly IN by Theorem 2.11. On the other hand, since R is not a clean ring, A is not clean by [4, Theorem 6.4]. Therefore the ring A has a cyclic A-module which is not strongly CS by Theorem 2.12.

Recall that a ring R is called *zero-dimensional* if every prime ideal of R is maximal.

Proposition 3.7. *For a ring R the following statements are equivalent:*
(1) $R/\mathfrak{p} \oplus R/\mathfrak{p}'$ is a weakly IN R-module for every distinct prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ of R;
(2) $R/\mathfrak{p} \oplus R/\mathfrak{p}'$ is a strongly CS R-module for every distinct prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ of R;
(3) R is a zero-dimensional ring.

Proof. (1) $\Rightarrow$ (3) Suppose that R has a nonmaximal prime ideal $\mathfrak{p}$ and let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. By hypothesis, $R/\mathfrak{p} \oplus R/\mathfrak{m}$ is a weakly IN R-module and hence $\mathfrak{p} + \mathfrak{m} = R$ (see Corollary 3.4(1)). Thus $\mathfrak{m} = R$, which is a contradiction. Therefore R is a zero-dimensional ring.

(3) $\Rightarrow$ (2) Let $\mathfrak{p}$ and $\mathfrak{p}'$ be two distinct prime ideals of R. Since R is zero-dimensional, $\mathfrak{p}$ and $\mathfrak{p}'$ are maximal. Moreover, note that R is a clean ring by [3, Corollary 11]. So $R/\mathfrak{p} \oplus R/\mathfrak{p}'$ is a strongly CS R-module by Theorem 3.5.

(2) $\Rightarrow$ (1) This follows from the fact that any strongly CS R-module is weakly IN (see Theorem 2.8). □

Recall that a ring R is called an *mp-ring* if every prime ideal contains a unique minimal prime ideal; equivalently, every maximal ideal of R contains a unique minimal prime ideal.

Replacing the term “prime" in Proposition 3.7 by “minimal prime", we obtain the following characterizations.

Proposition 3.8. *The following are equivalent for a ring R:*

(1) *$R/\mathfrak{p} \oplus R/\mathfrak{p}'$ is a weakly IN R-module for every distinct minimal prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ of R;*

(2) *$\mathfrak{p} + \mathfrak{p}' = R$ for every distinct minimal prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ of R;*

(3) *R is an mp-ring.*

Proof. (1) ⇔ (2) Use Corollary 3.4(1).
(2) ⇔ (3) see [1, Theorem 6.2]. □

Following [1, Definition 8.1], a ring R is said to be a *purified* ring if for every distinct minimal prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ of R, there exists an idempotent e of R such that $e \in \mathfrak{p}$ and $1 - e \in \mathfrak{p}'$. Note that every purified ring is an mp-ring.

Theorem 3.9. For a ring R the following are equivalent:
(1) $R/\mathfrak{p} \oplus R/\mathfrak{p}'$ is a strongly CS R-module for every distinct minimal prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ of R;
(2) R is an mp-ring and idempotents lift modulo $\mathfrak{p} \cap \mathfrak{p}'$ for every distinct minimal prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ of R;
(3) R is a purified ring.

Proof. This follows by combining Corollary 3.4(2) with Proposition 3.8. □

4 Modules over Dedekind Domains

This short section is devoted to the description of the structure of both weakly IN and strongly CS modules over Dedekind domains. Recall that for an R-module M, $Ass(M)$ denotes the set of prime ideals of R associated to M, that is, $Ass(M) = \{\mathfrak{p} \in Spec(R) \mid \mathfrak{p} = Ann_R(x) \text{ for some } 0 \neq x \in M\}$.

Theorem 4.1. Let R be a Dedekind domain with field of fractions K and let M be a nonzero R-module. Then the following are equivalent:
(1) M is a strongly CS R-module;
(2) M is a uniform R-module;
(3) M is isomorphic to an R-submodule of K or there exists a maximal ideal $\mathfrak{p}$ of R such that $M \cong E(R/\mathfrak{p})$ or $M \cong R/\mathfrak{p}^n$ for some positive integer n.

Proof. (1) ⇔ (2) This follows from the fact that R is indecomposable (see Remark 2.3(3)).
(2) ⇒ (3) Let M be a nonzero uniform R-module. Since R is noetherian, $E(M) \cong E(R/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of R (see [14, Corollary of Theorem 2.32]). Hence M is isomorphic to a submodule of $E(R/\mathfrak{p})$ and $Ass(M) = \{\mathfrak{p}\}$. If $\mathfrak{p} = 0$, then M is isomorphic to an R-submodule of $E(R) \cong K$. Now assume that $\mathfrak{p} \neq 0$. Then $\mathfrak{p}$ is a maximal ideal of R as R is a Dedekind domain. Clearly, $E = E(R/\mathfrak{p})$ is a torsion R-module. Moreover, M is indecomposable since E is uniform. Using [10, Theorem 10], we infer that $M \cong E(R/\mathfrak{p})$ or $M \cong R/\mathfrak{q}^n$ for some maximal ideal $\mathfrak{q}$ of R and some positive integer n. In the latter case we have $\mathfrak{q} \in Ass(M)$ and consequently $\mathfrak{q} = \mathfrak{p}$.
(3) ⇒ (2) Let $\mathfrak{p}$ be a nonzero prime ideal of R. It is clear that K and $E(R/\mathfrak{p})$ are uniform R-modules. Also, $R/\mathfrak{p}^n$ is uniform since it is a uniserial R-module by [14, Lemma 6.8]. □

Let R be a Dedekind domain with field of fractions K. Recall that an R-submodule F of K is called a fractional ideal of R if there is a nonzero element r of R such that $rF \subseteq R$. Note that every finitely generated R-submodule of K is a fractional ideal of R (see [14, Lemma 6.15]). It is well known that every injective R-module has no maximal submodules. The following corollary is an immediate consequence of the preceding theorem.

Corollary 4.2. *Let R be a Dedekind domain and let M be a nonzero finitely generated R-module. Then the following are equivalent:*
(1) *M is strongly CS;*
(2) *$M \cong I$ where I is a nonzero fractional ideal of R or there exists a maximal ideal $\mathfrak{p}$ of R such that $M \cong R/\mathfrak{p}^n$ for some positive integer n.*

Theorem 4.3. Let R be a Dedekind domain with field of fractions K and let M be a nonzero R-module. Then the following are equivalent:
(1) M is a weakly IN R-module;
(2) M is isomorphic to an R-submodule of K or $M \cong E(R/\mathfrak{p})$ for some maximal ideal $\mathfrak{p}$ of R or $M \cong R/\mathfrak{p}_1^{n_1} \oplus \cdots \oplus R/\mathfrak{p}_k^{n_k}$ for some positive integers n_i $(1 \le i \le k)$ and distinct maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ of R.

Proof. (1) $\Rightarrow$ (2) Let M be a nonzero weakly IN R-module. If $Ann_R(M) = 0$, then M is strongly CS by Corollary 2.10. From Theorem 4.1, it follows that M is isomorphic to an R-submodule of K or $M \cong E(R/\mathfrak{p})$ for some maximal ideal $\mathfrak{p}$ of R. Now suppose that $Ann_R(M) \neq 0$. By [14, Theorem 6.11], there exists a family $\{\mathfrak{p}_i, n_i\}_{i \in I}$ such that $\mathfrak{p}_i$ $(i \in I)$ are maximal ideals of R and there are only finitely many distinct ones, n_i $(i \in I)$ are positive integers and $M = \oplus_{i \in I} M_i$ where $M_i \cong R/\mathfrak{p}_i^{n_i}$ for every $i \in I$. Let $j \neq k \in I$. Since $M_j \cap M_k = 0$ and M is weakly IN, we have $Ann_R(M_j) + Ann_R(M_k) = R$. Thus, $\mathfrak{p}_j^{n_j} + \mathfrak{p}_k^{n_k} = R$. So $\mathfrak{p}_j \neq \mathfrak{p}_k$ for all $j \neq k$ in I. Consequently, I is a finite set.

(2) $\Rightarrow$ (1) This follows from Theorems 3.1 and 4.1. □

Remark 4.4. Let R be a Dedekind domain and let I be a nonzero ideal of R. By [14, Lemma 6.12 and Theorem 6.14], $I = \mathfrak{p}_1^{n_1}\mathfrak{p}_2^{n_2}\ldots\mathfrak{p}_k^{n_k} = \mathfrak{p}_1^{n_1} \cap \mathfrak{p}_2^{n_2} \cap \cdots \cap \mathfrak{p}_k^{n_k}$ for some distinct maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ of R and some positive integers n_i $(1 \le i \le k)$. By the Chinese Remainder Theorem, we have $R/I \cong R/\mathfrak{p}_1^{n_1} \oplus \cdots \oplus R/\mathfrak{p}_k^{n_k}$. Now using Theorem 4.3, we conclude that every cyclic R-module is weakly IN (see also Theorem 2.11).

Combining Theorem 4.3 and Remark 4.4, we obtain the following corollary.

Corollary 4.5. *Let R be a Dedekind domain and let M be a nonzero finitely generated R-module. Then the following are equivalent:*
(1) *M is weakly IN;*
(2) *M is cyclic or $M \cong I$ where I is a nonzero fractional ideal of R.*

5 CS Trivial Extensions

Let R be a ring and let M be an R-module. The abelian group $R \oplus M$ can be endowed with the following product: $(a,x)(b,y) = (ab, ay + bx)$. The result is a ring called the trivial extension of R by M denoted by $R \propto M$. R becomes a subring of $R \propto M$ and M an ideal such that $M^2 = 0$. If I is an ideal of R and N is a submodule of M such that $IM \subseteq N$, then $(I,N) = \{(a,x) \in R \propto M \mid a \in I,\ x \in N\}$ is an ideal of $A = R \propto M$ and we have $Ann_A(I,N) = (Ann_R(I) \cap Ann_R(N), Ann_M(I))$. In this section our main result is a characterization of CS trivial extensions. To prove it, we need the following three lemmas.

Lemma 5.1. *Let R be a ring and let M be an R-module such that $A = R \propto M$ is a CS ring. Then M is a weakly IN R-module.*

Proof. Let N and L be two submodules of M such that $N \cap L = 0$. Then $(0,N)$ and $(0,L)$ are two ideals of A satisfying $(0,N) \cap (0,L) = 0$. By Proposition 2.1, $Ann_A(0,N) + Ann_A(0,L) = A$. It follows that $(Ann_R(N), M) + (Ann_R(L), M) = A$ and hence $Ann_R(N) + Ann_R(L) = R$. Therefore M is weakly IN. □

Lemma 5.2. *Let R be a ring and let M be a faithful R-module. Then the following are equivalent:*
(1) *$A = R \propto M$ is a CS ring;*
(2) *M is a weakly IN R-module;*
(3) *M is a strongly CS R-module.*

Proof. (1) $\Rightarrow$ (2) See Lemma 5.1.

(2) $\Leftrightarrow$ (3) This follows from Corollary 2.10.

(3) $\Rightarrow$ (1) Let I be an ideal of A and let $V = \{x \in M \mid (0,x) \in I\}$. Then V is a submodule of M and $I \cap (0,M) = (0,V)$. Since M is strongly CS, there exists $e = e^2 \in R$ such that $V \subseteq^{ess} eM$. First let us show that $I \subseteq (e,0)A = (eR,eM)$. Consider an element (a,x) of I. Then, for any element z of M, $(a,x)(0,z) = (0,az) \in I \cap (0,M) = (0,V)$. Thus $az \in V$ and hence $aM \subseteq V \subseteq eM$. Consequently, $(1-e)aM = 0$. But M is a faithful R-module, so $(1-e)a = 0$ and hence $a = ea \in eR$. Moreover, $(a,x)(1-e,0) = (a(1-e),(1-e)x) = (0,(1-e)x) \in I \cap (0,M) = (0,V)$. This gives $(1-e)x \in V \subseteq eM$. Thus $x - ex \in eM$ and so $x \in eM$. Therefore $I \subseteq (eR,eM)$. Now let us show that $I \subseteq^{ess} (eR,eM)$. Let $(0,0) \neq (ea,ex) \in (eR,eM)$, where $a \in R$ and $x \in M$. If $ea \neq 0$, then $eaM \neq 0$ since M is faithful. So there exists $y \in M$ such that $eay \neq 0$. But $0 \neq eay \in eM$ and $V \subseteq^{ess} eM$, so there exists $t \in R$ such that $0 \neq teay \in V$. Hence $(0,0) \neq (0,ty)(ea,ex) = (0,teay) \in (0,V) = I \cap (0,M)$. Therefore $(0,0) \neq (0,ty)(ea,ex) \in I$. Now suppose that $ea = 0$. Then $ex \neq 0$. Since $V \subseteq^{ess} eM$, there exists $u \in R$ such that $0 \neq uex \in V$. So $(0,0) \neq (u,0)(ea,ex) = (0,uex) \in (0,V) = I \cap (0,M)$. Hence $0 \neq (u,0)(ea,ex) \in I$. It follows that $I \subseteq^{ess} (e,0)A$. Note that $(e,0)^2 = (e,0)$. Consequently, A is a CS ring. □

Lemma 5.3. *Let R be a ring and let M be a nonzero R-module. Let $e^2 = e \in R$. Then the following hold true:*
(1) *eR is strongly CS as R-module if and only if eR is a CS ring.*
(2) *Assume that $Ann_R(M) = eR$. Then $(1-e)M$ is a faithful $(1-e)R$-module. Moreover, the rings $A = R \propto M$ and $eR \times ((1-e)R \propto (1-e)M)$ are isomorphic.*

Proof. (1) Note that the R-submodules and the eR-submodules of eR are the same.

($\Rightarrow$) Let I be an ideal of the ring eR. Since eR is a strongly CS R-module, there exists $f = f^2 \in R$ such that I is an essential R-submodule of feR. Thus I is an essential eR-submodule of $(fe)eR$. Moreover, $fe = (fe)^2 \in eR$. Therefore eR is a CS ring.

($\Leftarrow$) Let I be an ideal of R contained in eR. Then $I = eI$ is an ideal of eR. Since eR is a CS ring, there exists an idempotent $f \in eR$ such that I is an essential eR-submodule of feR. It is clear that I is also an essential R-submodule of feR.

(2) It is clear that $R = eR \oplus (1-e)R$ and $M = eM \oplus (1-e)M$. By [4, Theorem 4.4], $A = R \propto M \cong (eR \propto eM) \times ((1-e)R \propto (1-e)M)$ (as rings). But $eM = 0$, so the ring A is isomorphic to the ring $eR \times ((1-e)R \propto (1-e)M)$. The first assertion is obvious. □

Theorem 5.4. The following are equivalent for a ring R and an R-module M:
(1) $A = R \propto M$ is a CS ring;
(2) M satisfies the following two conditions:
(a) $Ann_R(M)$ is a direct summand of R which is a CS ring, and
(b) M is a weakly IN R-module;
(3) M satisfies the following two conditions:
(a) $Ann_R(M)$ is a direct summand of R which is a CS ring, and
(b) M is a strongly CS R-module;
(4) $M \oplus Ann_R(M)$ is a weakly IN R-module;
(5) $M \oplus Ann_R(M)$ is a strongly CS R-module.

Proof. It is easily seen that $\{(e,0) \in A \mid e^2 = e \in R\}$ is the set of idempotents of A.

(1) $\Rightarrow$ (2) (a) Suppose that $Ann_R(M) \neq 0$. Clearly, $(Ann_R(M),0)$ is an ideal of A. Since A is a CS ring, there exists an idempotent e of R such that $(Ann_R(M),0) \subseteq^{ess} (e,0)A = (eR,eM)$. Hence, $Ann_R(M) \subseteq eR$. We claim that $eM = 0$. Suppose, on the contrary, that $eM \neq 0$. Then $(0,eM)$ is a nonzero ideal of A which is contained in (eR,eM). Thus $(0,eM) \cap (Ann_R(M),0) \neq 0$, a contradiction. Therefore $eR \subseteq Ann_R(M)$. It follows that $Ann_R(M) = eR$ is a direct summand of R. Moreover, eR is a CS ring by Lemmas 3.2(2) and 5.3(2).

(b) follows from Lemma 5.1.

(2) $\Rightarrow$ (3) Use Corollary 2.10.

(3) $\Rightarrow$ (5) By hypothesis, there exists $e^2 = e \in R$ such that $Ann_R(M) = eR$. Thus $Ann_R(M) = eAnn_R(M)$. Moreover, since $R = eR \oplus (1-e)R$, we have $M = (1-e)M$. Also, M and $Ann_R(M)$ are strongly CS R-modules by Lemma 5.3(1). So, by Theorem 3.3, $M \oplus Ann_R(M)$ is a strongly CS R-module.

(5) $\Rightarrow$ (4) This follows from Theorem 2.8.

(4) $\Rightarrow$ (1) By Theorem 3.1, $Ann_R(M) + Ann_R(Ann_R(M)) = R$. Then there exist $a \in Ann_R(M)$ and $b \in Ann_R(Ann_R(M))$ such that $a + b = 1$. Let $c \in Ann_R(M) \cap Ann_R(Ann_R(M))$. Thus $c = ca + cb = 0$ and hence $Ann_R(M) \cap Ann_R(Ann_R(M)) = 0$. It follows that $Ann_R(M)$ is a direct summand of R. So $Ann_R(M) = eR$ for some idempotent e of R. Note that eR is a weakly IN R-module by Proposition 2.4. Moreover, the R-submodules and the eR-submodules of eR are exactly the same. Also, we have $Ann_{eR}(L) = eAnn_R(L)$ for any R-submodule L of eR. We thus deduce that eR is a CS ring by Proposition 2.1. In addition, using Proposition 2.4 again, we see that $(1-e)M$ is a weakly IN $(1-e)R$-module. Note that $(1-e)M$ is a faithful $(1-e)R$-module (see Lemma 5.3(2)). Then $(1-e)R \propto (1-e)M$ is a CS ring by Lemma 5.2. Since the rings $R \propto M$ and $eR \times ((1-e)R \propto (1-e)M)$ are isomorphic (see Lemma 5.3(2)), it follows that $R \propto M$ is a CS ring by Lemma 3.2(2). □

Remark 5.5. Let R be a ring and let M be an R-module such that $R \propto M$ is a CS ring. By Theorem 5.4, M is a strongly CS R-module. However, the ring R need not be a CS ring, in general, as it is shown in the following example. Let R be a local ring which is not uniform (for example, we can take $R = K \propto (K \oplus K)$, where K is a field) and let $\mathfrak{m}$ be the maximal ideal of R. Consider the R-module $M = E(R/\mathfrak{m})$ and the ring $A = R \propto M$. By [14, Corollary 2 of Proposition 2.26], M is a faithful R-module. Moreover, since M is a uniform R-module, M is strongly CS. Thus A is a CS ring by Lemma 5.2 (indeed, A is uniform). Since R is local, R is indecomposable and so R cannot be a CS ring since it is not uniform (see Remark 2.3(3)). However, when the R-module M is flat, then the condition $R \propto M$ is CS forces R to become CS as it is shown in the following result.

Corollary 5.6. *Let R be a ring and let M be a flat R-module. Then the following are equivalent:*
(1) $A = R \propto M$ is a CS ring;
(2) R and M are weakly IN R-modules and $Ann_R(M)$ is a direct summand of R;
(3) R and M are strongly CS R-modules and $Ann_R(M)$ is a direct summand of R.

Proof. (1) $\Rightarrow$ (3) By Theorem 5.4, M is a strongly CS R-module and $Ann_R(M)$ is a direct summand of R. To show that R is CS, take two ideals I and J of R such that $I \cap J = 0$. Since M is a flat R-module, we have $IM \cap JM = (I \cap J)M$ by [16, Proposition 8.5]. Thus $IM \cap JM = 0$. Consider the ideals $I' = (I, IM)$ and $J' = (J, JM)$ of A. Since $I' \cap J' = 0$ and A is CS, we have $Ann_A(I') + Ann_A(J') = A$ (see Proposition 2.1). It follows that

$$(Ann_R(I), Ann_M(I)) + (Ann_R(J), Ann_M(J)) = A.$$

So $Ann_R(I) + Ann_R(J) = R$. Using again Proposition 2.1, we conclude that R is a CS ring. Hence R is strongly CS.

(3) $\Rightarrow$ (2) This follows from Theorem 2.8.

(2) $\Rightarrow$ (1) Since R is weakly IN, R is a strongly CS ring by Remark 2.3(1). As $Ann_R(M)$ is a direct summand of R, it follows that $Ann_R(M)$ is a CS ring (Proposition 2.4 and Lemma 5.3(1)). Now use Theorem 5.4. □

Corollary 5.7. *Let H be a faithful ideal of R and let $T(R)$ be the total ring of fractions of R. Then the following are equivalent:*
(1) $R \propto H$ is a CS ring;
(2) $R \propto T(R)$ is a CS ring;
(3) R is a CS ring.

Proof. Note that $T(R)$ is a flat R-module (see [5, Corollary 3.6]).

(1) ⇒ (3) Let I and J be two ideals of R such that $I \cap J = 0$. So $IH \cap JH = 0$. But IH and JH are two R-submodules of the faithful R-module H. Moreover, H is a weakly IN R-module by Lemma 5.2. Therefore $Ann_R(IH) + Ann_R(JH) = R$. But $Ann_R(IH) = Ann_R(I)$ and $Ann_R(JH) = Ann_R(J)$ as H is faithful. Then $Ann_R(I) + Ann_R(J) = R$. By Proposition 2.1, we see that R is a CS ring.

(3) ⇒ (2) It is clear that $Ann_R(T(R)) = 0$. To prove that $R \propto T(R)$ is a CS ring, we only need to show that $T(R)$ is a strongly CS R-module (see Corollary 5.6). Let S denote the set of all non-zero-divisors in R. Then $T(R) = S^{-1}R$. Let I' be an ideal of $T(R)$. Then $I' = S^{-1}I$ for some ideal I of R. But R is a CS ring, so there exists $e^2 = e \in R$ such that $I \subseteq^{ess} eR$. It is easily seen that $I' = S^{-1}I \subseteq^{ess} eT(R)$. Consequently, $T(R)$ is a strongly CS R-module.

(2) ⇒ (1) By Corollary 5.6, R is a weakly IN R-module. Since H is an R-submodule of R, H is also a weakly IN R-module. According to Lemma 5.2, it follows that $R \propto H$ is a CS ring. □

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